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CONVERGENCE OF A SINGULAR EULER-POISSON APPROXIMATION OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. In this note, we rigorously justify a singular approximation of the incompressible Navier-Stokes equations. Our approximation combines two classical approximations of the incompressible Euler equations: a standard relaxation approximation, but with a diffusive scaling, and the Euler-Poisson equations in the quasineutral regime.

1. Introduction

We shall consider the following system:

$$\begin{cases} \partial_t u^\varepsilon + \text{ div } V^\varepsilon = \nabla\varphi^\varepsilon, \\ \varepsilon\partial_t V^\varepsilon + a^2\nabla u^\varepsilon = u^\varepsilon \otimes u^\varepsilon - V^\varepsilon, \\ \partial_t \rho^\varepsilon + \text{ div } (\rho^\varepsilon u^\varepsilon) = 0, \\ \Delta\varphi^\varepsilon = \frac{\rho^\varepsilon - 1}{\varepsilon}, \end{cases} \tag{1}$$

for $t > 0$, $x \in \mathbb{T}^d$, $u^\varepsilon \in \mathbb{R}^d$, $V^\varepsilon \in \mathbb{R}^{d,d}$, $\rho^\varepsilon \in \mathbb{R}$. Here $\varepsilon > 0$ is a small parameter and a is a non zero constant value. To uniquely solve the Poisson equation, we add the condition $\int_{\mathbb{T}^d} \rho = 1$. Passing to the limit when ε goes to zero, it is easy to see, at least at a very formal level, that $(\rho^\varepsilon, u^\varepsilon, V^\varepsilon)$ tends to $(\rho^{NS}, u^{NS}, V^{NS})$, where

$$\rho^{NS} = 1, \ V^{NS} = u^{NS} \otimes u^{NS} - a^2\nabla u^{NS} \tag{2}$$

and

$$\begin{cases} \partial_t u^{NS} + \text{ div } (u^{NS} \otimes u^{NS}) = a^2\Delta u^{NS} + \nabla\varphi^{NS}, \\ \text{div } u^{NS} = 0. \end{cases} \tag{3}$$

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In other words, u^{NS} is a solution of the incompressible Navier-Stokes equations. The aim of this note is to give a rigorous justification to this formal computation. We shall prove the following result.

Theorem 1. *Let u^{NS} be a solution of the Navier-Stokes equations* (3) *such that $u^{NS} \in C([0,T], H^{m+3}(\mathbb{T}^d))$ and $\int_{\mathbb{T}^d} u^{NS} = 0$ for $d \leq 3$ and $m > 1 + d/2$. Assume that the initial value $(u_0^\varepsilon, V_0^\varepsilon, \rho_0^\varepsilon) \in H^{m+1}$ of* (1) *is such that $\int_{\mathbb{T}^d} \rho_0^\varepsilon = 1$, $\int_{\mathbb{T}^d} u_0^\varepsilon = 0$ and*

$$\alpha_m(\varepsilon) := ||u_0^\varepsilon - u^{NS}_{/t=0}||^2_{H^{m+1}} + \frac{1}{\varepsilon}||\rho_0^\varepsilon - 1||^2_{H^m} + \varepsilon||V_0^\varepsilon - V^{NS}_{/t=0}||^2_{H^{m+1}} \to 0$$

when ε goes to zero. Then there exist ε_0 and C_T such that for $0 < \varepsilon \leq \varepsilon_0$ there is a strong solution $(\rho^\varepsilon, u^\varepsilon, V^\varepsilon) \in C([0,T], H^{m+1})$ of (1) *such that*

$$||u^\varepsilon(t) - u^{NS}(t)||^2_{H^{m+1}} + \frac{1}{\varepsilon}||\rho^\varepsilon(t) - 1||^2_{H^m} \leq C_T(\varepsilon^2 + \alpha_m(\varepsilon)),\ \forall t \in [0,T]. \tag{4}$$

System (1) combines the features of the diffusive relaxation approximation of parabolic systems and of the quasineutral limit of the Euler-Poisson equation towards the incompressible Euler equation. These two problems were extensively studied by various methods.

