

## A COUNTEREXAMPLE CONCERNING LINE-FREE GROUPS AND COMPLETE ERDŐS SPACE

JAN J. DIJKSTRA AND JAN VAN MILL

(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. We present a weakly closed, one-dimensional, line-free subgroup of the separable Banach space  $c$  that is not homeomorphic to complete Erdős space. The existence of this example disproves a conjecture of Dobrowolski, Grabowski, and Kawamura.

Complete Erdős space was first featured by Erdős in [8], who proved that it is totally disconnected and one-dimensional. It can be represented by, for instance,

$$\mathfrak{E}_c = \{z \in \ell^2 : z_i \in \mathbb{R} \setminus \mathbb{Q} \text{ for each } i \in \mathbb{N}\},$$

where  $\ell^2$  is the Hilbert space of square summable real sequences.  $\mathfrak{E}_c$  is a universal element of the class of almost zero-dimensional spaces; for background information see [11, 9, 3, 4, 5]. A subset of a topological space is called a *C-set* if it can be written as an intersection of clopen subsets of the space. A topological space is called *almost zero-dimensional* if every point has a neighbourhood basis consisting of C-sets. Every almost zero-dimensional space is at most one-dimensional; see [11, 10, 1].

An additive subgroup of a vector space is called *line-free* if it does not contain nontrivial linear subspaces. It is remarked in [2] that a topological classification of the line-free closed subgroups of Banach spaces produces a classification of all closed subgroups of Banach spaces. Let  $G$  be an arbitrary nondiscrete, weakly closed, line-free, additive subgroup of a separable Banach space  $E$ . Dobrowolski, Grabowski, and Kawamura [7] proved that  $G$  is homeomorphic to complete Erdős space whenever  $E$  is reflexive. In addition, Ancel, Dobrowolski, and Grabowski [2] showed that  $E$  contains zero-dimensional examples of such groups  $G$  precisely if  $E$  contains an isomorphic copy of  $c_0$ . These results prompted Dobrowolski, Grabowski, and Kawamura [7] to formulate the following

**Conjecture.** *Every separable, nondiscrete, weakly closed, one-dimensional, line-free subgroup of a Banach space is homeomorphic to  $\mathfrak{E}_c$ .*

We present a counterexample to this conjecture, thereby finding a new topological type that closed subgroups of Banach spaces can have. We shall distinguish our example from  $\mathfrak{E}_c$  by the following property of  $\mathfrak{E}_c$ . A topological space is called *somewhere zero-dimensional* if it contains a point at which the space is

---

Received by the editors December 8, 2004 and, in revised form, February 25, 2005.

2000 *Mathematics Subject Classification.* Primary 46B25, 54F45.

*Key words and phrases.* Banach space, line-free group, weakly closed, complete Erdős space, almost zero-dimensional.

©2006 American Mathematical Society  
Reverts to public domain 28 years from publication

zero-dimensional, that is, the point has a clopen neighbourhood basis. Dijkstra, van Mill, and Steprāns [6] have shown that  $\mathfrak{C}_c$  has the property that every point  $x \in \mathfrak{C}_c$  has a neighbourhood  $U$  such that every closed subset of  $U$  is either empty or somewhere zero-dimensional.

**Counterexample.** We construct our counterexample in the Banach space  $c$ . We find it convenient to represent  $c$  as the space of all continuous real-valued functions  $f$  on the convergent sequence  $\{0\} \cup \{1/n : n \in \mathbb{N}\}$ . The norm  $\|f\| = \sup\{|f(1/n)| : n \in \mathbb{N}\}$  makes  $c$  a separable Banach space. For  $n \in \mathbb{N}$  let  $\varphi_n$  be the element of the dual of  $c$  that is given by  $\varphi_n(f) = 2^n f(1/n)$ . Since  $\{\varphi_n : n \in \mathbb{N}\}$  is clearly a total sequence of functionals, we have that

$$G = \{f \in c : \varphi_n(f) \in \mathbb{Z} \text{ for each } n \in \mathbb{N}\}$$

is a line-free, weakly closed, additive subgroup of  $c$ . We first verify that  $G$  is almost zero-dimensional and hence that  $\dim G \leq 1$ . Consider an arbitrary closed  $\varepsilon$ -ball  $B_\varepsilon(f) = \{g \in c : \|g - f\| \leq \varepsilon\}$  in  $c$ . Let  $g \in G \setminus B_\varepsilon(f)$  and note that  $|g(1/n) - f(1/n)| > \varepsilon$  for some  $n \in \mathbb{N}$ . Then  $\{h \in G : h(1/n) = g(1/n)\}$  is an obviously clopen subset of  $G$  that is disjoint from  $B_\varepsilon(f)$ . Thus  $G \cap B_\varepsilon(f)$  is a  $C$ -set in  $G$ , proving the almost zero-dimensionality of  $G$ . The fact  $\dim G \leq 1$  also follows from [2, Theorem 3.1], when we note that the  $\varphi_n$ 's form a norming sequence; see Dijkstra and van Mill [5, Remark 30].

