ON EXTENDED EIGENVALUES AND EXTENDED EIGENVECTORS OF SOME OPERATOR CLASSES

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Abstract. We give a complete description of the set of extended eigenvectors of the Volterra integration operator $V$, $V f(x) = \int_0^x f(t) dt$, on $L^2[0, 1]$, which strengthens the result of a paper by Biswas, Lambert, and Petrovic (2002). We also introduce the concept of a well splitting operator and study its extended eigenvalues and extended eigenvectors.

1. Introduction and preliminaries

Let $E$ be a Banach space, and denote by $B(E)$ the algebra of all bounded linear operators on $E$. If $A$ is an operator in $B(E)$ and $\lambda$ is a complex number, then following Biswas, Lambert and Petrovic [1] we say that a complex number $\lambda$ is an extended eigenvalue of $A$ if there exists a nonzero operator $X$ in $B(E)$ such that $AX = \lambda X A$; such an operator $X$ is called extended eigenvector corresponding to $\lambda$. The set of all extended eigenvalues of $A$ will be called the extended point spectrum, and it will be denoted as $\sigma_{ext}^p(A)$. For a given $\lambda \in \sigma_{ext}^p(A)$ we define $\{A\}_{\lambda}$ as the set of all $\lambda$-extended eigenvectors for $A$ ($\{A\}_1 = \{A\}_{\lambda}$, the commutant of $A$). The set of all extended eigenvectors for $A$ will be denoted as $\{A\}_{ext}$, i.e., $\{A\}_{ext} = \bigcup_{\lambda \in \sigma_{ext}^p(A)} \{A\}_{\lambda}$. The basic facts about the extended eigenvalues and extended eigenvectors of operators can be found in [1–3].

In what follows, $V$ will denote the simple Volterra integration operator on $L^2[0, 1]$ defined as

$$(V f)(x) = \int_0^x f(t) dt.$$ 

It was established in [1] that the set of extended eigenvalues of the Volterra integration operator $V$ is precisely the set $(0, \infty)$. Moreover, it was shown in [1], Theorem 6, that for each such extended eigenvalue $\lambda$, the appropriate extended eigenvector can be found in the class of integral operators. In other words, for each $\lambda > 0$, the equation

$$X V = \lambda V X$$

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has a nonzero integral operator as a solution. In this paper, we give a complete
description of the set \( \{ V \}_{\text{ext}} \) of extended eigenvectors of the integration operator \( V \)
on \( L^2 [0, 1] \) (see Theorem 1 in Section 2) which strengthens Theorem 6 of the paper [1]. In Section 3 we study the set of extended eigenvalues and extended eigenvectors of so-called well splitting operators on a Banach space \( E \). In Section 4 we give an application of Deddens algebras and Shulman subspaces to the “extended” spectral theory.

2. Extended eigenvectors for \( V \)

Let \( \varphi : [0, 1] \rightarrow [0, 1] \) be a measurable function. A composition operator \( C_\varphi \) is
defined as \( C_\varphi f(x) = f(\varphi(x)) \). The composition operator \( C_{\lambda x} \), where \( \lambda \in [0, 1] \), will
be denoted simply as \( C_\lambda \). The so-called Duhamel operator \( D_f \) on \( L^2 [0, 1] \) is defined
as
\[
D_f g \overset{\text{def}}{=} (f \ast g)(x) = \frac{d}{dx} \int_0^x f(x-t)g(t)dt.
\]

The following result describes the set \( \{ V \}_{\text{ext}} \) of Volterra integration operator \( V \) on \( L^2 [0, 1] \).

**Theorem 1.** Let \( \lambda \in (0, \infty) \) and let \( X \in B \left( L^2 [0, 1] \right) \) be a nonzero operator. Then:

(i) if \( \lambda \leq 1 \), then \( XV = \lambda VX \) if and only if \( X = D_1 C_\lambda \), that is,
\[
(Xf)(x) = \frac{d}{dx} \int_0^x (X1)(x-t)f(\lambda t)dt, \quad f \in L^2 [0, 1] ;
\]

(ii) if \( \lambda > 1 \), then \( XV = \lambda VX \) if and only if \( XC_{1/\lambda} = D_1 \).

