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# RADIX REPRESENTATIONS, SELF-AFFINE TILES, AND MULTIVARIABLE WAVELETS

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Abstract. We investigate the connection between radix representations for $\mathbb{Z}^n$ and self-affine tilings of $\mathbb{R}^n$. We apply our results to show that Haar-like multivariable wavelets exist for all dilation matrices that are sufficiently large.

## 1. Introduction

We investigate the connections between radix representations for $\mathbb{Z}^n$, self-affine tilings of $\mathbb{R}^n$, and Haar-like scaling functions for multiresolution analyses and associated wavelet sets.

In a separate paper, we investigate the idea, also introduced by Jeong [5], of radix representations for vectors in $\mathbb{Z}^n$ or general point lattices $\Gamma = M(\mathbb{Z}^n)$ ($M$ a nondegenerate $n \times n$ matrix) [1]. We wish to consider expanding matrices which preserve $\Gamma$. Without loss of generality, we may assume $\Gamma = \mathbb{Z}^n$. A matrix which preserves $\mathbb{Z}^n$ must have integer entries.

**Definition 1.** A *dilation matrix* for $\mathbb{Z}^n$ is an $n \times n$ matrix $A$ with integer entries, all of whose eigenvalues $\lambda$ satisfy $|\lambda| > 1$.

Note that for a dilation matrix $A$, $q := |\det A|$ is an integer, with $q > 1$. Then $\mathbb{Z}^n/A(\mathbb{Z}^n)$ has a nontrivial cokernel. Let $D$ be a complete set of coset representatives of $\mathbb{Z}^n/A(\mathbb{Z}^n)$. We call the elements of $D$ *digits*.

We may associate a sequence of digits with each $x \in \mathbb{Z}^n$ by the Euclidean algorithm, as follows. Each $x \in \mathbb{Z}^n$ is in a unique coset of $\mathbb{Z}^n/A(\mathbb{Z}^n)$, thus there exist unique $x_1 \in \mathbb{Z}^n$ and $r_0 \in D$ such that

$$x = Ax_1 + r_0.$$

Similarly, for each $x_j$, $j \geq 1$, there exist unique $x_{j+1} \in \mathbb{Z}^n$ and $r_j \in D$ such that

$$x_j = Ax_{j+1} + r_j.$$

Formally, we write

$$x \sim \sum_{j=0}^{\infty} A^j r_j.$$

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If there exists a nonnegative integer $N$ such that $r_j = \mathbf{0}$ for all $j > N$, then the Euclidean algorithm terminates and we say that $x$ has a radix representation with radix $A$ and digit set $D$.

**Definition 2.** Let $A$ be a dilation matrix. We say that the matrix $A$ *yields a radix representation with digit set* $D$ if for every $x \in \mathbb{Z}^n$ there exists a nonnegative integer $N = N(x)$ and a sequence of digits $d_0, d_1, \ldots, d_N$ in $D$ such that

$$x = \sum_{j=0}^{N} A^j d_j.$$

That is, a dilation matrix $A$ yields a radix representation with digit set $D$ if *every* $x \in \mathbb{Z}^n$ has a radix representation with radix $A$ and digit set $D$.

Let $A$ be a dilation matrix, and define

$$\mu = \min \{\sigma : \ \sigma \text{ a singular value of } A\}.$$

Let $F$ be a fundamental domain for $\mathbb{Z}^n$, centered at the origin,

$$F = \left[-\frac{1}{2}, \frac{1}{2}\right)^n.$$

In [1], we give the following two results about radix representations.

**Theorem 3.** *Let $A$ be an $n \times n$ dilation matrix. If $\mu > 2$, then $A$ yields a radix representation of $\mathbb{Z}^n$ with digit set $D = A(F) \cap \mathbb{Z}^n$.*

**Corollary 4.** *For every dilation matrix $A$, there exists a positive integer $\beta \geq 1$ such that $A^\beta$ yields a radix representation with digit set $D_\beta = A^\beta(F) \cap \mathbb{Z}^n$.*

## 2. Radix representations and tilings

Radix representations are closely related to self-affine tilings of $\mathbb{R}^n$.

