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A MATRICIAL CORONA THEOREM

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Abstract. We show that a usual corona-type theorem on a space of functions automatically extends to a matrix version.

In this paper we give a simple algebraic argument to extend the corona theorem for a given algebra of bounded functions to a one-sided infinite matrix corona theorem. In addition, we provide estimates for the size of solutions. For the algebra $H^\infty(D)$, this was established by Fuhrmann [5] for the finite matrix case and by Vasyunin (see Nikolski [7]) for the one-sided infinite case. We note that an important result of Treil [11] shows that a complete extension of the corona theorem to two-sided infinite matrices is not possible, in general. Our technique is similar to that of Fuhrmann [5], but our proofs are different. We prove a general version which can be applied to such algebras as the multipliers on Dirichlet space. Moreover, there is an H^p version for $H^\infty(D^n)$. Our results provide sharper estimates than those obtained by one of the authors, Zhang [15].

For ease of notation, we will assume that $\mathcal{A}$ denotes a multiplier algebra for a reproducing kernel Hilbert space of functions on Ω. That is, $H(\Omega)$ is a Hilbert space of functions on Ω, such that, for each $w \in \Omega$, there exists a unique $k_w \in H(\Omega)$, the reproducing kernel, which satisfies $f(w) = \langle f, k_w \rangle_{H(\Omega)}$ for all $f \in H(\Omega)$. Then $\mathcal{A} = \{g \in H(\Omega) \mid M_g \in B(H(\Omega))$, where $(M_g f)(w) = g(w)f(w)$ for all $w \in \Omega\}$. If B is an algebra of bounded functions, which is not such a multiplier algebra, we only consider finite matrices, so the arguments appearing below are easily modified in this case.

For $\{f_j\}_{j=1}^\infty \subset \mathcal{A}$ and $z \in \Omega$, we let $F(z) = (f_1(z), f_2(z), \dots)$ and define $M_F(\{g_j\}_{j=1}^\infty) = \sum_{j=1}^\infty f_j g_j$, an operator acting from $\bigoplus_1^\infty H(\Omega)$ to $H(\Omega)$. M_F^T will denote the transpose of M_F acting from $H(\Omega)$ to $\bigoplus_1^\infty H(\Omega)$.

Since $M_F^*(k_w) = (f_1\overline{(w)}k_w, f_2\overline{(w)}k_w, \dots)^T$, we see that

$$\sum_{j=1}^\infty |f_j(w)|^2 \leq \|M_F\|^2 \text{ for all } w \in \Omega.$$

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We assume that $\mathcal{A}$ satisfies a "corona theorem"; that is,

Theorem (CT). *Assume that* $\{f_j\}_{j=1}^{\infty} \subset \mathcal{A}$ *and* $F = (f_1, f_2, \dots)$ *satisfies*

$$(1) \quad 0 < \delta^2 \leq F(z)F(z)^*$$

and

$$(2) \quad \max\{\|M_F\|, \|M_F^T\|\} = 1.$$

Then there exists $\{g_j\}_{j=1}^{\infty} \subset \mathcal{A}$*, such that if* $G = (g_1, g_2, \dots)$,

$$\begin{aligned} &\text{(a)} \quad F\,G^T = 1 \ \textit{ in } \Omega, \\ &\text{(b)} \quad \max\{\|M_G\|, \|M_G^T\|\} < \infty. \end{aligned}$$

When **CT** holds for $\mathcal{A}$ and F satisfies (1) and (2), we define

$$C_1(\delta) = \inf\{\max\{\|M_G\|, \|M_G^T\|\} \,:\, FG^T = 1 \text{ and } G = (g_1, \dots) \text{ with } g_i \in \mathcal{A}\}.$$

Note. (i) In some algebras such as multipliers on Dirichlet space

$$\|M_G^T\| < \infty \Rightarrow \|M_G\| < \infty,$$

so (2) above may be replaced by

$$(2') \ \|M_F^T\| = 1.$$

(See Trent [12].)

