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ON EXTENDED EIGENVALUES AND EXTENDED EIGENVECTORS OF SOME OPERATOR CLASSES

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ABSTRACT. We give a complete description of the set of extended eigenvectors of the Volterra integration operator V, $Vf(x) = \int_0^x f(t)dt$, on $L^2[0,1]$, which strengthens the result of a paper by Biswas, Lambert, and Petrovic (2002). We also introduce the concept of a well splitting operator and study its extended eigenvalues and extended eigenvectors.

1. INTRODUCTION AND PRELIMINARIES

Let E be a Banach space, and denote by $\mathcal{B}(E)$ the algebra of all bounded linear operators on E. If A is an operator in $\mathcal{B}(E)$ and λ is a complex number, then following Biswas, Lambert and Petrovic [1] we say that a complex number λ is an extended eigenvalue of A if there exists a nonzero operator X in $\mathcal{B}(E)$ such that $AX=\lambda XA$; such an operator X is called extended eigenvector corresponding to λ. The set of all extended eigenvalues of A will be called the extended point spectrum, and it will be denoted as $\sigma_p^{ext}(A)$. For a given $\lambda \in \sigma_p^{ext}(A)$ we define $\{A\}_\lambda^{'}$ as the set of all λ-extended eigenvectors for A ($\{A\}_1^{'}$ is $\{A\}^{'}$, the commutant of A). The set of all extended eigenvectors for A will be denoted as $\{A\}_{ext}^{'}$, i.e., $\{A\}_{ext}^{'} = \bigcup_{\lambda\in\sigma_p^{ext}(A)} \{A\}_\lambda^{'}$. The basic facts about the extended eigenvalues and extended eigenvectors of operators can be found in [1]–[3].

In what follows, V will denote the simple Volterra integration operator on $L^2[0,1]$ defined as

$$(Vf)(x) = \int_0^x f(t)dt.$$

It was established in [1] that the set of extended eigenvalues of the Volterra integration operator V is precisely the set $(0,\infty)$. Moreover, it was shown in [1], Theorem 6, that for each such extended eigenvalue λ, the appropriate extended eigenvector can be found in the class of integral operators. In other words, for each $\lambda > 0$, the equation

$$XV = \lambda VX$$

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has a nonzero integral operator as a solution. In this paper, we give a complete description of the set $\{V\}_{ext}^{'}$ of extended eigenvectors of the integration operator V on $L^2[0,1]$ (see Theorem 1 in Section 2) which strengthens Theorem 6 of the paper [1]. In Section 3 we study the set of extended eigenvalues and extended eigenvectors of so-called well splitting operators on a Banach space E. In Section 4 we give an application of Deddens algebras and Shulman subspaces to the "extended" spectral theory.

2. Extended eigenvectors for V

Let $\varphi : [0,1] \to [0,1]$ be a measurable function. A composition operator C_φ is defined as $C_\varphi f(x) = f(\varphi(x))$. The composition operator $C_{\lambda x}$, where $\lambda \in [0,1]$, will be denoted simply as C_λ. The so-called Duhamel operator $\mathcal{D}_f$ on $L^2[0,1]$ is defined as

$$\mathcal{D}_f\, g \overset{def}{=} (f \circledast g)(x) \overset{def}{=} \frac{d}{dx}\int_0^x f(x-t)g(t)dt. \tag{1}$$

The following result describes the set $\{V\}_{ext}^{'}$ of Volterra integration operator V on $L^2[0,1]$.