**Proof.** (i) Let \( XV = \lambda VX \). Then
\[
 XV^n = \lambda^n VX^n
\]
for any \( n \geq 1 \), that is,
\[
 XV^n f = \lambda^n VX^n f
\]
for all \( f \in L^2 [0, 1] \). In particular,
\[
 XV^n 1 = \lambda^n VX^n 1,
\]
that is,
\[
 X \left( \frac{x^{n-1}}{(n-1)!} * 1 \right) = \lambda^n \frac{x^{n-1}}{(n-1)!} * X1,
\]
or
\[
 \frac{x^n}{n!} = \lambda^n \frac{x^{n-1}}{(n-1)!} * X1, \quad n \geq 1.
\]
* − multiplying the last equality by \( 1 \) we obtain that
\[
1 * X \frac{x^n}{n!} = \lambda^n \left( 1 * \frac{x^{n-1}}{(n-1)!} \right) * X1 ,
\]
that is,
\[
1 * X \frac{x^n}{n!} = \left( \frac{\lambda x}{n!} \right) * X1 ,
\]
for any \( n \geq 1 \). Hence
\[
1 * X p(x) = p(\lambda x) * X1.
\]
for all polynomials $p$. Since $L^2[0,1]$ is a Banach algebra with respect to the convolution product $\ast$, it follows from the last equality that

$$1 \ast Xf(x) = f(\lambda x) \ast X1$$

for all $f \in L^2[0,1]$. Since $Vf(x) = 1 \ast f$ for every $f \in L^2[0,1]$, the last equality means that

$$VXf(x) = \mathcal{K}_{X1}C_\lambda f(x),$$

where $\mathcal{K}_{X1}$ denotes the usual convolution operator on $L^2[0,1]$, $\mathcal{K}_{X1}g(x) \overset{def}{=} X1 \ast g = \int_0^x (X1)(x-t)g(t)dt$, $g \in L^2[0,1]$. From this, by applying operator $\frac{d}{dx}$ we obtain that

$$(Xf)(x) = \frac{d}{dx}\mathcal{K}_{X1}C_\lambda f(x),$$

that is,

$$Xf = D_{X1}C_\lambda f$$

for all $f \in L^2[0,1]$, which means that $X = D_{X1}C_\lambda$. Now it suffices to show that if $X = D_{X1}C_\lambda$, then $VX = \lambda VX$. In fact, for all $f \in L^2[0,1]$ we have

$$\begin{align*}
XVf(x) &= D_{X1}C_\lambda f(x) = D_{X1}(Vf)(\lambda x) \\
&= X1 \ast (Vf)(\lambda x) = X1 \ast (\lambda x \ast f(\lambda x)) \\
&= \lambda x \ast (X1 \ast f(\lambda x)) = \lambda x \ast D_{X1}C_\lambda f(x) \\
&= \lambda(x \ast D_{X1}C_\lambda f(x)) = \lambda VXf(x) \\
&= \lambda VXf(x),
\end{align*}$$

which completes the proof of (i).

(ii) Let $XV = \lambda VX$. Clearly $VX = \frac{1}{\lambda}XV$, which implies that $V^nX = X\left(\frac{1}{\lambda}V\right)^n$ for every $n \geq 1$. The same arguments as in the proof of (i) yield the equality

$$Xf\left(X1 \ast \frac{x}{\lambda}\right) = X1 \ast f, \quad f \in L^2[0,1],$$

where $\ast$ is the Duhamel product defined by (1). Conversely, if $XC_{1/\lambda} = D_{X1}$, then since $Vf(x) = x \ast f$ and $D_{X1}V = VD_{X1}$, for all polynomials $p$ we have

$$\begin{align*}
\lambda VXp(x) &= \lambda VX1/\lambda C_\lambda p(x) \\
&= \lambda VD_{X1}C_\lambda p(x) = \lambda VD_{X1}p(\lambda x) \\
&= \lambda D_{X1}Vp(\lambda x) = \lambda XC_{1/\lambda}Vp(\lambda x) \\
&= \lambda XC_{1/\lambda}(x \ast p(\lambda x)) = XC_{1/\lambda}(\lambda x \ast p(\lambda x)) \\
&= XC_{1/\lambda}(Vp)(\lambda x) = XVP(x),
\end{align*}$$

which completes the proof of (ii) (because the set of polynomials is dense in $L^2[0,1]$). Theorem 1 is proved.