(ii) For $\mathcal{A} = H^{\infty}(\Omega), \Omega \subset \mathbb{C}^n$, bounded and open,

$$\|M_F\| = \|M_F^T\| = \sup_{z\in\Omega} (\sum_{j=1}^{\infty} |f_j(z)|^2)^{\frac{1}{2}}.$$

So (2) above is equivalent to $(2')\sup\limits_{z\in\Omega} (\sum_{j=1}^{\infty} |f_j(z)|^2) = 1$.

(iii) For $F = (f_1, \dots, f_j)$, a finite number of elements of $\mathcal{A}$, (2) may be omitted from the hypotheses.

[From (i), (ii), and (iii), we see that variations of **CT** involving changing hypothesis (2) are possible. Extensions of **CT** for such versions follow from our techniques below, but we will stick to the above formulation of CT.]

We establish the "matricial corona theorem" for such algebras $\mathcal{A}$, which satisfy **CT**.

Theorem (MCT). *Assume that **CT** holds for* $\mathcal{A}$*. Let* F *denote an* $m \times \infty$ *matrix of elements of* $\mathcal{A}$ *satisfying:*

$$(1) \ 0 < \epsilon^2 I_m \leq F(z)F(z)^* \ \textit{ for all } z \in \Omega$$

and

$$(2) \ \max\{\|M_F\|, \|M_F^T\|\} = 1.$$

Then there exists an $m \times \infty$ *matrix* G *with entries in* $\mathcal{A}$ *satisfying*

$$\begin{aligned} &\text{(a)} \ F\,G^T = I_m, \\ &\text{(b)} \ \max\{\|M_G\|, \|M_G^T\|\} < \infty. \end{aligned}$$

Moreover, we may choose G, as above, so that

$$\max\{\|M_G\|, \|M_G^T\|\} \leq \sqrt{m}\, m!\, C_1 \left(\frac{\epsilon^{2m}}{m!}\right).$$

Denote the rows of F satisfying (1) and (2) of **MCT** by $f_1, \dots, f_m$. For $1 \leq j \leq m$, we will denote the jth term of the Koszul complex with 0th element f_p by $Q_p^{(j)}$. This means that for each $z \in \Omega$

$$\begin{aligned} \ker\, f_p(z) &\supseteq \text{ ran } Q_p^{(1)}(z), \\ \ker\, Q_p^{(1)}(z) &\supseteq \text{ ran } Q_p^{(2)}(z) \\ &\vdots \end{aligned} \tag{1}$$

and, moreover,

$$\begin{aligned} f_p(z)^* f_p(z) + Q_p^{(1)}(z) Q_p^{(1)}(z)^* &= (f_p(z) f_p(z)^*)\, I, \\ Q_p^{(1)}(z)^* Q_p^{(1)}(z) + Q_p^{(2)}(z) Q_p^{(2)}(z)^* &= (f_p(z) f_p(z)^*)\, I \\ &\vdots \end{aligned} \tag{2}$$

Since at each $z \in \Omega$, $f_p(z) f_p(z)^* \geq \epsilon^2 > 0$, we deduce that

$$\begin{aligned} \ker\, f_p(z) &= \text{ ran } Q_p^{(1)}(z), \\ \ker\, Q_p^{(1)}(z) &= \text{ ran } Q_p^{(2)}(z) \\ &\vdots \end{aligned} \tag{3}$$

In addition,

$$Q_p^{(j)}(z) Q_q^{(j+1)}(z) = -Q_q^{(j)}(z) Q_p^{(j+1)}(z) \tag{4}$$

and

$$f_1(z) Q_2^{(1)}(z) \cdots Q_m^{(m-1)}(z) Q_m^{(m-1)}(z)^* \cdots f_1(z)^* = \det(F(z)F(z)^*). \tag{5}$$

See the Appendix for the existence of $\{Q_p^{(j)}(z)\}_{j=1}^{\infty}$ and the above properties.

We will need a lemma.