Theorem 1. *Let $\lambda \in (0,\infty)$ and let $X \in \mathcal{B}\left(L^2[0,1]\right)$ be a nonzero operator. Then:*
(i) *if $\lambda \le 1$, then $XV = \lambda VX$ if and only if $X = \mathcal{D}_{X\mathbf{1}}C_\lambda$, that is,*

$$(Xf)(x) = \frac{d}{dx}\int_0^x (X\mathbf{1})(x-t)f(\lambda t)dt, \qquad f \in L^2[0,1] \ ;$$

(ii) *if $\lambda > 1$, then $XV = \lambda VX$ if and only if $XC_{1/\lambda} = \mathcal{D}_{X\mathbf{1}}$.*

Proof. (i) Let $XV = \lambda VX$. Then

$$XV^n = \lambda^n V^n X$$

for any $n \ge 1$, that is,

$$XV^n f = \lambda^n V^n X f$$

for all $f \in L^2[0,1]$. In particular,

$$XV^n\mathbf{1} = \lambda^n V^n X\mathbf{1},$$

that is,

$$X\left(\frac{x^{n-1}}{(n-1)!} * \mathbf{1}\right) = \lambda^n \frac{x^{n-1}}{(n-1)!} * X\mathbf{1},$$

or

$$X\frac{x^n}{n!} = \lambda^n \frac{x^{n-1}}{(n-1)!} * X\mathbf{1}, \qquad n \ge 1.$$

$*-$ multiplying the last equality by $\mathbf{1}$ we obtain that

$$\mathbf{1} * X\frac{x^n}{n!} = \lambda^n\left(\mathbf{1} * \frac{x^{n-1}}{(n-1)!}\right) * X\mathbf{1}\ ,$$

that is,

$$\mathbf{1} * X\frac{x^n}{n!} = \frac{(\lambda x)^n}{n!} * X\mathbf{1}\ ,$$

for any $n \ge 1$. Hence

$$\mathbf{1} * Xp(x) = p(\lambda x) * X\mathbf{1}$$

for all polynomials p. Since $L^2[0,1]$ is a Banach algebra with respect to the convolution product $*$, it follows from the last equality that

$$\mathbf{1} * Xf(x) = f(\lambda x) * X\mathbf{1}$$

for all $f \in L^2[0,1]$. Since $Vf(x) = \mathbf{1} * f$ for every $f \in L^2[0,1]$, the last equality means that

$$VXf(x) = \mathcal{K}_{X\mathbf{1}} C_\lambda f(x),$$

where $\mathcal{K}_{X\mathbf{1}}$ denotes the usual convolution operator on $L^2[0,1]$, $\mathcal{K}_{X\mathbf{1}} g(x) \overset{def}{=} X\mathbf{1} * g = \int\limits_0^x (X\mathbf{1})(x-t)\, g(t)dt$, $g \in L^2[0,1]$. From this, by applying operator $\frac{d}{dx}$ we obtain that

$$(Xf)(x) = \frac{d}{dx} \mathcal{K}_{X\mathbf{1}} C_\lambda f(x),$$

that is,

$$(Xf)(x) = \frac{d}{dx} \int\limits_0^x (X\mathbf{1})(x-t)\, f(\lambda t) dt \ , \tag{2}$$

or

$$Xf = \mathcal{D}_{X\mathbf{1}} C_\lambda f$$

for all $f \in L^2[0,1]$, which means that $X = \mathcal{D}_{X\mathbf{1}} C_\lambda$. Now it suffices to show that if $X = \mathcal{D}_{X\mathbf{1}} C_\lambda$, then $XV = \lambda VX$. In fact, for all $f \in L^2[0,1]$ we have

$$\begin{aligned} XVf(x) &= \mathcal{D}_{X\mathbf{1}} C_\lambda f(x) = \mathcal{D}_{X\mathbf{1}} (Vf)(\lambda x) \\ &= X\mathbf{1} \circledast (Vf)(\lambda x) = X\mathbf{1} \circledast (\lambda x \circledast f(\lambda x)) \\ &= \lambda x \circledast (X\mathbf{1} \circledast f(\lambda x)) = \lambda x \circledast \mathcal{D}_{X\mathbf{1}} C_\lambda f(x) \\ &= \lambda (x \circledast \mathcal{D}_{X\mathbf{1}} C_\lambda f(x)) = \lambda V \mathcal{D}_{X\mathbf{1}} C_\lambda f(x) \\ &= \lambda VXf(x) \ , \end{aligned}$$

which completes the proof of (i).