The following result was proved by a different method in [1], Theorem 7.

**Corollary 2.** The composition operator $C_\varphi$ satisfies the equation $C_\varphi V = \lambda VC_\varphi$, where $\lambda \in (0,1]$, if and only if $\varphi(x) = \lambda x$, $x \in [0,1]$. 

Proof. It is obvious that $C_\varphi 1 = 1$. Then according to part (i) of Theorem 1 we have that $C_\varphi V = \lambda VC_\varphi$, if and only if

$$C_\varphi f(x) = \frac{d}{dx} \int_0^x f(\lambda t) dt,$$

that is,

$$C_\varphi f(x) = f(\lambda x) = C_\lambda f(x)$$

for all $f \in L^2[0,1]$, i.e., $C_\varphi = C_\lambda$, or $\varphi(x) = \lambda x$. This completes the proof. □

3. Well splitting operators

Let $X$ be a separable Banach space and $A \in B(X)$. An operator $A$ is called a splitting operator in $X$ if for every $x \in X$ there exists a linear densely defined operator $B_x$ (generally nonbounded) such that

$$A^n x = B_x y_n$$

for every $n$, $n = 0, 1, 2, \ldots$, and for some complete system $\{y_n\}_{n \geq 0}$ of the space $X$. We say that the splitting operator $A$ is well splitting if for every $x \in X$ the corresponding operators $B_x$ in (3) are bounded in $X$. It is immediate from the last definition that a well splitting operator is cyclic if for some $x \in X$ an operator $B_x$ has dense range in $X$. It is easy to see that the concept of splitting operator is a generalization of the so-called basis operator introduced by Nikolski [4].

Example 1. If $X$ denotes any of the spaces $C^{(n)}[0,1]$ and $W_p^{(n)}[0,1]$ (Sobolev space), $1 \leq p < +\infty$, then it is not difficult to show that the Volterra integration operator $V$ acting in $X$ is a well splitting operator. Indeed, let us denote by $\odot$ the Duhamel product defined as

$$(f \odot g)(x) = \frac{d}{dx} \int_0^x f(x-t)g(t)dt$$

for all $f, g \in X$. Simple calculations show that $X$ is a Banach algebra with respect to the Duhamel product $\odot$. In particular, for any $f \in X$ the “Duhamel operator” $D_f$, $D_f g \overset{df}{=} f \odot g$, is a bounded operator in $X$. Now it follows immediately from (4) that

$$V^n f = \frac{x^n}{n!} \odot f, \ n \geq 0,$$

that is,

$$V^n f = D_f \left( \frac{x^n}{n!} \right), \ n \geq 0,$$

for all $f \in X$. By Weierstrass’s approximation theorem $\{\frac{x^n}{n!}\}_{n \geq 0}$ is a complete system in $X$, and hence, formula (5) means that $V$ is a well splitting operator in $X$.

In what follows, for every complete system $\{y_n\}_{n \geq 0}$ in $X$ we will denote by the symbol $\Lambda_{\{y_n\}}$ the set of all complex numbers $\lambda$ for which the diagonal operator $D_{\{\lambda\}}, D_{\{\lambda\}} y_n \overset{df}{=} \lambda^n y_n, \ n \geq 0,$ is bounded in $X$.

The following result gives, in particular, more general examples of well splitting operators.
Theorem 3. Let $X$ be a Banach space with basis $\{e_n\}_{n \geq 0}$, and $T, Te_n = \lambda_n e_{n+1}$, $\lambda_n \neq 0, n \geq 0$, be a weighted shift operator continuously acting in $X$. Let us set $w_n \overset{def}{=} \lambda_0 \lambda_1 \cdots \lambda_{n-1}$, $w_0 \overset{def}{=} 1$. Suppose that the following conditions are satisfied:

1) There exists an integer $N \geq 0$ such that

$$\sum_{n,m \geq N} \left| \frac{w_{n+m}}{w_n w_m} \right| < +\infty;$$

2) $\|e_{n+m}\| \leq c \|e_n\| \|e_m\|$ for all $n, m \geq 0$, and for some number $c > 0$.