Lemma. *Let* $A = (a_1, a_2, \dots), a_i \in \mathcal{A}$, *and assume that*

$$\max\{\|M_A\|, \|M_A^T\|\} < \infty.$$

Fix $z \in \Omega$ *and let*

$$Q^{(j)}(z) = Q_A^{(j)}(z).$$

Then for $1 \leq j$

$$\|M_{Q^{(j)}}\| \leq (j+1)\|M_A\| \text{ and } \|M_{Q^{(j)}}^T\| \leq (j+1)\|M_A^T\|.$$

Proof. We will show that the first inequality holds. The second follows in a similar manner. See the Appendix for notation. Let $\mathcal{I}_j$ denote increasing j-tuples of positive integers. Now

$$\begin{aligned} Q^{(j)}(z)^*\underline{x}(z) &= \overline{A(z)} \wedge \underline{x}(z) \\ &= (\sum_{j=1}^{\infty} \overline{a_j(z)}\, e_j) \wedge (\sum_{\pi \in \mathcal{I}_j} x_\pi(z)\, e_\pi) \\ &= \sum_{j=1}^{\infty} \sum_{\pi \in \mathcal{I}_j} \overline{a_j(z)}\, x_\pi(z)\, e_j \wedge e_\pi \\ &= \sum_{1 \le i_1 < i_2 < \dots < i_{j+1}} \sum_{r=1}^{j+1} (-1)^{r-1}\, \overline{a_r(z)}\, x_{i_1,\dots,\widehat{i_r},\dots,i_{j+1}}\, e_{i_1,\dots,i_{j+1}}. \end{aligned}$$

Thus

$$\begin{aligned} \|M^*_{Q^{(j)}}\underline{x}\|^2 &= \sum_{\substack{\sigma \in \mathcal{I}_{j+1} \\ (\sigma = (i_1,\dots,i_{j+1}))}} \|\sum_{r=1}^{j+1} (-1)^{r-1} M a_r^* x_{\sigma - \{i_r\}}\|^2 \\ &\le (j+1) \sum_{i_1 < \dots < i_{j+1}} \sum_{r=1}^{j+1} \|M a_{i_r}^* x_{\sigma - \{i_r\}}\|^2 \\ &= (j+1) \sum_{i_1 < \dots < i_{j+1}} (\|M a_{i_1}^* x_{\sigma - \{j_1\}}\|^2 + \dots \\ &\qquad + \|M a_{i_{j+1}}^* x_{\sigma - \{i_{j+1}\}}\|^2) \\ &\le (j+1)\, \|M_A^*\|^2 (\sum_{i_2 < \dots < i_{j+1}} \|x_{i_2,\dots,i_{j+1}}\|^2 + \dots \\ &\qquad + \sum_{i_1 < \dots < i_j} \|x_{i_1,\dots,i_j}\|^2) \\ &\le (j+1)^2 \|M_A\|^2 \|\underline{x}\|^2. \end{aligned}$$

So

$$\|M_{Q^{(j)}}\| \le (j+1)\, \|M_A\|.$$

$\square$

We are ready to begin the proof of **MCT**.

Proof. (**MCT**) Since $\epsilon^2 I_m \le F(z)F(z)^*$ for $z \in \Omega$, we have $\epsilon^{2m} \le \det(F(z)F(z)^*)$ for all $z \in \Omega$. Thus, if we define

$$H(z) = f_1(z) Q_2^{(1)}(z) \cdots Q_m^{(m-1)}(z), \text{ then } H = (h_1, h_2, \dots)$$

with $h_i \in \mathcal{A}$ and

$$\epsilon^{2m} \le \det(F(z)F(z)^*) = H(z)H(z)^*$$

by (5). From the lemma, we have

$$\begin{aligned}\|M_H\| \leq \|M_{f_1}\| \|M_{Q_2^{(1)}}\| \cdots \|M_{Q_m^{(m-1)}}\| &\leq \|M_{f_1}\| (2\,\|M_{f_2}\|) \cdots (m\,\|M_{f_m}\|) \\ &= m!\prod_{j=1}^{m} \|M_{f_j}\| \\ &\leq m!\,\|M_F\|^m \\ &\leq m!\,,\text{ since } \|M_F\| \leq 1.\end{aligned}$$

