

## INTERPOLATION THEOREM ON LORENTZ SPACES OVER WEIGHTED MEASURE SPACES

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ABSTRACT. In 1997 Ferreyra proved that it is impossible to extend the Stein-Weiss theorem in the context of Lorentz spaces. In this paper we obtain an interpolation theorem on Lorentz spaces over weighted measure spaces.

### 1. INTRODUCTION

The aim of the paper is to prove an interpolation theorem on Lorentz spaces over weighted measure spaces.

In 1966, Calderón [2] and Hunt [4] proved an interpolation theorem on Lorentz spaces  $L_{p,q}$ . On the other hand, in 1958, Stein and Weiss [7] proved an interpolation theorem on  $L_p$  spaces which allows one to change measures simultaneously with changing exponents. Thus it is natural to seek an interpolation theorem on  $L_{p,q}$  spaces which allows one to change measures simultaneously with changing exponents. However, in 1997, Ferreyra [3] gave a negative result for such an interpolation.

In [5], the following result was obtained; notation is given in section 2.

Let  $i = 0, 1$ . Suppose that  $0 < p_0 < p_1 \leq \infty$ ,  $0 < r_i \leq \infty$  with  $r_0 \neq r_1$ ,  $0 < q_i \leq \infty$ ,  $0 < s_i \leq \infty$  and  $\mathbb{D} = (-\alpha, -\alpha) \in \mathbb{R}^2$  with  $\alpha > 1$ . Put  $1/p = (1 - \theta)/p_0 + \theta/p_1$ ,  $1/r = (1 - \theta)/r_0 + \theta/r_1$  for  $0 < \theta < 1$ . Let  $v_i, w_i$  be nonnegative measurable functions. If  $T$  is a quasi-linear operator bounded from  $L_{p_i, q_i; v_i}$  to  $L_{r_i, s_i; w_i}$ , then there exists a constant  $C$  such that  $\|Tf\|_{r, s; \mathbb{D}; w} \leq C\|f\|_{p, q; v}$  for every linear combination  $f$  of two characteristic functions with disjoint supports, where  $q \leq s$  and  $v^{1/p} = v_0^{(1-\theta)/p_0} v_1^{\theta/p_1}$ ,  $w^{1/r} = w_0^{(1-\theta)/r_0} w_1^{\theta/r_1}$ .

In this paper, we prove an interpolation theorem of the above type. Under the hypothesis that  $T$  is nonnegative, and bounded from  $L_{p_i, q_i; v}$  to  $L_{r_i, s_i; w_i}$ , we obtain the inequality  $\|Tf\|_{r, \infty; w} \leq C\|f\|_{p, 1; v}$ . Note that the domain space is a Lorentz space  $L_{p, 1; v}$  with fixed weight  $v$ . This conclusion is weak in the sense that the target  $L_{r, \infty}$  is large and the domain  $L_{p, 1}$  is small in the framework of  $L_{p, q}$  classes.

The plan of the paper is as follows. In preliminary section 2 we give definitions and lemmas. In section 3 we state our main theorem and give its proof.

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## 2. PRELIMINARIES

Let  $(M, \mathfrak{M}, \mu)$  be a resonant measure space and let  $f$  be a complex-valued or real-valued measurable function defined on  $M$ .

**Definition 2.1.** Let  $w$  be a nonnegative measurable function on  $M$ . Then the distribution function of  $f$  with respect to  $w$  is defined as follows:

$$\lambda_{f,w}(y) = w(\{x \in M : |f(x)| > y\}), \quad y > 0,$$

where  $w(A) = \int_A w d\mu$  for every  $A \in \mathfrak{M}$ . Then the nonincreasing rearrangement of  $f$  with respect to  $w$  is defined as follows:

$$f_w^*(t) = \inf\{y : \lambda_{f,w}(y) \leq t\}, \quad t > 0.$$

The average function of  $f$  with respect to  $w$  is defined as follows:

$$f_w^{**}(t) = \frac{1}{t} \int_0^t f_w^*(y) dy, \quad t > 0.$$

**Definition 2.2.** Let  $w$  be a nonnegative measurable function on  $M$ . Then the weighted Lorentz space  $L_{p,q;w}$  is defined as the set of all measurable functions  $f$  on  $M$  such that  $\|f\|_{p,q;w} < \infty$ , where

$$\|f\|_{p,q;w} = \begin{cases} \left( \frac{q}{p} \int_0^\infty [t^{1/p} f_w^*(t)]^q \frac{dt}{t} \right)^{1/q}, & 1 \leq p < \infty, 0 < q < \infty, \\ \sup_{t>0} t^{1/p} f_w^*(t), & 1 \leq p \leq \infty, q = \infty. \end{cases}$$