Then:

(i) $T$ is well splitting operator on a $X$;

(ii) If $\lambda \in \Lambda_{\{e_n\}}$ is a nonzero number and $A \in \mathcal{B}(X)$ is a nonzero operator, then $A \in \{T\}_\lambda$ if and only if $AD_{\{\lambda\}} = DA_{e_0}$.

Proof. For the arbitrarily chosen elements $x = \sum_{n \geq 0} x_n e_n$ and $y = \sum_{n \geq 0} y_n e_n$ of the space $X$, let us define a discrete analogy of the Duhamel product by the following formula:

$$x \circ y \overset{def}{=} \sum_{n,m \geq 0} \frac{w_{n+m}}{w_n w_m} x_n y_m e_{n+m}. \tag{6}$$

By virtue of the conditions of theorem, formula (6) is correctly defined. By setting $\tau_n(x) \overset{def}{=} \sum_{k \geq n} x_k e_k$ and using the conditions of the theorem we have:

$$x \circ y = \sum_{n,m \geq 0} \frac{w_{n+m}}{w_n w_m} x_n y_m e_{n+m}$$

$$= \sum_{n \geq 0} x_n \sum_{m \geq 0} \frac{w_{n+m}}{w_m} y_m e_{n+m}$$

$$= x_0 y + \frac{x_1}{w_1} Ty + \ldots + \frac{x_N-1}{w_{N-1}} T^{N-1} y$$

$$+ y_0 \tau_N(x) + \frac{y_1}{w_1} \tau_N(Tx) + \ldots + \frac{y_{N-1}}{w_{N-1}} \tau_N(T^{N-1}x)$$

$$+ \sum_{n \geq N} \sum_{m \geq N} \frac{w_{n+m}}{w_n w_m} x_n y_m e_{n+m}.$$
Hence
\[
\| x \odot y \| \leq \| x_0 \| \| y \| + \frac{x_1}{w_1} \| Ty \| + \ldots + \frac{x_{N-1}}{w_{N-1}} \| T^{N-1} y \|
\]
\[
+ \| y_0 \| \tau_N (x) + \frac{y_1}{w_1} \| \tau_N (Tx) \| + \ldots + \frac{y_{N-1}}{w_{N-1}} \| \tau_N (T^{N-1} x) \|
\]
\[
+ \sum_{n \geq N} \sum_{m \geq N} \left| \frac{w_{n+m}}{w_n w_m} \right| |x_n| |y_m| \| e_{n+m} \|
\]
\[
\leq C \| x \| \| y \| ,
\]
that is,
\[(7) \quad \| x \odot y \| \leq C \| x \| \| y \|
\]
for all $x, y \in X$. It follows from (6) and (7) that $X$ is a Banach algebra with respect to $\odot$ and with unit $e_0$. For every $x \in X$ we define the following operator:
\[
D_x y = x \odot y , \quad y \in X.
\]
It is clear from (6) that
\[
T^n y = w_n e_n \odot y
\]
for every $y \in X$ and any $n \geq 0$, that is,
\[(8) \quad T^n y = D_y (w_n e_n).
\]
In fact, for every $y \in X$ and any $n \geq 0$ we have
\[
T^n y = T^n \left( \sum_{m \geq 0} y_m e_m \right) = \sum_{m \geq 0} y_m T^n e_m
\]
\[
= \sum_{m \geq 0} y_m \lambda_m \lambda_{m+1} \ldots \lambda_{m+n-1} e_{m+n} = \sum_{m \geq 0} y_m \frac{w_{n+m}}{w_m} e_{m+n}
\]
\[
= \sum_{m \geq 0} w_n y_m \frac{w_{n+m}}{w_n w_m} e_{n+m} = \sum_{m \geq 0} w_n y_m (e_n \odot e_m)
\]
\[
= w_n e_n \odot \sum_{m \geq 0} y_m e_m = w_n e_n \odot y = D_y (w_n e_n),
\]
as desired. Hence, formula (8) means that $T$ is a well splitting operator in $X$.