(ii) Let $XV = \lambda VX$. Clearly $VX = \frac{1}{\lambda} XV$, which implies that $V^n X = X\left(\frac{1}{\lambda} V\right)^n$ for every $n \geq 1$. The same arguments as in the proof of (i) yield the equality

$$Xf\left(\frac{x}{\lambda}\right) = X\mathbf{1} \circledast f, \qquad f \in L^2[0,1] \ ,$$

where $\circledast$ is the Duhamel product defined by (1). Conversely, if $XC_{1/\lambda} = \mathcal{D}_{X\mathbf{1}}$, then since $Vf(x) = x \circledast f$ and $\mathcal{D}_{X\mathbf{1}} V = V \mathcal{D}_{X\mathbf{1}}$, for all polynomials p we have

$$\begin{aligned} \lambda VXp(x) &= \lambda VXC_{1/\lambda} C_\lambda p(x) \\ &= \lambda V \mathcal{D}_{X\mathbf{1}} C_\lambda p(x) = \lambda V \mathcal{D}_{X\mathbf{1}} p(\lambda x) \\ &= \lambda \mathcal{D}_{X\mathbf{1}} Vp(\lambda x) = \lambda XC_{1/\lambda} Vp(\lambda x) \\ &= \lambda XC_{1/\lambda} (x \circledast p(\lambda x)) = XC_{1/\lambda} (\lambda x \circledast p(\lambda x)) \\ &= XC_{1/\lambda} (Vp)(\lambda x) = XVp(x) \ , \end{aligned}$$

which completes the proof of (ii) (because the set of polynomials is dense in $L^2[0,1]$). Theorem 1 is proved. □

The following result was proved by a different method in [1], Theorem 7.

Corollary 2. *The composition operator C_φ satisfies the equation $C_\varphi V = \lambda V C_\varphi$, where $\lambda \in (0,1]$, if and only if $\varphi(x) = \lambda x$, $x \in [0,1]$.*

Proof. It is obvious that $C_\varphi \mathbf{1} = \mathbf{1}$. Then according to part (i) of Theorem 1 we have that $C_\varphi V = \lambda V C_\varphi$ if and only if

$$C_\varphi f(x) = \frac{d}{dx}\int_0^x f(\lambda t)dt,$$

that is,

$$C_\varphi f(x) = f(\lambda x) = C_\lambda f(x)$$

for all $f \in L^2[0,1]$, i.e., $C_\varphi = C_\lambda$, or $\varphi(x) = \lambda x$. This completes the proof. $\square$

3. Well splitting operators

Let X be a separable Banach space and $A \in \mathcal{B}(X)$. An operator A is called a splitting operator in X if for every $x \in X$ there exists a linear densely defined operator B_x (generally nonbounded) such that

$$A^n x = B_x y_n \tag{3}$$

for every $n, n = 0, 1, 2, ...,$ and for some complete system $\{y_n\}_{n\geq 0}$ of the space X. We say that the splitting operator A is well splitting if for every $x \in X$ the corresponding operators B_x in (3) are bounded in X. It is immediate from the last definition that a well splitting operator is cyclic if for some $x \in X$ an operator B_x has dense range in X. It is easy to see that the concept of splitting operator is a generalization of the so-called basis operator introduced by Nikolski [4].

Example 1. If X denotes any of the spaces $C^{(n)}[0,1]$ and $W_p^{(n)}[0,1]$ (Sobolev space), $1 \leq p < +\infty$, then it is not difficult to show that the Volterra integration operator V acting in X is a well splitting operator. Indeed, let us denote by $\circledast$ the Duhamel product defined as

$$(f \circledast g)(x) = \frac{d}{dx}\int_0^x f(x-t)g(t)dt \tag{4}$$

for all $f, g \in X$. Simple calculations show that X is a Banach algebra with respect to the Duhamel product $\circledast$. In particular, for any $f \in X$ the "Duhamel operator" $\mathcal{D}_f$, $\mathcal{D}_f\, g \stackrel{def}{=} f \circledast g$, is a bounded operator in X. Now it follows immediately from (4) that

$$V^n f = \frac{x^n}{n!} \circledast f \ , \ n \geq 0,$$

that is,

$$V^n f = \mathcal{D}_f\left(\frac{x^n}{n!}\right) \ , \ n \geq 0, \tag{5}$$

for all $f \in X$. By Weierstrass's approximation theorem $\left\{\frac{x^n}{n!}\right\}_{n\geq 0}$ is a complete system in X, and hence, formula (5) means that V is a well splitting operator in X.