In general, however,  $\|\cdot\|_{p,q;w}$  is not a norm since the Minkowski inequality may fail. But by replacing  $f_w^*$  with  $f_w^{**}$  in the above definition of  $\|f\|_{p,q;w}$ , we obtain a norm  $\|\cdot\|_{(p,q);w}$  for all  $q \geq 1$ :

$$\|f\|_{(p,q);w} = \begin{cases} \left( \frac{q}{p} \int_0^\infty [t^{1/p} f_w^{**}(t)]^q \frac{dt}{t} \right)^{1/q}, & 1 < p < \infty, 0 < q < \infty, \\ \sup_{t>0} t^{1/p} f_w^{**}(t), & 1 < p \leq \infty, q = \infty. \end{cases}$$

We state the following from Bennett and Sharpley [1].

**Lemma 2.3.** *If  $1 < p \leq \infty$  and  $1 \leq q \leq \infty$ , then*

$$\|f\|_{p,q;w} \leq \|f\|_{(p,q);w} \leq \frac{p}{p-1} \|f\|_{p,q;w}.$$

*Proof.* The first inequality is an immediate consequence of Definition 2.2 and the fact that  $f_w^* \leq f_w^{**}$ . The second follows from the Hardy inequality.  $\square$

As mentioned in the introduction, we include the definition of Lorentz-Zygmund spaces with norm  $\|\cdot\|_{p,q;\mathbb{D};w}$ . We note that the case where  $\mathbb{D} = (0, 0)$  corresponds to the weighted Lorentz spaces given in Definition 2.2.

**Definition 2.4.** Let  $w$  be a nonnegative measurable function on  $M$ . Then the weighted Lorentz-Zygmund space  $L_{p,q;w}(\log L)_{\mathbb{D}}$ ,  $\mathbb{D} = (\alpha, \beta) \in \mathbb{R}^2$ , is defined as the set of all measurable functions  $f$  on  $M$  such that  $\|f\|_{p,q;\mathbb{D};w} < \infty$ , where

$$\|f\|_{p,q;\mathbb{D};w} = \begin{cases} \left( \frac{q}{p} \int_0^\infty [t^{1/p} f_w^*(t)]^q (\log t)^{\mathbb{D}} \frac{dt}{t} \right)^{1/q}, & 0 < p, q < \infty, \\ \sup_{t>0} t^{1/p} f_w^*(t) (\log t)^{\mathbb{D}}, & 0 < p \leq \infty, q = \infty. \end{cases}$$

Here,

$$(\log t)^{\mathbb{D}} = \begin{cases} (1 - \log t)^{\alpha}, & 0 < t < 1, \\ (1 + \log t)^{\beta}, & 1 \leq t < \infty. \end{cases}$$

Let  $(N, \mathfrak{N}, \nu)$  be another resonant measure space. Let  $T$  be an operator from a class of measurable functions on  $M$  to that on  $N$ .

**Definition 2.5.** An operator  $T$  is called quasi-linear if there is a constant  $K \geq 1$  such that

$$|T(f + g)| \leq K(|Tf| + |Tg|), \quad |T(\lambda f)| = |\lambda| \cdot |Tf|$$

hold  $\nu$ -a.e. on  $N$  for all  $f$  and  $g$  in the domain of  $T$  and for all scalars  $\lambda$ . If we can take  $K = 1$ ,  $T$  is called sublinear.  $T$  is called nonnegative when  $Tf \geq 0$ ,  $\nu$ -a.e. for all  $f$  in the domain of  $T$ .

We recall the following result from Bennett and Sharpley [1, Lemma 5.1, p. 231]. This is necessary for the limiting argument given in the proof of Theorem 3.1 in the next section 3.

**Lemma 2.6.** *Let  $X$  and  $Y$  be rearrangement-invariant Banach function spaces over resonant measure spaces  $(M, \mu)$  and  $(N, \nu)$ , respectively. Let  $T$  be a nonnegative sublinear operator defined on a dense linear subspace  $D$  of  $X$  and taking values in  $Y$ . If*

$$(2.1) \quad \|Tf\|_Y \leq C\|f\|_X$$

for all  $f \in D$ , then  $T$  has a unique extension to a nonnegative sublinear operator from  $X$  to  $Y$ , for which (2.1) holds for all  $f \in X$ .

Finally, we prove the following.

**Lemma 2.7.** *There are constants  $C_1$  and  $C_2$  such that*

$$C_1 \sum_{k=-\infty}^{\infty} 2^{k/p} f_w^*(2^k) \leq \|f\|_{p,1;w} \leq C_2 \sum_{k=-\infty}^{\infty} 2^{k/p} f_w^*(2^k).$$

*Proof.* Let  $t \in [2^k, 2^{k+1}]$ . Since  $f_w^*$  is nonincreasing,  $f_w^*(2^{k+1}) \leq f_w^*(t) \leq f_w^*(2^k)$ .