(ii) Clearly, if $\lambda A T = T A$, then $\lambda^n A T^n = T^n A$, $n \geq 0$. In particular,
\[
A \lambda^n T^n e_0 = T^n A e_0 , \quad n \geq 0.
\]
Using formula (8), from this we obtain that
\[
A w_n \lambda^n e_n = A (w_n \lambda^n e_n \odot e_0) = (w_n e_n \odot A e_0)
\]
or
\[
A \lambda^n e_n = (e_n \odot A e_0),
\]
that is,
\[
A \lambda^n e_n = A e_0 \odot e_n , \quad n \geq 0.
\]
From this
\[
A P_\lambda = A e_0 \odot P
\]
for all polynomials $P = \sum_{n \geq 0} P_n e_n \in X$, where $P_{\lambda} \overset{\text{def}}{=} \sum_{n \geq 0} \lambda^n P_n e_n = D(\lambda) P$, that is,

$$AD(\lambda) P = Ae_0 \odot P.$$  

Since $(X, \odot)$ is a Banach algebra (see inequality $(7)$), from this we deduce that

$$AD(\lambda)x = Ae_0 \odot x$$

for all $x \in X$, i.e.,

$$AD(\lambda) = DA_{e_0},$$

where $DA_{e_0}$ is the Duhamel operator.

Conversely, let us prove that every nonzero operator $A$ satisfying (10) belongs to $\{T\}^{\prime}_\lambda$. Indeed, for all vector polynomials $P = \sum_{m=0}^{\deg P} P_m e_m$ we have

$$TAP = TAD(\lambda)D\{\frac{1}{\lambda}\} P = TAD(\lambda)P\{\frac{1}{\lambda}\} = TDA_{e_0}P\{\frac{1}{\lambda}\}$$

$$ = w_1 e_1 \odot DA_{e_0}P\{\frac{1}{\lambda}\} = D_{e_1}DA_{e_0}P\{\frac{1}{\lambda}\} = DA_{e_0}D_{e_1}P\{\frac{1}{\lambda}\}$$

$$ = DA_{e_0}\left(\frac{w_1 e_1}{\lambda} \odot P\{\frac{1}{\lambda}\}\right) = \lambda DA_{e_0}\left(\frac{w_1 e_1}{\lambda} \odot \sum_{m=0}^{\deg P} P_m \frac{1}{\lambda^m} e_m\right)$$

$$ = \lambda DA_{e_0}\left[\frac{1}{\lambda} w_1 \sum_{m=0}^{\deg P} \frac{w_m+1}{w_m} \frac{P_m}{\lambda^m} e_{m+1}\right]$$

$$ = \lambda DA_{e_0}\left[\frac{1}{\lambda} \sum_{m=0}^{\deg P} \frac{P_m}{\lambda^m} e_{m+1}\right]$$

$$ = \lambda DA_{e_0}\left[\frac{1}{\lambda} \sum_{m=0}^{\deg P} \frac{P_m}{\lambda^{m+1}} e_{m+1}\right]$$

$$ = \lambda DA_{e_0}\left[\frac{1}{\lambda} \sum_{m=0}^{\deg P} \frac{P_m}{\lambda^{m+1}} e_{m+1}\right]$$

$$ = \lambda DA_{e_0}\left[D\{\frac{1}{\lambda}\} \sum_{m=0}^{\deg P} \frac{P_m}{\lambda^{m+1}} e_{m+1}\right]$$

$$ = \lambda DA_{e_0}\left[D\{\frac{1}{\lambda}\} T \sum_{m=0}^{\deg P} P_m e_m\right]$$

$$ = \lambda ATP,$$

and so $TAx = \lambda ATx$ for all $x \in X$, that is, $A \in \{T\}^{\prime}_\lambda$. The theorem is proved. □