**Definition 5.** A measurable set $Q \subset \mathbb{R}^n$ gives a self-affine tiling of $\mathbb{R}^n$ under translation by $\mathbb{Z}^n$ if

1. $\bigcup_{k \in \mathbb{Z}^n} (Q + k) = \mathbb{R}^n$, and the intersection $(Q + k_1) \cap (Q + k_2)$ has measure zero for any two distinct $k_1, k_2 \in \mathbb{Z}^n$ (tiling); and
2. there is a collection of $q = |\det A|$ vectors $k_1, \ldots, k_q \in \mathbb{Z}^n$ that are distinct coset representatives of $\mathbb{Z}^n / A(\mathbb{Z}^n)$ such that

$$A(Q) \simeq \bigcup_{i=1}^{q} (Q + k_i) \quad \text{(self-affine)}.$$

Set

$$T = T(A, D) := \{\xi \in \mathbb{R}^n : \ \xi = \sum_{j=1}^{\infty} A^{-j} d_j\}$$

with the digits $d_j \in D$ for some digit set $D$. One can easily check that $T$ is a self-affine set. We would like to be able to think of the elements of $T$ as the fractional parts of vectors in $\mathbb{R}^n$ in the same way that the fractional parts of real numbers lie in $[0, 1]$. This is an accurate interpretation if $T$ is congruent to $\mathbb{R}^n / \mathbb{Z}^n$. Thought of another way, we would like $T$ to tile $\mathbb{R}^n$ under translation by $\mathbb{Z}^n$.

**Theorem 6.** *Let $A$ be a dilation matrix, and let $D$ be a digit set for $A$. Then $A$ yields a radix representation with digit set $D$ if and only if the set $T(A, D)$ tiles $\mathbb{R}^n$ under translation by $\mathbb{Z}^n$ and the origin $\mathbf{0}$ is in the interior of $T$.*

We split the proof of this theorem into a few lemmas.

**Lemma 7.** *Let $A$ be a dilation matrix, and let $D$ be a digit set for $A$. If $A$ yields a radix representation with a digit set $D$, then the set $T(A, D)$ tiles $\mathbb{R}^n$ under translation by $\mathbb{Z}^n$.*

*Proof.* Applying Proposition 5.19 from [12] the dilation matrix $A$, digit set $D$, and corresponding set $T = T(A, D)$ satisfy

1. $T$ is a compact subset of $\mathbb{R}^n$;
2. $A(T) = \bigcup_{d\in D}(T + d)$;
3. $\bigcup_{x\in\mathbb{Z}^n}(T + x) = \mathbb{R}^n$; and
4. $T$ contains an open set.

To show that $T$ tiles $\mathbb{R}^n$ under translation by $\mathbb{Z}^n$, we must show that

$$m((T + x) \cap (T + y)) = 0 \text{ for all } x \neq y, x, y \in \mathbb{Z}^n$$

(where $m(\cdot)$ denotes Lebesgue measure). We extend an idea of Lagarias and Wang ([6], p. 31) to show that $m((T + x) \cap (T + y)) = 0$ for any distinct $x, y \in \mathbb{Z}^n$ for which there is a radix representation with radix $A$ and digit set $D$.

Note that for any subset $Q$ of $\mathbb{R}^n$, $m(A(Q)) = qm(Q)$ (where $q = |\det A|$). In particular,

$$\begin{aligned} qm(T) = m(A(T)) &= m\left(\bigcup_{d\in D}(T + d)\right) \text{ (by property (2) of } T) \\ &\leq \sum_{d\in D} m(T + d) = \sum_{d\in D} m(T) = qm(T). \end{aligned}$$

Additionally, property (2) implies that $A^{k+1}(T) = \bigcup_{d\in D}(A^k(T) + A^k d)$ for all $k \geq 0$. Then

$$q^{k+1}m(T) \leq \sum_{d\in D} m(A^k(T) + A^k d) = \sum_{d\in D} q^k m(T) = q^{k+1}m(T).$$

Thus

$$m\left((A^k(T) + A^k d_i) \cap (A^k(T) + A^k d_j)\right) = 0$$

for all distinct $d_i, d_j \in D$.