In what follows, for every complete system $\{y_n\}_{n\geq 0}$ in X we will denote by the symbol $\Lambda_{\{y_n\}}$ the set of all complex numbers λ for which the diagonal operator $D_{\{\lambda\}}$, $D_{\{\lambda\}} y_n \stackrel{def}{=} \lambda^n y_n$, $n \geq 0$, is bounded in X.

The following result gives, in particular, more general examples of well splitting operators.

Theorem 3. *Let X be a Banach space with basis $\{e_n\}_{n\geq 0}$, and T, $Te_n = \lambda_n e_{n+1}$, $\lambda_n \neq 0, n \geq 0$, be a weighted shift operator continuously acting in X. Let us set $w_n \overset{def}{=} \lambda_0\lambda_1...\lambda_{n-1}$, $w_0 \overset{def}{=} 1$. Suppose that the following conditions are satisfied:*

1) *There exists an integer $N \geq 0$ such that*

$$\sum_{n,m\geq N}\left|\frac{w_{n+m}}{w_n w_m}\right| < +\infty;$$

2) $\|e_{n+m}\| \leq c\,\|e_n\|\,\|e_m\|$ *for all $n, m \geq 0$, and for some number $c > 0$.*

Then:

(i) *T is well splitting operator on a X;*

(ii) *If $\lambda \in \Lambda_{\{e_n\}}$ is a nonzero number and $A \in \mathcal{B}(X)$ is a nonzero operator, then $A \in \{T\}_\lambda^{'}$ if and only if $AD_{\{\lambda\}} = \mathcal{D}_{Ae_0}$.*

Proof. For the arbitrarily chosen elements $x = \sum_{n\geq 0} x_n e_n$ and $y = \sum_{n\geq 0} y_n e_n$ of the space X, let us define a discrete analogy of the Duhamel product by the following formula:

$$x \circledast y \overset{def}{=} \sum_{n,m\geq 0}\frac{w_{n+m}}{w_n w_m}x_n y_m e_{n+m}. \tag{6}$$

By virtue of the conditions of theorem, formula (6) is correctly defined. By setting $\tau_n(x) \overset{def}{=} \sum_{k\geq n} x_k e_k$ and using the conditions of the theorem we have:

$$\begin{aligned}
x \circledast y &= \sum_{n,m\geq 0}\frac{w_{n+m}}{w_n w_m}x_n y_m e_{n+m}\\
&= \sum_{n\geq 0}\frac{x_n}{w_n}\sum_{m\geq 0}\frac{w_{n+m}}{w_m}y_m e_{n+m}\\
&= x_0\sum_{m\geq 0}y_m e_m + \frac{x_1}{w_1}\sum_{m\geq 0}\frac{w_{m+1}}{w_m}y_m e_{m+1}\\
&\quad +\frac{x_2}{w_2}\sum_{m\geq 0}\frac{w_{m+2}}{w_m}y_m e_{m+2} + ...\\
&\quad +\frac{x_{N-1}}{w_{N-1}}\sum_{m\geq 0}\frac{w_{m+N-1}}{w_m}y_m e_{m+N-1}\\
&\quad +\sum_{n\geq N}\frac{x_n}{w_n}\sum_{m\geq 0}\frac{w_{n+m}}{w_m}y_m e_{n+m}\\
&= x_0 y + \frac{x_1}{w_1}Ty + ... + \frac{x_{N-1}}{w_{N-1}}T^{N-1}y\\
&\quad +y_0\tau_N(x) + \frac{y_1}{w_1}\tau_N(Tx) + ... + \frac{y_{N-1}}{w_{N-1}}\tau_N\left(T^{N-1}x\right)\\
&\quad +\sum_{n\geq N}\sum_{m\geq N}\frac{w_{n+m}}{w_n w_m}x_n y_m e_{n+m}.
\end{aligned}$$