Let us recall that the diffusive scaling $(\frac{x}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon})$ has been largely investigated in the framework of hydrodynamic limits of the Boltzmann equations [4] and in the analysis of hyperbolic-parabolic relaxation limits for weak solutions of hyperbolic systems of balance laws with strongly diffusive source terms by means of compensated compactness techniques by Marcati and collaborators [15, 16, 9]. Under the same scaling, general (possibly degenerate) parabolic equations in multi-D have been approximated by semilinear hyperbolic equations; see [5, 2].

Concerning the quasineutral limit, which consists in the limit of vanishing viscosity for the Poisson part of the system, there are results for various specific models. In particular, this limit has been performed for the Vlasov-Poisson system by Brenier [6], Grenier [11, 12], and in the one-dimensional and isothermal Euler-Poisson system by Cordier and Grenier [8]. We refer to [14], [18] and references therein for more recent contributions.

The main focus in the present note is on the use of hyperbolic energy methods for studying incompressible fluids. In this regard, our system can also be seen as a refinement of the relaxation system studied in [7], whose techniques however were restricted to the two-dimensional case. In particular our result gives a rate of convergence in strong H^s norm of the solution of the singular system towards a strong solution of the incompressible Navier-Stokes equation. Hyperbolic ideas have also been used in the weak solutions framework by Donatelli and Marcati to deal with the diffusive relaxation of hyperbolic approximations towards Leray solutions of the incompressible Navier-Stokes equation [10]. Finally, similar approximations have been considered as reduced kinetic models in [3] and on a rigorous basis in [13], to design effective numerical schemes for incompressible fluids.

2. Proof of Theorem 1

First, let us set

$$u = u^\varepsilon - u^{NS},\ V = V^\varepsilon - V^{NS},\ \varphi = \varphi^\varepsilon - \varphi^{NS},\ \rho = \rho^\varepsilon - 1.$$

Since the pressure φ^{NS} in the incompressible Navier-Stokes equation is given by

$$\Delta\varphi^{NS} = \nabla(u^{NS}\cdot\nabla u^{NS}) = \nabla u^{NS}\cdot\nabla u^{NS},$$

the vector (ρ, u, V, φ) solves the system

$$\begin{cases} \partial_t u + \text{ div } V = \nabla\varphi, \\ \varepsilon\partial_t V + a^2\nabla u = u^{NS}\otimes u + u\otimes u^{NS} - V + u\otimes u + \varepsilon\partial_t V^{NS}, \\ \partial_t\rho + \text{ div } u + u^{NS}\cdot\nabla\rho + \text{ div } (\rho u) = 0, \\ \Delta\varphi = \frac{\rho}{\varepsilon} - \nabla u^{NS}\cdot\nabla u^{NS}. \end{cases} \tag{5}$$

As in [14], we make the following change of unknowns:

$$d = \operatorname{div} u,\ \omega = \operatorname{curl} u,$$
$$D_i = (\text{ div } V)_i = \sum_j \partial_j V_{ij},\ \Omega_{ij} = (\operatorname{curl} V)_{ij} = \partial_i V_j - \partial_j V_i.$$

By using the last equation in (5) and the fact that $\Delta u = \nabla d - \operatorname{curl}\omega$, we get the following system:

$$\begin{cases} \partial_t d + \text{ div } D &= \frac{\rho}{\varepsilon} - \nabla u^{NS}\cdot\nabla u^{NS}, \\ \partial_t\omega + \operatorname{curl} D &= 0, \\ \varepsilon\partial_t D + a^2(\nabla d - \operatorname{curl}\omega) &= -D + \text{ div }\Big(u^{NS}\otimes u + u\otimes u^{NS}\Big) \\ & \quad - \text{ div } (u\otimes u) + \varepsilon\,\partial_t(\text{ div } V^{NS}), \\ \varepsilon\,\partial_t\Omega &= -\Omega + \operatorname{curl}\Big(u^{NS}\otimes u + u\otimes u^{NS}\Big) \\ & \quad +\varepsilon\,\partial_t \operatorname{curl} V^{NS} - \operatorname{curl}(u\otimes u), \\ \partial_t\rho + u^{NS}\cdot\nabla\rho &= -d - \text{ div } (\rho u). \end{cases}$$