Since $\mathbf{0} \in D$, $A^{k+1}(T) \supset A^k(T)$ for all $k \geq 0$. So

$$(A^{k+1}(T) + A^{k+1}d) \supset (A^k(T) + A^{k+1}d)$$

for all $d \in D$. Now consider $T + x$ for any $x \in \mathbb{Z}^n$. By hypothesis,

$$x = A^N d_N + \sum_{j=0}^{N-1} A^j d_j$$

for some $N \geq 0$, with $d_N \neq \mathbf{0} \in D$ and $d_j \in D$. Then

$$\begin{aligned} T + x = \{y \in \mathbb{R}^n :\ y = A^N d_N + \sum_{j=0}^{N-1} A^j d_j + \sum_{j=-\infty}^{-1} A^j d_j,\ \text{all } d_j \in D\} \\ \subset A^{N-1}(T) + A^N d_N \subset A^N(T) + A^N d_N \end{aligned}$$

and

$$(T \cap (T + x)) \subseteq (A^N(T) \cap (A^N(T) + A^N d_N)).$$

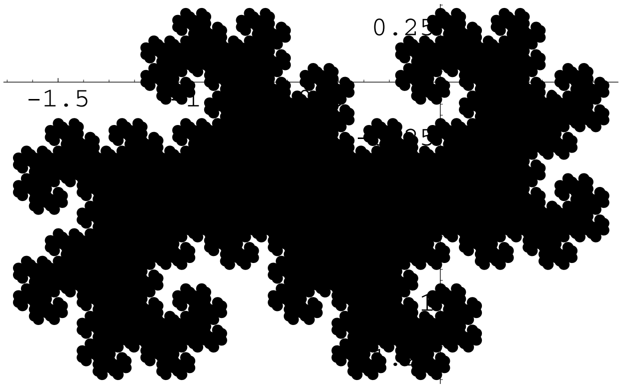

FIGURE 1. The tile $T$ for the twin dragon matrix.

Then $d_N \neq \mathbf{0} \in D$ implies that

$$0 = m\left(A^N(T) \cap (A^N(T) + A^N d_N)\right) \geq m\left(T \cap (T + x)\right).$$

$\square$

If the set $T = T(A, D)$ tiles $\mathbb{R}^n$, it is not necessarily true that $A$ yields a radix representation. For example, in the case where $A = 2$, we can find a radix representation for all nonnegative integers with the digit set $D = \{0, 1\}$, or for all non-positive integers with the digit set $D = \{0, -1\}$, but we cannot represent all integers with a radix representation using any digit set [10]. Yet $T = [0, 1]$ (with digit set $D = \{0, 1\}$) does tile $\mathbb{R}$ under translation by $\mathbb{Z}$. Similarly, if $A$ is the twin dragon matrix [12],

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

then $A$ is a dilation matrix. A digit set for $A$ is

$$D = \{d_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, d_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\},$$

and the set $T$ generated by the twin dragon matrix with this digit set tiles $\mathbb{R}^2$ under translation by $\mathbb{Z}^n$ [12], yet $A$ does not yield a radix representation of $\mathbb{Z}^n$ (for example, the vector

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

does not have a radix representation) [1].

In both of the examples, the origin $\mathbf{0}$ is on the boundary of the tile $T$, so that $\bigcup_{k=0}^{\infty} A^k(T) \subsetneq \mathbb{R}^n$. We claim in Theorem 6 that this must be the case for all dilation matrices $A$ which give tiles $T$ but do not yield radix representations. We first introduce a technical lemma.