**Remark 1.** It can be proved that if $\lambda \in \Lambda_{\{e_n\}}$, then $TA = \lambda AT$ if and only if

$$DA_{x} = AD_{x}D(\lambda).$$

Indeed, if $TA = \lambda AT$, then we obtain that

$$AD_{x}D(\lambda) w_n e_n = AD_{x} \lambda^n w_n e_n = A(x \odot \lambda^n w_n e_n)$$

$$ = A(\lambda^n T^n x) = T^n Ax = w_n e_n \odot Ax = DA_{x} w_n e_n,$$

for all $n \geq 0$, which obviously implies (11).
Conversely, if an operator $A$ has the property (11), then we have

$$ATx = A(w_1e_1 \otimes x) = AD_xw_1e_1 = \frac{1}{\lambda} AD_xw_1\lambda e_1$$

$$= \frac{w_1}{\lambda} AD_xD_{\{\lambda\}}e_1 = \frac{w_1}{\lambda} D_{Ax}e_1 = \frac{1}{\lambda} D_{Ax}w_1e_1$$

$$= \frac{1}{\lambda}(Ax \otimes w_1e_1) = \frac{1}{\lambda} TAx$$

for all $x \in X$, and hence $TA = \lambda AT$, which completes the proof.

Our next result shows that the property (11) is shared by any well splitting operator.

**Theorem 4.** Let $T$ be a well splitting operator on a separable Banach space $X$ defined by a complete system $\{y_n\}_{n \geq 0}$, i.e.,

$$T^nx = B_x y_n, \quad n \geq 0,$$

for each $x \in X$. Let $\lambda \in \Lambda_{\{y_n\}}$ be any nonzero number. Then $\lambda AT = TA$ if and only if

$$B_{Ax} = AB_{x} D_{\{\lambda\}}$$

for each $x \in X$.

**Proof.** Let $T$ be a well splitting operator on $X$, and let $A$ be a bounded linear operator on $X$ such that $\lambda AT = TA$. Then it is obvious that

$$B_{Ax} y_n = T^n Ax = A(\lambda^n T^n) x = AB_{x} \lambda^n y_n$$

$$= AB_{x} D_{\{\lambda\}} y_n$$

for each $x \in X$ and $n$. Since the system $\{y_n\}_{n \geq 0}$ is complete in $X$, it follows that $B_{Ax} = AB_{x} D_{\{\lambda\}}$ for each $x \in X$. Conversely, if an operator $A \in \mathcal{B}(X)$ satisfies $B_{Ax} = AB_{x} D_{\{\lambda\}}$ for each $x \in X$, then we have that $B_{Ax} y = AB_{x} D_{\{\lambda\}} y$ for each $y \in X$; in particular, $B_{Ax} y_1 = AB_{x} D_{\{\lambda\}} y_1$. Then we have

$$TAx = B_{Ax} y_1 = AB_{x} D_{\{\lambda\}} y_1 = AB_{x} \lambda y_1$$

$$= \lambda AB_{x} y_1 = \lambda ATx,$$

for each $x \in X$, so it follows that $TA = \lambda AT$, that is, $A \in \{T\}_{\lambda}$, as desired. The proof is completed. \hfill $\square$

**Corollary 5.** $TA = AT$ if and only if $B_{Ax} = AB_{x}$ for each $x \in X$.

4. **An application of Deddens algebras and Shulman subspaces**

In this short section Deddens operator algebras and Shulman subspaces (which were introduced in [5]) are used for the investigation of extended eigenvalues of operators. Throughout the text, $H$ will signify a Hilbert space of generally unspecified dimension. The following definition is due to Deddens [6] (see [5] and [7] for general definitions).

**Definition 1.** Let $A$ be invertible in $\mathcal{B}(H)$:

$$\mathcal{B}_A \overset{def}{=} \{X \in \mathcal{B}(H) : \|A^n X A^{-n}\| \text{ bounded for } n \geq 0\}.$$
Thus, $B_A$ is the collection of operators having bounded conjugation orbits, conjugation being by $A$. In general, Deddens algebra $B_A$ clearly contains $\{A\}'$ and is an algebra because

$$
\|A^n X_1 X_2 A^{-n}\| = \|A^n X_1 A^{-n} A^n X_2 A^{-n}\| 
\leq \|A^n X_1 A^{-n}\| \|A^n X_2 A^{-n}\|.
$$

In [5], $B_A$ was introduced as an alternative for the description of nest algebras, and this result suggests that the boundedness condition defining $B_A$ is of interest for any invertible $A$.