This last system can be written as a singular perturbation of a quasilinear symmetrizable hyperbolic system. Setting $\mathcal{U} = (d, \omega, D, \Omega, \rho)$ yields

$$\mathcal{A}_0^\varepsilon\partial_t\mathcal{U} + \mathcal{A}(t,x,\partial_x)\mathcal{U} + \frac{1}{\varepsilon}\mathcal{K}^\varepsilon\mathcal{U} + \mathcal{D}\mathcal{U} + \mathcal{N}(\mathcal{U},\partial_x\mathcal{U}) = \mathcal{L}(t,x,\partial_x)\mathcal{U} + \varepsilon R, \tag{6}$$

where

$$\mathcal{A}_0^\varepsilon = \mathrm{diag}(1,1,\varepsilon,\varepsilon,1),\ \mathcal{D} = \mathrm{diag}(0,0,1,1,0),$$

$$\mathcal{A}(t,x,\partial_x) = \begin{pmatrix} 0 & 0 & \mathrm{div} & 0 & 0 \\ 0 & 0 & \mathrm{curl} & 0 & 0 \\ a^2\nabla & -a^2\,\mathrm{curl} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u^{NS}\cdot\nabla \end{pmatrix},$$

$$\mathcal{K}^\varepsilon = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{L}(t,x,\partial_x)\mathcal{U} = \begin{pmatrix} 0 \\ 0 \\ \mathrm{div}\ (u^{NS}\otimes u + u\otimes u^{NS}) \\ \mathrm{curl}(u^{NS}\otimes u + u\otimes u^{NS}) \\ 0 \end{pmatrix},$$

$$\mathcal{N}(\mathcal{U},\partial_x\mathcal{U}) = \begin{pmatrix} 0 \\ 0 \\ \mathrm{div}\ (u\otimes u) \\ \mathrm{curl}(u\otimes u) \\ \mathrm{div}\ (\rho u) \end{pmatrix},\ R = \begin{pmatrix} -\nabla u^{NS}\cdot\nabla u^{NS} \\ 0 \\ \partial_t\ \mathrm{div}\ (V^{NS}) \\ \partial_t\,\mathrm{curl}(V^{NS}) \\ 0 \end{pmatrix}.$$

Let us remark that $\mathcal{L}$ is a zero order operator when it acts on $\mathcal{U}$. The nonlinear term $\mathcal{N}$ can be split into

$$\mathcal{N}(\mathcal{U},\partial_x\mathcal{U}) = \mathcal{S}(\mathcal{U}) + \mathcal{Q}(\mathcal{U},\,\partial_x)\mathcal{U},$$

where

$$\mathcal{S}(\mathcal{U}) = \begin{pmatrix} 0 \\ 0 \\ \mathrm{div}\ (u\otimes u) \\ \mathrm{curl}(u\otimes u) \\ \rho\ \mathrm{div}\ u \end{pmatrix}$$

is a semilinear term (still considered as acting on $\mathcal{U}$) and $\mathcal{Q}(\mathcal{U},\partial_x)\mathcal{U}$ is a quasilinear term with

$$\mathcal{Q}(\mathcal{U},\partial_x)\mathcal{U} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u\cdot\nabla\rho \end{pmatrix}.$$

Let us set

$$\Sigma^\varepsilon = \ \mathrm{diag}\ (a^2, a^2 I_d, I_d, I_{d\times d}, a^2/\varepsilon),$$

and for $|\alpha| \leq m$

(7)
$$\begin{aligned} E^{\varepsilon}_{\alpha,m}(t) &= \tfrac{1}{2}\Big(\Sigma^{\varepsilon}\mathcal{A}^{\varepsilon}_0\partial^{\alpha}\mathcal{U}, \partial^{\alpha}\mathcal{U}\Big) \\ &= \tfrac{1}{2}\Big(a^2||\partial^{\alpha}d||^2 + a^2||\partial^{\alpha}\omega||^2 + \varepsilon||\partial^{\alpha}D||^2 + \varepsilon||\partial^{\alpha}\Omega||^2 + \tfrac{a^2}{\varepsilon}||\partial^{\alpha}\rho||^2\Big), \end{aligned}$$

$$E^{\varepsilon}_m(t) = \sum_{|\alpha|\leq m} E^{\varepsilon}_{\alpha,m}. \tag{8}$$