On the one hand, we obtain

$$\begin{aligned} \|f\|_{p,1;w} &= \frac{1}{p} \int_0^{\infty} t^{1/p} f_w^*(t) \frac{dt}{t} \\ &= \frac{1}{p} \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} t^{1/p} f_w^*(t) \frac{dt}{t} \\ &\leq \frac{1}{p} \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} t^{1/p} f_w^*(2^k) \frac{dt}{t} \\ &= \sum_{k=-\infty}^{\infty} f_w^*(2^k) (2^{(k+1)/p} - 2^{k/p}) \\ &= (2^{1/p} - 1) \sum_{k=-\infty}^{\infty} 2^{k/p} f_w^*(2^k). \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} \|f\|_{p,1;w} &\geq \frac{1}{p} \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} t^{1/p} f_w^*(2^{k+1}) \frac{dt}{t} \\ &= \sum_{k=-\infty}^{\infty} f_w^*(2^{k+1}) (2^{(k+1)/p} - 2^{k/p}) \\ &= (1 - 2^{-1/p}) \sum_{k=-\infty}^{\infty} 2^{(k+1)/p} f_w^*(2^{k+1}). \end{aligned}$$

□

### 3. MAIN THEOREM

We can now prove the following.

**Theorem 3.1.** *Let  $i = 0, 1$ . Suppose that  $1 \leq p_0 < p_1 \leq \infty$ ,  $1 \leq r_i \leq \infty$  with  $r_0 \neq r_1$ ,  $0 < q_i, s_i \leq \infty$ . Put  $1/p = (1 - \theta)/p_0 + \theta/p_1$ ,  $1/r = (1 - \theta)/r_0 + \theta/r_1$  for  $0 < \theta < 1$ . Let  $v, w_i$  be nonnegative measurable functions. If  $T$  is a nonnegative sublinear operator from  $L_{p_i, q_i; v}$  to  $L_{r_i, s_i; w_i}$ , then there exists a constant  $C$  such that*

$$\|Tf\|_{r, \infty; w} \leq C \|f\|_{p, 1; v},$$

where  $w^{1/r} = w_0^{(1-\theta)/r_0} w_1^{\theta/r_1}$ .

*Remark 3.2.* We do not know if this theorem is true in the case where  $T$  is not assumed to be nonnegative.

*Proof of Theorem 3.1.* In our proof, the letter  $C$  will denote a constant not necessarily the same at each occurrence. Let  $f$  be a measurable function on  $M$ . We consider pairwise disjoint measurable subsets of  $M$ ,

$$E_n = \{x \in M : f_v^*(2^{n+1}) < |f(x)| \leq f_v^*(2^n)\}, \quad n = 0, \pm 1, \pm 2, \dots$$

We may assume that  $|f(x)| < \infty$ ,  $\mu$ -a.e.  $x \in M$ . If  $|f(x)| \equiv \infty$  on some set of positive measure, then we have  $f_v^*(t) \equiv \infty$  near  $t = 0$ , which implies that  $\|f\|_{p,1;v} = \infty$ . Moreover, we may also assume that  $f_v^*(t)$  tends to 0 as  $t \rightarrow \infty$ . If  $f_v^*(t)$  does not tend to 0, then  $\|f\|_{p,1;v} = \infty$ .

Let  $f_n$  be defined by

$$f_n(x) = \begin{cases} f(x), & x \in E_n, \\ 0, & \text{otherwise.} \end{cases}$$

Then we consider the following three cases according to the value of  $|f(x)|$ . When  $|f(x)| = 0$ ,  $x$  does not belong to any  $E_n$ , so  $f_n(x) = 0$  for every  $n$ . Next, when  $0 < |f(x)| < \infty$ ,  $x$  belongs to some  $E_n$  because we have assumed that  $f_v^*(t)$  tends to 0 as  $t \rightarrow \infty$ . In this case,  $f_n(x) = f(x)$  and  $f_m(x) = 0$  for  $m \neq n$ . Finally, when  $|f(x)| = \infty$ ,  $x$  does not belong to any  $E_n$  because we have assumed that  $|f(x)| < \infty$ ,  $\mu$ -a.e.  $x \in M$ . In this final case,  $f_n(x) = 0$  for every  $n$ . Therefore, we may describe the function  $f$  as the following sum of the functions  $f_n$  (see [6]):

$$(3.1) \quad f(x) = \sum_{n=-\infty}^{\infty} f_n(x), \quad \mu\text{-a.e. } x \in M.$$

We remark that if  $E_n = \emptyset$ , then  $f_n(x) \equiv 0$  on  $M$ . Another important remark is that we have  $v(E_n) \leq 2 \cdot 2^n$  because of the construction of the sets  $E_n$ , which is used in the estimate (3.6) below. Because of Lemma 2.6, we may assume that all but finitely many  $f_n$ 's in (3.1) are identically zero. The set of such functions  $f$  is a dense linear subspace of  $L_{p,1;v}$ ,  $1 < p < \infty$ , and we can apply a limiting argument.