**Lemma 8.** *Let $A$ be a dilation matrix, and let $D$ be a digit set for $A$. If $\mathbf{0} \notin T^\circ$, then there exists an increasing subsequence $\{\ell_j\}_{j \geq 1}$ of the positive integers and a sequence of vectors $\{\zeta_j : \zeta_j \in A^{-\ell_j}(\mathbb{Z}^n)\}_{j \geq 1}$ converging to $\mathbf{0}$ such that $\zeta_j \notin T$ for all $j \geq 1$.*

*Proof.* In the proof of Lemma 7, we note that the set $T$ is compact. Thus for any $\omega \in \mathbb{R}^n$ such that $\omega \notin T$, $d(\omega, T) > 0$, and there exists an open ball centered at $\omega$, $B_\omega$, such that $\overline{B_\omega} \cap T = \emptyset$.

Let $\{y_j\}_{j\geq 1}$ be a sequence of vectors in $\mathbb{R}^n$ converging to $\mathbf{0}$ with $y_j \notin T$ for all $j \geq 1$. Set $\epsilon_j = \|y_j\|_{l^2}$, and note that $\epsilon_j \geq d(y_j, T) > 0$, with $\lim_{j\to\infty} \epsilon_j = 0$.

Set $r_1 = \frac{d(y_1,T)}{2}$ and

$$r_j = \min\left\{\frac{r_{j-1}}{2}, \frac{d(y_j, T)}{2}\right\}$$

for $j \geq 2$. Then $\{r_j\}_{j\geq 1}$ is a decreasing sequence of positive numbers, $\lim_{j\to\infty} r_j = 0$, and the open balls $B_j$ of radius $r_j$ centered at the vectors $y_j$ satisfy $\overline{B_j} \cap T = \emptyset$ for all $j \geq 1$. By construction, if $y_j^*$ is any point in the ball $B_j$ for each $j \geq 1$, then $\lim_{j\to\infty} y_j^* = \mathbf{0}$ and $y_j^* \notin T$ for all $j \geq 1$.

In particular, there exists an increasing subsequence $\{\ell_j\}_{j\geq 1}$ of the positive integers such that $A^{-\ell_j}$ is a fine enough lattice to ensure that $A^{-\ell_j}(\mathbb{Z}^n) \cap B_j \neq \emptyset$ for each $j \geq 1$. We may choose some $\zeta_j \in A^{-\ell_j}(\mathbb{Z}^n) \cap B_j$ for each $j \geq 1$. By construction, $\zeta_j \notin T$ for each $j \geq 1$, and the sequence $\{\zeta_j\}_{j\geq 1}$ converges to $\mathbf{0}$. □

**Lemma 9.** *Let $A$ be a dilation matrix, and let $D$ be a digit set for $A$. If $A$ yields a radix representation with digit set $D$, then $\mathbf{0}$ is in the interior of the set $T$ generated by $A$ and $D$.*

*Proof.* We prove the desired result by contradiction. Assume that $\mathbf{0} \notin T^\circ$. By Lemma 8, let $\{\zeta_j\}_{j\geq 1}$ be a sequence of vectors in $\mathbb{R}^n$ converging to $\mathbf{0}$ such that $\zeta_j \in A^{-\ell_j}(\mathbb{Z}^n)$ (for some increasing subsequence $\{\ell_j\}_{j\geq 1}$ of the positive integers) but $\zeta_j \notin T$ for all $j \geq 1$. We may write $\zeta_j = A^{-\ell_j} x_j$ for some $x_j \in \mathbb{Z}^n$ for each $j$. Since $A$ yields a radix representation, there exists an integer $N_j$ for each $x_j$ and digits $d_0^{(j)}, \ldots, d_{N_j}^{(j)} \in D$ such that

$$x_j = \sum_{i=0}^{N_j} A^i d_i^{(j)}.$$

Thus

$$\zeta_j = \sum_{i=0}^{N_j} A^{i-\ell_j} d_i^{(j)} = k_j + \xi_j$$

with $k_j \in \mathbb{Z}^n$ and $\xi_j \in T$.