Note that since $\Sigma^{\varepsilon}\mathcal{Q}(\mathcal{U},\xi)$ and $\Sigma^{\varepsilon}\mathcal{A}(t,x,\xi)$ are symmetric, we can consider (6) as a quasilinear hyperbolic system with zero order terms (a linear one coming from $\mathcal{L}$ and a semilinear one coming from $\mathcal{S}$ which are pseudo-differential operators of order zero when acting on $\mathcal{U}$. Consequently, for $\varepsilon > 0$ fixed, we have a result of local existence and uniqueness of strong solutions in $C_T(H^m)$; see for instance [1, 17]. This allows us to define T^{ε} as the largest time such that

$$E^{\varepsilon}_m(t) \leq M_{\varepsilon},\ \forall t \in [0, T^{\varepsilon}), \tag{9}$$

where M_{ε} which is such that $M_{\varepsilon} \to 0$ when ε goes to zero will be chosen carefully later. To achieve the proof of Theorem 1, and in particular inequality (4), it is sufficient to establish that $T^{\varepsilon} \geq T$, which will be proved by showing that in (9) the equality cannot be reached for $T^{\varepsilon} < T$ thanks to a good choice of M_{ε}.

Before performing the energy estimate, we apply the operator ∂^{α} for $|\alpha| \leq m$ to (6), to obtain

$$\begin{aligned} \mathcal{A}^{\varepsilon}_0\partial_t\partial^{\alpha}\mathcal{U} + \mathcal{A}(t,x,\partial_x)\partial^{\alpha}\mathcal{U} \quad &+\tfrac{1}{\varepsilon}\mathcal{K}^{\varepsilon}\partial^{\alpha}\mathcal{U} + \mathcal{D}\partial^{\alpha}\mathcal{U} + \partial^{\alpha}\mathcal{N}(\mathcal{U}) \\ &= \partial^{\alpha}\mathcal{L}(t,x,\partial_x)\mathcal{U} + \varepsilon R + [\partial^{\alpha}, \mathcal{A}(t,x,\partial_x)]\mathcal{U}. \end{aligned} \tag{10}$$

Now, we proceed to perform the energy estimates for (10) in a classical way by taking the scalar product of system (10) with $\Sigma^{\varepsilon}\partial^{\alpha}\mathcal{U}$. Along the proof, we shall denote by C a number independent of ε, which actually may change from line to line, and by $\mathcal{C}(\cdot)$ a nondecreasing function. Moreover $(\cdot,\cdot)$ and $||\cdot||$ stand for the usual L^2 scalar product and norm, $||\cdot||_m$ is the usual H^m Sobolev norm and $||\cdot||_{m,\infty}$ is the usual $W^{m,\infty}$ norm.

Let us start the estimate of each term. First, since $\Sigma^{\varepsilon}\mathcal{A}(x,\xi)$ is symmetric and div $u^{NS} = 0$, we have that

$$\Big|\Big(\Sigma^{\varepsilon}\mathcal{A}(x,\partial_x)\partial^{\alpha}\mathcal{U}, \partial^{\alpha}\mathcal{U}\Big)\Big| = \Big|\frac{a^2}{\varepsilon}\int \text{ div } (u^{NS})|\partial^{\alpha}\rho|^2\,dx\Big| = 0. \tag{11}$$

Next, since $\Sigma^{\varepsilon}\mathcal{K}^{\varepsilon}$ is skew-symmetric, we have that

$$\Big(\Sigma^{\varepsilon}\mathcal{K}^{\varepsilon}\,\partial^{\alpha}\mathcal{U}, \partial^{\alpha}\mathcal{U}\Big) = 0. \tag{12}$$