By using the sublinearity of  $T$  (Definition 2.5), the Minkowski inequality (Definition 2.2) and Lemma 2.3, we have

$$(3.2) \quad \|Tf\|_{(r,\infty);w} \leq \sum_{n=-\infty}^{\infty} \|Tf_n\|_{(r,\infty);w} \leq \frac{r}{(r-1)} \sum_{n=-\infty}^{\infty} \|Tf_n\|_{r,\infty;w}.$$

(We note again that the summations in the inequalities in (3.2) have only finitely many nonzero terms.)

We now estimate the quantity  $\|Tf_n\|_{r,\infty;w}$ . Because  $T$  is assumed to be a sublinear operator from  $L_{p_i,q_i;v}$  to  $L_{r_i,s_i;w_i}$ ,

$$t^{1/r_i} (Tf_n)_{w_i}^*(t) \leq C \|f_n\|_{p_i,q_i;v}, \quad i = 0, 1.$$

By the change of variables  $t = \lambda_{Tf_n,w_i}(s)$ , we have

$$(3.3) \quad s(\lambda_{Tf_n,w_i}(s))^{1/r_i} \leq C \|f_n\|_{p_i,q_i;v}, \quad i = 0, 1.$$

We use the Hölder inequality and the relation that  $1/r = (1-\theta)/r_0 + \theta/r_1$ ,  $w^{1/r} = w_0^{(1-\theta)/r_0} w_1^{\theta/r_1}$  to obtain

$$(3.4) \quad \begin{aligned} & s(\lambda_{Tf_n,w}(s))^{1/r} \\ &= s \left( \int_{\{x;|Tf_n(x)|>s\}} w(x) d\mu(x) \right)^{1/r} \\ &\leq s \left( \int_{\{x;|Tf_n(x)|>s\}} w_0(x) d\mu(x) \right)^{(1-\theta)/r_0} \left( \int_{\{x;|Tf_n(x)|>s\}} w_1(x) d\mu(x) \right)^{\theta/r_1} \\ &= \left[ s(\lambda_{Tf_n,w_0}(s))^{1/r_0} \right]^{1-\theta} \left[ s(\lambda_{Tf_n,w_1}(s))^{1/r_1} \right]^{\theta}. \end{aligned}$$

From (3.3) and (3.4),

$$s(\lambda_{Tf_n,w}(s))^{1/r} \leq C \|f_n\|_{p_0,q_0;v}^{1-\theta} \|f_n\|_{p_1,q_1;v}^{\theta}.$$

Thus we obtain

$$(3.5) \quad \|Tf_n\|_{r,\infty;w} \leq C \|f_n\|_{p_0,q_0;v}^{1-\theta} \|f_n\|_{p_1,q_1;v}^{\theta}.$$

As for the quantity  $\|f_n\|_{p_0,q_0;v}$ , we have

$$\begin{aligned} \|f_n\|_{p_0,q_0;v} &= \left( \frac{q_0}{p_0} \int_0^\infty [t^{1/p_0} (f_n)_v^*(t)]^{q_0} \frac{dt}{t} \right)^{1/q_0} \\ &\leq \left( \frac{q_0}{p_0} \int_0^{v(E_n)} [t^{1/p_0} f_v^*(2^n)]^{q_0} \frac{dt}{t} \right)^{1/q_0} \\ &= f_v^*(2^n) (v(E_n))^{1/p_0}. \end{aligned}$$

Similarly as for the quantity  $\|f_n\|_{p_1,q_1;v}$ ,

$$\|f_n\|_{p_1,q_1;v} \leq f_v^*(2^n) (v(E_n))^{1/p_1}.$$

By using the relation that  $1/p = (1 - \theta)/p_0 + \theta/p_1$ , we obtain from (3.5),

$$(3.6) \quad \|Tf_n\|_{r,\infty;w} \leq C f_v^*(2^n) (v(E_n))^{1/p} \leq C f_v^*(2^n) \cdot (2 \cdot 2^n)^{1/p}.$$

Consequently, from (3.2), (3.6) and Lemma 2.7 we obtain

$$\|Tf\|_{r,\infty;w} \leq C \sum_{n=-\infty}^{\infty} f_v^*(2^n) \cdot 2^{n/p} \leq C \|f\|_{p,1;v}.$$

□

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