Hence

$$\begin{aligned}\|x \circledast y\| &\leq |x_0| \|y\| + \left|\frac{x_1}{w_1}\right| \|Ty\| + ... + \left|\frac{x_{N-1}}{w_{N-1}}\right| \left\|T^{N-1}y\right\| \\ &\quad + |y_0| \|\tau_N(x)\| + \left|\frac{y_1}{w_1}\right| \|\tau_N(Tx)\| + ... + \left|\frac{y_{N-1}}{w_{N-1}}\right| \left\|\tau_N\left(T^{N-1}x\right)\right\| \\ &\quad + \sum_{n\geq N}\sum_{m\geq N}\left|\frac{w_{n+m}}{w_n w_m}\right| |x_n| |y_m| \|e_{n+m}\| \\ &\leq \widetilde{c}\|x\| \|y\| + c\sum_{n\geq N}\sum_{m\geq N}\left|\frac{w_{n+m}}{w_n w_m}\right| |x_n| \|e_n\| |y_m| \|e_m\| \\ &\leq C\|x\| \|y\|,\end{aligned}$$

that is,

$$\|x \circledast y\| \leq C\|x\| \|y\| \tag{7}$$

for all $x, y \in X$. It follows from (6) and (7) that X is a Banach algebra with respect to $\circledast$ and with unit e_0. For every $x \in X$ we define the following operator:

$$\mathcal{D}_x y = x \circledast y \ , \ y \in X.$$

It is clear from (6) that

$$T^n y = w_n e_n \circledast y$$

for every $y \in X$ and any $n \geq 0$, that is,

$$T^n y = \mathcal{D}_y\left(w_n e_n\right). \tag{8}$$

In fact, for every $y \in X$ and any $n \geq 0$ we have

$$\begin{aligned}T^n y &= T^n\left(\sum_{m\geq 0} y_m e_m\right) = \sum_{m\geq 0} y_m T^n e_m \\ &= \sum_{m\geq 0} y_m \lambda_m \lambda_{m+1} ... \lambda_{m+n-1} e_{m+n} = \sum_{m\geq 0} y_m \frac{w_{m+n}}{w_m} e_{m+n} \\ &= \sum_{m\geq 0} w_n y_m \frac{w_{n+m}}{w_n w_m} e_{n+m} = \sum_{m\geq 0} w_n y_m \left(e_n \circledast e_m\right) \\ &= w_n e_n \circledast \sum_{m\geq 0} y_m e_m = w_n e_n \circledast y = \mathcal{D}_y\left(w_n e_n\right),\end{aligned}$$

as desired. Hence, formula (8) means that T is a well splitting operator in X.

(ii) Clearly, if $\lambda AT = TA$, then $\lambda^n AT^n = T^n A$, $n \geq 0$. In particular,

$$A\lambda^n T^n e_0 = T^n A e_0 \ , \ n \geq 0.$$

Using formula (8), from this we obtain that

$$A w_n \lambda^n e_n = A\left(w_n \lambda^n e_n \circledast e_0\right) = \left(w_n e_n \circledast A e_0\right)$$

or

$$A\lambda^n e_n = \left(e_n \circledast A e_0\right),$$

that is,

$$A\lambda^n e_n = A e_0 \circledast e_n \ , \ n \geq 0.$$

From this

$$A P_\lambda = A e_0 \circledast P$$

for all polynomials $P = \sum_{n\geq 0} P_n e_n \in X$, where $P_\lambda \overset{def}{=} \sum_{n\geq 0} \lambda^n P_n e_n = D_{\{\lambda\}}P$, that is,

$$AD_{\{\lambda\}}P = Ae_0 \circledast P.$$

Since $(X, \circledast)$ is a Banach algebra (see inequality (7)), from this we deduce that

$$AD_{\{\lambda\}}x = Ae_0 \circledast x \tag{9}$$

for all $x \in X$, i.e.,

$$AD_{\{\lambda\}} = \mathcal{D}_{Ae_0}, \tag{10}$$

where $\mathcal{D}_{Ae_0}$ is the Duhamel operator.