Also, we have

$$\Big(\Sigma^{\varepsilon}\mathcal{D}\,\partial^{\alpha}\mathcal{U}, \partial^{\alpha}\mathcal{U}\Big) = ||\partial^{\alpha}D||^2 + ||\partial^{\alpha}\Omega||^2. \tag{13}$$

Now we observe that, since $\int_{\mathbb{T}^d} u = 0$, we have the inequality

$$||u||^2_{m+1} \leq C(||d||^2_m + ||\omega||^2_m) \leq C||\nabla u||^2_m.$$

Hence, the Young inequality gives

(14)
$$\begin{aligned}\big(\Sigma^\varepsilon\partial^\alpha\mathcal{L},\partial^\alpha\mathcal{U}\big) &= \int\partial^\alpha\ \mathrm{div}\ \big(u^{NS}\otimes u+u\otimes u^{NS}\big)\,\partial^\alpha D\\ &+\int\partial^\alpha\,\mathrm{curl}\big(u^{NS}\otimes u+u\otimes u^{NS}\big)\,\partial^\alpha\Omega\\ &\le \tfrac14\big(||\partial^\alpha D||^2+||\partial^\alpha\Omega||^2\big)+\mathcal{C}(||u^{NS}||_{m+1,\infty})\big(||\omega||_m^2+||d||_m^2\big)\\ &\le \tfrac14\big(||\partial^\alpha D||^2+||\partial^\alpha\Omega||^2\big)+\mathcal{C}(||u^{NS}||_{m+1,\infty})E_m^\varepsilon.\end{aligned}$$

To give the estimate of the term $(\Sigma^\varepsilon\partial^\alpha\mathcal{N},\partial^\alpha\mathcal{U})$, we split it in several terms. The first one is given by, for $m>d/2$,

$$\begin{aligned}&\int\Big(\partial^\alpha\ \mathrm{div}\ (u\otimes u)\quad\partial^\alpha D+\partial^\alpha\,\mathrm{curl}(u\otimes u)\,\partial^\alpha\,\Omega\Big)\\ &\qquad\le \tfrac14\Big(||\partial^\alpha D||^2+||\partial^\alpha\Omega||^2\Big)+C||u\otimes u||_{m+1}^2\\ &\qquad\le \tfrac14\Big(||\partial^\alpha D||^2+||\partial^\alpha\Omega||^2\Big)+(E_m^\varepsilon)^2.\end{aligned}\tag{15}$$

The second one is

$$\begin{aligned}&\tfrac1\varepsilon\int\partial^\alpha\ \mathrm{div}\ (\rho u)\,\partial^\alpha\rho\\ &=\tfrac1\varepsilon\int u\cdot\nabla\partial^\alpha\rho\ \partial^\alpha\rho+[\partial^\alpha,u\cdot\nabla]\rho\,\partial^\alpha\rho+\partial^\alpha(\rho\ \mathrm{div}\ u)\,\partial^\alpha\rho\\ &=I+II+III.\end{aligned}$$

By an integration by parts, we have $I=-\frac1\varepsilon\int\ \mathrm{div}\ u\,|\partial^\alpha\rho|^2$, hence

$$I\le||\ \mathrm{div}\ \ u||_{0,\infty}\,E_m^\varepsilon\le(E_m^\varepsilon)^{\frac32}.$$

The terms II and III are easily estimated by

$$\begin{aligned}II &\le C||u||_m\,\frac{||\rho||_m^2}{\varepsilon}\le C(E_m^\varepsilon)^{\frac32},\\ III &\le C\Big(||u||_m+||\nabla u||_m\Big)\frac{||\rho||_m^2}{\varepsilon}\le C(||d||_m+||\omega||_m)\frac{||\rho||_m^2}{\varepsilon}\le C(E_m^\varepsilon)^{\frac32}.\end{aligned}$$

Therefore we obtain the estimate

$$\frac1\varepsilon\int\partial^\alpha\ \mathrm{div}\ (\rho u)\,\partial^\alpha\rho\le C(E_m^\varepsilon)^{\frac32}.\tag{16}$$