Since $T$ is compact, $\|\xi_j\|_{l^2}$ is bounded above by $b$ for some $b > 0$. Then, since $\zeta_j$ converges to $\mathbf{0}$, $\|k_j\|_{l^2}$ is also bounded above for sufficiently large $j$. We use the rough estimate that there exists an integer $M \geq 1$ such that for all $j \geq M$, $\|k_j\|_{l^2} \leq 2b$. Thus for $j \geq M$, the integer vectors $k_j$ all belong to a finite subset of $\mathbb{Z}^n$. This implies that there exists an integer $N \geq 0$ such that $N_j \leq N$ for all $j \geq M$. For $j > \max\{M, N\}$, $N - \ell_j \leq N - j < 0$, and thus $\zeta_j \in T$ for all sufficiently large $j$. This contradicts the choice of $\zeta_j$, thus our assumption that $\mathbf{0} \notin T^\circ$ must be false. □

We have shown that if a dilation matrix $A$ yields a radix representation with digit set $D$, then the set $T = T(A, D)$ tiles $\mathbb{R}^n$ under translation by $\mathbb{Z}^n$, and $\mathbf{0} \in T^\circ$. Next we prove the converse.

**Lemma 10.** *Let $A$ be a dilation matrix, and let $D$ be a digit set for $A$. If the set $T = T(A, D)$ tiles $\mathbb{R}^n$ under translation by $\mathbb{Z}^n$ and if $\mathbf{0} \in T^\circ$, then $A$ yields a radix representation with digit set $D$.*

*Proof.* Following the notation of [6], set

$$D_{A,k} := \{x \in \mathbb{Z}^n : \ x = \sum_{j=0}^{k-1} A^j d_j, d_j \in D\},$$

the set of vectors that can be expressed with a radix representation of length less than or equal to $k$. Note that $D_{A,1} = D$. By construction, $A(T) = \bigcup_{d \in D}(T + d)$. Thus

$$A^k(T) = \bigcup_{x \in D_{A,k}} (T + x).$$

By the tiling hypothesis, $(T^\circ + x) \cap (T^\circ + y) = \emptyset$ for all distinct $x, y \in \mathbb{Z}^n$. Thus $(T^\circ + y) \cap A^k(T^\circ) = \emptyset$ for all $y \in \mathbb{Z}^n$ with $y \notin D_{A,k}$, and

$$D_{A,k} \supseteq (A^k(T^\circ) \cap \mathbb{Z}^n).$$

Let $B$ be an open ball centered at the origin such that $B \subseteq T^o$. Then

$$D_{A,k} \supseteq (A^k(B) \cap \mathbb{Z}^n).$$

The sets $A^{k+1}(B)$ are expanding, with $\bigcup_{k \geq 0} A^{k+1}(B) = \mathbb{R}^n$. Thus

$$\bigcup_{k \geq 0} D_{A,k} \supseteq \bigcup_{k \geq 0} (A^k(B) \cap \mathbb{Z}^n) = \mathbb{Z}^n.$$

The opposite containment is true as well, since $D_{A,k} \subset \mathbb{Z}^n$ for each $k$. Thus $\bigcup_{k \geq 0} D_{A,k} = \mathbb{Z}^n$. □

We have now completed the proof of Theorem 6.

Recall that $\mu$ is the smallest singular value of the dilation matrix $A$ and that $F$ is our canonical fundamental domain of $\mathbb{Z}^n$, $F = [-\frac{1}{2}, \frac{1}{2})^n$. Combining Theorems 3 and Lemma 7, we also have the following corollary.

**Corollary 11.** *Let $A$ be a dilation matrix, and let $D = A(F) \cap \mathbb{Z}^n$. If $\mu > 2$, then the set*

$$T = \{x \in \mathbb{R}^n : \ x = \sum_{j=-\infty}^{-1} A^j d_j, d_j \in D\}$$

*tiles $\mathbb{R}^n$ under translation by $\mathbb{Z}^n$.*

## 3. HAAR-LIKE WAVELETS

Self-affine tiles allow us to construct multivariable wavelet sets associated with multiresolution analyses. Here we review some basic definitions from wavelet theory.