Conversely, let us prove that every nonzero operator A satisfying (10) belongs to $\{T\}'_\lambda$. Indeed, for all vector polynomials $P = \sum_{m=0}^{\deg P} P_m e_m$ we have

$$\begin{aligned}
TAP &= TAD_{\{\lambda\}}D_{\{\frac{1}{\lambda}\}}P = TAD_{\{\lambda\}}P_{\{\frac{1}{\lambda}\}} = T\mathcal{D}_{Ae_0}P_{\{\frac{1}{\lambda}\}} \\
&= w_1e_1 \circledast \mathcal{D}_{Ae_0}P_{\{\frac{1}{\lambda}\}} = \mathcal{D}_{w_1e_1}\mathcal{D}_{Ae_0}P_{\{\frac{1}{\lambda}\}} = \mathcal{D}_{Ae_0}\mathcal{D}_{w_1e_1}P_{\{\frac{1}{\lambda}\}} \\
&= \mathcal{D}_{Ae_0}(w_1e_1 \circledast P_{\{\frac{1}{\lambda}\}}) = \mathcal{D}_{Ae_0}\lambda(\frac{w_1e_1}{\lambda} \circledast P_{\{\frac{1}{\lambda}\}}) \\
&= \lambda\mathcal{D}_{Ae_0}(\frac{w_1e_1}{\lambda} \circledast P_{\{\frac{1}{\lambda}\}}) = \lambda AD_{\{\lambda\}}(\frac{w_1e_1}{\lambda} \circledast \sum_{m=0}^{\deg P} P_m \frac{1}{\lambda^m} e_m) \\
&= \lambda AD_{\{\lambda\}}[\frac{1}{w_1}\frac{w_1}{\lambda}\sum_{m=0}^{\deg P}\frac{w_{m+1}}{w_m}\frac{P_m}{\lambda^m}e_{m+1}] \\
&= \lambda AD_{\{\lambda\}}[\frac{1}{\lambda}\sum_{m=0}^{\deg P}\lambda_m\frac{P_m}{\lambda^m}e_{m+1}] \\
&= \lambda AD_{\{\lambda\}}[\frac{1}{\lambda}\sum_{m=0}^{\deg P}\lambda\lambda_m\frac{P_m}{\lambda^{m+1}}e_{m+1}] \\
&= \lambda AD_{\{\lambda\}}[\sum_{m=0}^{\deg P}\lambda_m P_m\frac{1}{\lambda^{m+1}}e_{m+1}] \\
&= \lambda AD_{\{\lambda\}}[D_{\{\frac{1}{\lambda}\}}\sum_{m=0}^{\deg P}\lambda_m P_m e_{m+1}] \\
&= \lambda AD_{\{\lambda\}}D_{\{\frac{1}{\lambda}\}}T\sum_{m=0}^{\deg P}P_m e_m \\
&= \lambda ATP,
\end{aligned}$$

and so $TAx = \lambda ATx$ for all $x \in X$, that is, $A \in \{T\}'_\lambda$. The theorem is proved. □

Remark 1. It can be proved that if $\lambda \in \Lambda_{\{e_n\}}$, then $TA = \lambda AT$ if and only if

$$\mathcal{D}_{Ax} = A\mathcal{D}_x D_{\{\lambda\}}. \tag{11}$$

Indeed, if $TA = \lambda AT$, then we obtain that

$$\begin{aligned}
A\mathcal{D}_x D_{\{\lambda\}} w_n e_n &= A\mathcal{D}_x \lambda^n w_n e_n = A(x \circledast \lambda^n w_n e_n) \\
&= A(\lambda^n T^n x) = T^n Ax = w_n e_n \circledast Ax = \mathcal{D}_{Ax} w_n e_n,
\end{aligned}$$

for all $n \geq 0$, which obviously implies (11).