To estimate the commutator, we have

$$\begin{aligned}\Big(\Sigma^\varepsilon\,[\partial^\alpha,\mathcal{A}(x,\partial_x)]\,\mathcal{U},\mathcal{U}\Big)&=\tfrac1\varepsilon\int[\partial^\alpha,u^{NS}\cdot\nabla]\rho\,\partial^\alpha\rho\,dx\\ &\le C\tfrac{||\rho||_m}{\varepsilon}\Big(||u^{NS}||_m\,||\nabla\rho||_{0,\infty}+||\nabla u^{NS}||_{0,\infty}||\nabla\rho||_{m-1}\Big)\\ &\le\mathcal{C}\Big(||u^{NS}||_{1,\infty}+||u^{NS}||_m\Big)E_m^\varepsilon.\end{aligned}\tag{17}$$

To get the last line, we have used the fact that $m>1+d/2$.

Finally, we have that

$$
\begin{aligned}
\varepsilon\big|\big(\Sigma^\varepsilon\partial^\alpha R, \partial^\alpha \mathcal{U}\big)\big| &\le \varepsilon||\nabla u^{NS}\cdot\nabla u^{NS}||_m\,||d||_m + ||\partial_t \operatorname{div} V^{NS}||_m\,||D||_m \\
&\quad + ||\partial_t \operatorname{curl} V^{NS}||_m\,||\Omega||_m \\
&\le ||d||_m^2 + \tfrac{1}{4}\big(||D||_m^2 + ||\Omega||_m^2\big) + \varepsilon^2\mathcal{C}(||u^{NS}||_{m+1} + ||\partial_t V^{NS}||_m) \\
&\le E_m^\varepsilon + \tfrac{1}{4}\Big(||D||_m^2 + ||\Omega_m||^2\Big) + \varepsilon^2\mathcal{C}(||u^{NS}||_{m+3}).
\end{aligned}
\tag{18}
$$

Note that, to get the last line, we have used the explicit expression (2) of V^{NS} and the Navier-Stokes equations to replace the time derivative by spatial ones.

Now, we collect all the previous estimates (11)–(18), and we sum over α to find

$$
\begin{aligned}
\tfrac{d}{dt}E_m^\varepsilon + \tfrac{1}{4}\Big(||D||_m^2 + ||\Omega||_m^2\Big) &\le \varepsilon^2\mathcal{C}(||u^{NS}||_{m+3}) + \mathcal{C}(||u^{NS}||_{m+3})E_m^\varepsilon \\
&\quad + (E_m^\varepsilon)^2 + C(E_m^\varepsilon)^{\frac{3}{2}} \\
&\le \varepsilon^2\mathcal{C}(||u^{NS}||_{m+3}) + \mathcal{C}(||u^{NS}||_{m+3})\,E_m^\varepsilon + (E_m^\varepsilon)^2.
\end{aligned}
$$

By using (9), we get with $M_\varepsilon \le 1$ that

$$
\frac{d}{dt}E_m^\varepsilon \le \mathcal{C}_{m+3}\varepsilon^2 + \mathcal{C}_{m+3}E_m^\varepsilon,\ \forall t\in[0,T^\varepsilon)
$$

where $\mathcal{C}_{m+3} = \mathcal{C}(||u^{NS}||_{m+3})$, and hence by the Gronwall inequality we get that

$$
E_m^\varepsilon(t) \le \Big(\alpha_m(\varepsilon) + t\,\mathcal{C}_{m+3}\varepsilon^2\Big)e^{\mathcal{C}_{m+3}t},\ \forall t\in[0,T^\varepsilon). \tag{19}
$$

Consequently, if we choose $M_\varepsilon = (\alpha_m(\varepsilon) + T\,\mathcal{C}_{m+3}\varepsilon^2)^{\frac{1}{2}}$, we see that we cannot reach equality in (9) for $T^\varepsilon < T$. This proves that $T^\varepsilon \ge T$ and that (19) is valid on $[0,T]$.

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