For two operators $L, M \in \mathcal{B}(H)$ let us denote by $\mathcal{U}(L, M)$ the Shulman subspaces of the space $\mathcal{B}(H)$ (see [5]), defined by

$$
\mathcal{U}(L, M) \overset{\text{def}}{=} \{L\}' + \{L\}' M.
$$

Such subspaces have been studied in detail by Shulman in relation with nontransitivity of root algebras (see [5], [9]).

The relation between Deddens algebras and Shulman subspaces is established in the next theorem. (Below the number $\lambda \in \mathbb{C}$ is assumed to be such that $L_{\lambda} \overset{\text{def}}{=} \lambda I + L$ is an invertible operator.)

**Theorem 6 ([5]).** Let the operators $L, M \in \mathcal{B}(H)$ satisfy the Kleinecke-Shirokov condition, i.e., $X \overset{\text{def}}{=} [M, L] \in \{L\}'$. Then the intersection of Deddens algebra $B_{L_{\lambda} + L}$ and the weak closure of Shulman subspace $\mathcal{U}(L, M)$ coincide with a commutant of the operator $L$, that is,

$$
B_{L_{\lambda} + L} \cap \overline{\mathcal{U}(L, M)}^w = \{L\}'.
$$

In particular, since $[T, V] = V^2$, we have

$$
B_{I_+ V} \cap \overline{\mathcal{U}(V, T)}^w = \{V\}',
$$

where $T$ is the multiplication operator in $L^2 [0, 1]$, $(Tf)(x) = xf(x)$.

Here we prove the following theorem.

**Theorem 7.** Let $\lambda$ be a nontrivial scalar in the unit circle $\mathbb{T}$ of a complex plane, i.e., $\lambda \in \mathbb{T} \setminus \{1\}$, and let $A \in \mathcal{B}(H)$ be an invertible operator satisfying the Kleinecke-Shirokov condition for some operator $M \in \mathcal{B}(H)$. Suppose that $B_{A^{-1}} \subseteq \overline{\mathcal{U}(A, M)}^w$. If $\lambda X A = AX$, then $X = 0$.

**Proof.** Let $\lambda X A = AX$. Then $\lambda^n X A^n = A^n X$, $n \geq 0$. From this $A^n X A^{-n} = \lambda^n X$, $n \geq 0$, and hence, $\|A^n X A^{-n}\| = \|X\|$, $n \geq 0$, which means that $X \in B_A$. On the other hand, since $X A^n = \frac{1}{\lambda^n} A^n X$, we have that $X A^n = \frac{1}{\lambda^n} A^n X$, $n \geq 0$, from which it follows that $A^{-n} X A^n = \frac{1}{\lambda^n} X$, $n \geq 0$, that is, $\|A^{-n} X A^n\| = \|X\|$, $n \geq 0$, which implies that $X \in B_{A^{-1}}$. Since $B_{A^{-1}} \subseteq \overline{\mathcal{U}(A, M)}^w$, we have that $X \in \overline{\mathcal{U}(A, M)}^w$. Thus, $X \in B_A \cap \overline{\mathcal{U}(A, M)}^w = \{A\}'$ (see Theorem 6), which yields that $(\lambda - 1) AX = 0$, that is, $X = 0$ as desired. The theorem is proved. \(\Box\)

**Corollary 8.** Let $N \in \mathcal{B}(H)$ be a square-zero operator satisfying the Kleinecke-Shirokov condition for some operator $M \in \mathcal{B}(H)$. If $\lambda \in \mathbb{T} \setminus \{1\}$, and $\lambda X (I + N) = (I + N) X$, then $X = 0$. 

Proof. It is known (see, for instance, [5, 6]) that $B_{I+N}^{-1} = B_{I-N} = \{N\} \subset \mathcal{U}(I+N, M)^{w}$, that is, the operator $I + N$ satisfies condition of Theorem 7, and the desired result is obtained. □

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References


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