We shall now show with the method of Erdős [8] that for each  $\varepsilon > 0$  the set  $A = G \cap B_\varepsilon(\mathbf{0})$ , where  $\mathbf{0}$  stands for the zero function, is not zero-dimensional at each of its points. We may then conclude that  $\dim G \geq 1$  and that  $G$  is not homeomorphic to  $\mathfrak{C}_c$ . (However, according to [4, Propositions 6.3 and 6.10]  $G$  is homeomorphic to a dense subset of  $\mathfrak{C}_c$ .) Let  $f \in A$  be arbitrary. Since  $A = -A$  we may assume that  $f(0) \leq 0$ . Let  $U$  be a subset of  $A \cap B_{\varepsilon/3}(f)$  such that  $f \in U$ . We show that  $U$  has boundary points in  $A$ . For each  $n \in \mathbb{N}$  we let  $\alpha_n \in G$  be defined by  $\alpha_n(x) = 2^{-n}$  for  $x \leq 1/n$  and  $\alpha_n = 0$  for  $x > 1/n$ . Note that  $\|\alpha_n\| = \alpha_n(0) = 2^{-n}$ . We construct by recursion a sequence  $g_1, g_2, g_3, \dots$  in  $U$  as follows. We put  $g_1 = f$ . Assume that  $g_{n-1}$  has been found. Since  $U$  is bounded and  $g_{n-1} \in U$ , there is a  $k \in \{0\} \cup \mathbb{N}$  such that

$$g_{n-1} + k\alpha_n \in U \quad \text{and} \quad g_{n-1} + (k+1)\alpha_n \notin U.$$

Defining  $g_n = g_{n-1} + k\alpha_n$  we trivially have the following properties:

- (1)  $g_n \geq g_{n-1}$ ,
- (2)  $g_n + \alpha_n \in G \setminus U$ , and
- (3)  $\|g_n - g_{n-1}\| = g_n(0) - g_{n-1}(0)$ .

Since the sequence  $g_1(0), g_2(0), \dots$  is nondecreasing and bounded by  $\varepsilon$ , we have that it converges, say, to  $L$ . By property (3) we have  $\sum_{n=1}^{\infty} \|g_{n+1} - g_n\| = L - g_1(0)$ , thus  $g_1, g_2, \dots$  is a Cauchy sequence. Put  $g = \lim_{n \rightarrow \infty} g_n$  and note that  $g$  is in the closure of  $U$  in  $A$  because  $A$  is closed. Since the closure of  $U$  is contained in  $B_{\varepsilon/3}(f)$ , we have  $g(0) \leq f(0) + \varepsilon/3 \leq \varepsilon/3$ . Select an  $N \in \mathbb{N}$  such that  $2^{-N} < \varepsilon/3$  and  $g(1/n) < 2\varepsilon/3$  for each  $n > N$ . Let  $n > N$ . If  $i < n$ , then  $(g_n + \alpha_n)(1/i) = g_n(1/i) \in [-\varepsilon, \varepsilon]$ . If  $i \geq n$ , then

$$-\varepsilon \leq g_n(1/i) \leq (g_n + \alpha_n)(1/i) = g_n(1/i) + 2^{-n} \leq g(1/i) + \varepsilon/3 \leq \varepsilon.$$

Thus  $g_n + \alpha_n \in A$  for every  $n > N$ . With property (2) we have that  $g = \lim_{n \rightarrow \infty} (g_n + \alpha_n)$  is also in the closure of  $A \setminus U$ , thus  $U$  is not clopen in  $A$  and  $A$  is not zero-dimensional at  $f$ .

Since  $c$  is isomorphic to  $c_0$ , our construction also applies to that space and, by the Hahn-Banach Theorem, to every locally convex space that contains an isomorphic copy of  $c_0$ .

## REFERENCES

- [1] M. Abry and J. J. Dijkstra, *On topological Kadec norms*, Math. Ann. **332** (2005), 759–765. MR2179775
- [2] F. D. Ancel, T. Dobrowolski, and J. Grabowski, *Closed subgroups in Banach spaces*, Studia Math. **109** (1994), 277–290. MR1274013 (95d:46009)
- [3] J. J. Dijkstra and J. van Mill, *Homeomorphism groups of manifolds and Erdős space*, Electron. Res. Announc. Amer. Math. Soc. **10** (2004), 29–38. MR2048429 (2005c:57025)
- [4] J. J. Dijkstra and J. van Mill, *Erdős space and homeomorphism groups of manifolds*, preprint.
- [5] J. J. Dijkstra and J. van Mill, *Characterizing complete Erdős space*, preprint.
- [6] J. J. Dijkstra, J. van Mill and J. Steprāns, *Complete Erdős space is unstable*, Math. Proc. Cambridge Philos. Soc. **137** (2004), 465–473. MR2092071
- [7] T. Dobrowolski, J. Grabowski, and K. Kawamura, *Topological type of weakly closed subgroups in Banach spaces*, Studia Math. **118** (1996), 49–62. MR1373624 (97d:46013)
- [8] P. Erdős, *The dimension of the rational points in Hilbert space*, Ann. of Math. **41** (1940), 734–736. MR0003191 (2:178a)
- [9] K. Kawamura, L. G. Oversteegen, and E. D. Tymchatyn, *On homogeneous totally disconnected 1-dimensional spaces*, Fund. Math. **150** (1996), 97–112. MR1391294 (97d:54060)
- [10] M. Levin and R. Pol, *A metric condition which implies dimension  $\leq 1$* , Proc. Amer. Math. Soc. **125** (1997), 269–273. MR1389528 (97e:54033)
- [11] L. G. Oversteegen and E. D. Tymchatyn, *On the dimension of certain totally disconnected spaces*, Proc. Amer. Math. Soc. **122** (1994), 885–891. MR1273515 (95b:54040)

FACULTEIT DER EXACTE WETENSCHAPPEN / AFDELING WISKUNDE, VRIJE UNIVERSITEIT, DE BOELELAAN 1081<sup>a</sup>, 1081 HV AMSTERDAM, THE NETHERLANDS  
*E-mail address:* `dijkstra@cs.vu.nl`

FACULTEIT DER EXACTE WETENSCHAPPEN / AFDELING WISKUNDE, VRIJE UNIVERSITEIT, DE BOELELAAN 1081<sup>a</sup>, 1081 HV AMSTERDAM, THE NETHERLANDS  
*E-mail address:* `vanmill@cs.vu.nl`