Conversely, if an operator A has the property (11), then we have

$$\begin{aligned} ATx &= A(w_1e_1 \circledast x) = A\mathcal{D}_x w_1 e_1 = \frac{1}{\lambda} A\mathcal{D}_x w_1 \lambda e_1 \\ &= \frac{w_1}{\lambda} A\mathcal{D}_x D_{\{\lambda\}} e_1 = \frac{w_1}{\lambda}\mathcal{D}_{Ax} e_1 = \frac{1}{\lambda}\mathcal{D}_{Ax} w_1 e_1 \\ &= \frac{1}{\lambda}(Ax \circledast w_1 e_1) = \frac{1}{\lambda} TAx \end{aligned}$$

for all $x \in X$, and hence $TA = \lambda AT$, which completes the proof.

Our next result shows that the property (11) is shared by any well splitting operator.

Theorem 4. *Let T be a well splitting operator on a separable Banach space X defined by a complete system $\{y_n\}_{n\geq 0}$, i.e.,*

$$T^n x = B_x y_n,\ n \geq 0,$$

for each $x \in X$. Let $\lambda \in \Lambda_{\{y_n\}}$ be any nonzero number. Then $\lambda AT = TA$ if and only if

$$B_{Ax} = AB_x D_{\{\lambda\}}$$

for each $x \in X$.

Proof. Let T be a well splitting operator on X, and let A be a bounded linear operator on X such that $\lambda AT = TA$. Then it is obvious that

$$\begin{aligned} B_{Ax} y_n &= T^n Ax = A\left(\lambda^n T^n\right) x = AB_x \lambda^n y_n \\ &= AB_x D_{\{\lambda\}} y_n \end{aligned}$$

for each $x \in X$ and n. Since the system $\{y_n\}_{n\geq 0}$ is complete in X, it follows that $B_{Ax} = AB_x D_{\{\lambda\}}$ for each $x \in X$. Conversely, if an operator $A \in \mathcal{B}(X)$ satisfies $B_{Ax} = AB_x D_{\{\lambda\}}$ for each $x \in X$, then we have that $B_{Ax} y = AB_x D_{\{\lambda\}} y$ for each $y \in X$; in particular, $B_{Ax} y_1 = AB_x D_{\{\lambda\}} y_1$. Then we have

$$\begin{aligned} TAx &= B_{Ax} y_1 = AB_x D_{\{\lambda\}} y_1 = AB_x \lambda y_1 \\ &= \lambda AB_x y_1 = \lambda ATx, \end{aligned}$$

for each $x \in X$, so it follows that $TA = \lambda AT$, that is, $A \in \{T\}_\lambda$, as desired. The proof is completed. □

Corollary 5. *$TA = AT$ if and only if $B_{Ax} = AB_x$ for each $x \in X$.*

4. An application of Deddens algebras and Shulman subspaces

In this short section Deddens operator algebras and Shulman subspaces (which were introduced in [5]) are used for the investigation of extended eigenvalues of operators. Throughout the text, H will signify a Hilbert space of generally unspecified dimension. The following definition is due to Deddens [6] (see [5] and [7] for general definitions).

Definition 1. *Let A be invertible in $\mathcal{B}(H)$:*

$$\mathcal{B}_A \overset{def}{=} \left\{X \in \mathcal{B}(H) : \left\|A^n X A^{-n}\right\| \text{ bounded for } n \geq 0\right\}.$$

Thus, $\mathcal{B}_A$ is the collection of operators having bounded conjugation orbits, conjugation being by A. In general, Deddens algebra $\mathcal{B}_A$ clearly contains $\{A\}^{'}$ and is an algebra because

$$\begin{aligned}\left\|A^n X_1 X_2 A^{-n}\right\| &= \left\|A^n X_1 A^{-n} A^n X_2 A^{-n}\right\| \\ &\leq \left\|A^n X_1 A^{-n}\right\| \left\|A^n X_2 A^{-n}\right\|.\end{aligned}$$

In [6], $\mathcal{B}_A$ was introduced as an alternative for the description of nest algebras, and this result suggests that the boundedness condition defining $\mathcal{B}_A$ is of interest for any invertible A.