**Definition 12.** A *multiresolution analysis (MRA)* associated with a dilation matrix $A$ is a nested sequences of subspaces $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$ of $L^2(\mathbb{R}^n)$ satisfying [12]:

(1) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$;
(2) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;
(3) $f(x) \in V_j$ if and only if $f(Ax) \in v_{j+1}$ for all $j \in \mathbb{Z}$;
(4) $f(x) \in V_0$ if and only if $f(x - k) \in V_0$ for all $k \in \mathbb{Z}^n$; and

(5) there exists a function $\phi(x) \in V_0$, called a *scaling function*, such that

$$\{\phi(x-k) : \ k \in \mathbb{Z}^n\}$$

is a complete orthonormal basis for $V_0$.

The existence of multiresolution analyses in dimension $n > 1$ has been studied by a number of authors. In [3], Gröchenig and Madych showed that if $\phi = (m(Q))^{1/2}\chi_Q$ is a scaling function for a multiresolution analysis, then $Q$ must be an affine image of a self-affine tiling of $\mathbb{R}^n$ under translation by $\mathbb{Z}^n$. They showed also that if $T$ is a set of the form

$$T = \{x \in \mathbb{R}^n : \ x = \sum_{j=-\infty}^{-1} A^j d_j, d_j \in D\}$$

with $A$ a dilation matrix and $D$ a digit set for $A$, then $\phi = \chi_T$ is the scaling function for a multiresolution analysis (note that $|T| = 1$). Lagarias and Wang noted that all self-affine tiles $T$ which tile $\mathbb{R}^n$ under translation by $\mathbb{Z}^n$ must be of this form [8]. A scaling function that is the characteristic function of some measurable set is called a Haar-like scaling function (after the Haar scaling function, which is $\chi_{[0,1]}$).

Lagarias and Wang studied necessary conditions for sets of the form $T(A, D)$ to tile $\mathbb{R}^n$ under translation by $\mathbb{Z}^n$ in a series of papers ([6], [7], [8], [9]). Much of their work in these papers concerned the question of when a set $T(A, D)$ tiles $\mathbb{R}^n$ under translation by a sublattice of $\mathbb{Z}^n$. In [6] and [7], they showed that if $D$ is a complete set of coset representatives of $\mathbb{Z}^n/A(\mathbb{Z}^n)$, then $T(A, D)$ is a self-affine tile of $\mathbb{R}^n$ under translation by some sublattice of $\mathbb{Z}^n$. They also studied some properties of the tiling set $T(A, D)$. He and Lau [4] studied sets $T(A, D)$ which tile $\mathbb{R}^n$ under translation by a sublattice of $\mathbb{Z}^n$ as well; in particular, they looked at possible digit sets $D$.

In order for $\chi_T$ to be a scaling function for a multiresolution analysis, however, we need $T$ to tile $\mathbb{R}^n$ under translation by all of $\mathbb{Z}^n$. Lagarias and Wang gave some necessary conditions for a dilation matrix $A$ to yield a Haar-like scaling function in [8] and [9]. They showed that all dilation matrices in dimensions $n = 2$ and 3 yield Haar-like scaling functions. Our results below give a sufficient condition for a dilation matrix $A$ to yield a Haar-like scaling function, in any dimension. Note that for dilation matrices that yield a Haar-like scaling function, Strichartz has shown that multiresolution analyses and associated wavelet bases with arbitrary regularity can be constructed [11].

The results of the previous section imply the following two theorems.

**Theorem 13.** *Let $A$ be a dilation matrix, and let $D$ be a digit set for $A$ such that $A$ yields a radix representation for $\mathbb{Z}^n$ with digit set $D$. Let $T$ be the set depending on $A$ and $D$ defined above. Then $\phi = \chi_T$ is the scaling function for a multiresolution analysis. In particular, if $A$ satisfies $\mu > 2$ and if $D$ is the set $D = A(F) \cap \mathbb{Z}^n$ with $F = [-\frac{1}{2}, \frac{1}{2})^n$, then $\phi = \chi_T$ is the scaling function for a multiresolution analysis.*

**Theorem 14.** *Let $A$ be a dilation matrix. Then there exists a positive integer $\beta \geq 1$ such that for all integers $k \geq \beta$ there exists a multiresolution analysis associated with the dilation matrix $A^k$.*

Thus Haar-like scaling functions and associated MRAs exist for a large class of dilation matrices.

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