For two operators $L, M \in \mathcal{B}(H)$ let us denote by $\mathcal{U}(L,M)$ the Shulman subspaces of the space $\mathcal{B}(H)$ (see [5]), defined by

$$\mathcal{U}(L,M) \stackrel{def}{=} \{L\}^{'} + \{L\}^{'} M.$$

Such subspaces have been studied in detail by Shulman in relation with nontransitivity of root algebras (see [8], [9]).

The relation between Deddens algebras and Shulman subspaces is established in the next theorem. (Below the number $\lambda \in \mathbb{C}$ is assumed to be such that $L_\lambda \stackrel{def}{=} \lambda I + L$ is an invertible operator.)

Theorem 6 ([5]). *Let the operators L, $M \in \mathcal{B}(H)$ satisfy the Kleinecke-Shirokov condition, i.e., $X \stackrel{def}{=} [M,L] \in \{L\}^{'}$. Then the intersection of Deddens algebra $\mathcal{B}_{\lambda I+L}$ and the weak closure of Shulman subspace $\mathcal{U}(L,M)$ coincide with a commutant of the operator L, that is,*

$$\mathcal{B}_{\lambda I+L} \cap \overline{\mathcal{U}(L,M)}^{w} = \{L\}^{'}.$$

In particular, since $[T,V] = V^2$, we have

$$\mathcal{B}_{I+V} \cap \overline{\mathcal{U}(V,T)}^{w} = \{V\}^{'},$$

where T is the multiplication operator in $L^2[0,1]$, $(Tf)(x) = xf(x)$.

Here we prove the following theorem.

Theorem 7. *Let λ be a nontrivial scalar in the unit circle $\mathbb{T}$ of a complex plane, i.e., $\lambda \in \mathbb{T}\setminus\{1\}$, and let $A \in \mathcal{B}(H)$ be an invertible operator satisfying the Kleinecke-Shirokov condition for some operator $M \in \mathcal{B}(H)$. Suppose that $\mathcal{B}_{A^{-1}} \subseteq \overline{\mathcal{U}(A,M)}^{w}$. If $\lambda XA = AX$, then $X = 0$.*

Proof. Let $\lambda XA = AX$. Then $\lambda^n XA^n = A^n X$, $n \geq 0$. From this $A^n XA^{-n} = \lambda^n X$, $n \geq 0$, and hence, $\|A^n XA^{-n}\| = \|X\|$, $n \geq 0$, which means that $X \in \mathcal{B}_A$. On the other hand, since $XA = \frac{1}{\lambda}AX$, we have that $XA^n = \frac{1}{\lambda^n}A^n X$, $n \geq 0$, from which it follows that $A^{-n}XA^n = \frac{1}{\lambda^n}X$, $n \geq 0$, that is, $\|A^{-n}XA^n\| = \|X\|$, $n \geq 0$, which implies that $X \in \mathcal{B}_{A-1}$. Since $\mathcal{B}_{A^{-1}} \subseteq \overline{\mathcal{U}(A,M)}^{w}$, we have that $X \in \overline{\mathcal{U}(A,M)}^{w}$. Thus, $X \in \mathcal{B}_A \cap \overline{\mathcal{U}(A,M)}^{w} = \{A\}^{'}$ (see Theorem 6), which yields that $(\lambda - 1)AX = 0$, that is, $X = 0$ as desired. The theorem is proved. □

Corollary 8. *Let $N \in \mathcal{B}(H)$ be a square-zero operator satisfying the Kleinecke-Shirokov condition for some operator $M \in \mathcal{B}(H)$. If $\lambda \in \mathbb{T}\setminus\{1\}$, and $\lambda X(I+N) = (I+N)X$, then $X = 0$.*

Proof. It is known (see, for instance, [5, 6]) that $\mathcal{B}_{(I+N)^{-1}} = \mathcal{B}_{I-N} = \{N\}^{'} \subset \overline{\mathcal{U}(I+N, M)}^{w}$, that is, the operator $I + N$ satisfies condition of Theorem 7, and the desired result is obtained. □

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