Applying **CT** to $\frac{H}{m!}$ with $\delta = \frac{\epsilon^m}{m!}$, we get

$$R = (g_1, g_2, \dots) \text{ with } g_i \in \mathcal{A}$$

so that

$$H\,R^T = 1$$

and

$$\max\{\|M_R\|, \|M_R^T\|\} \leq C_1(\frac{\epsilon^m}{m!}).$$

Define

$$\begin{aligned}G^T =& [Q_2^{(1)} \cdots Q_m^{(m-1)} R^T, -Q_1^{(1)} Q_3^{(2)} \cdots Q_m^{(m-1)} R^T, \dots, \\ & (-1)^{j-1} Q_1^{(1)} Q_2^{(2)} \cdots \widehat{Q_j^{(j)}} Q_{j+1}^{(j)} \cdots Q_m^{(m-1)} R^T, \dots, \\ & (-1)^{m-1} Q_1^{(1)} \cdots Q_{m-1}^{(m-1)} R^T].\end{aligned}$$

Then

$$\|M_{G^T}\| \leq \sqrt{m}\,m!\,C_1(\frac{\epsilon^m}{m!}).$$

Similarly,

$$\|M_G\| \leq \sqrt{m}\,m!\,C_1(\frac{\epsilon^m}{m!}).$$

We need only check that

$$F\,G^T = I_m.$$

But

$$f_j G^T = [f_j Q_2^{(1)} \cdots Q_m^{(m-1)} R^T, \dots, (-1)^{m-1} f_j Q_1^{(1)} \cdots Q_{m-1}^{(m-1)} R^T].$$

The indices of the Q's in each term consist of $I_p = \{1, \dots, m\} - \{p\}$ for $p = 1, \dots, m$. For $p \neq j$, we see that

$$(*) \overset{\text{def}}{=} f_j Q_1^{(1)} \cdots Q_{p-1}^{(p-1)} Q_{p+1}^{(p)} \cdots Q_m^{(m-1)} R^T = 0.$$

This follows since $f_j Q_j^{(1)} = 0$ and $Q_s^{(r)} Q_t^{(r+1)} = -Q_t^{(r)} Q_s^{(r+1)}$, so since $j \in \{1, \dots, m\} - \{p\}$, we may "anti-"commute the Q's until we get

$$(*) = f_j Q_j^{(1)} \cdots R^T = 0.$$

Otherwise, $p = j$, and we have

$$\begin{aligned}
&(-1)^{j-1} f_j Q_1^{(1)} \cdots Q_{j-1}^{(j-1)} Q_{j+1}^{(j)} \cdots Q_m^{(m-1)} R^T \\
=&(-1)^{j-2} f_1 Q_j^{(1)} Q_2^{(2)} \cdots Q_{j-1}^{(j-1)} Q_{j+1}^{(j)} \cdots Q_m^{(m-1)} R^T \\
=&(-1)^{j-3} f_1 Q_2^{(1)} Q_j^{(2)} Q_3^{(3)} \cdots Q_{j-1}^{(j-1)} Q_{j+1}^{(j)} \cdots Q_m^{(m-1)} R^T \\
&\vdots \\
=&f_1 Q_2^{(1)} Q_3^{(2)} \cdots Q_m^{(m-1)} R^T \\
=&H\, R^T \\
=&1.
\end{aligned}$$

This completes our proof of the matricial corona theorem. □

Applications:

(1) For $\mathcal{A} = H^\infty(D)$, the finite corona theorem is due to Carleson [2] and the infinite corona theorem is due to Rosenblum [8] and Tolokonnikov [10], independently. Our results give the matricial corona theorem due to Fuhrmann and Vasyunin. In this case, our conditions $(*)$ show that if $\epsilon^2 I_m \leq F(z)F(z)^* \leq I_m$, then $\|M_{Q_p^{(i)}}\| \leq 1$. Therefore, if F is an $m \times \infty$ matrix with entries $f_{ij} \in H^\infty(D)$ satisfying $\delta^2 I_m \leq F(z)F(z)^* \leq I_m$ for all $z \in D$, we get an $\infty \times m$ solution G with entries $f_j \in H^\infty(D)$ satisfying

$$\|M_G\| = \|M_G^T\| \leq \sqrt{m}\, C_1(\delta^m).$$

For $\delta \approx 0$, the best estimate of $C_1(\delta)$ is $\frac{C_0}{\delta^2} \ln \frac{1}{\delta}$ due to Uchiyama (see Nikolski [7]).

So we only get

$$\|M_G\| = \|M_G^T\| \leq \sqrt{m}\, \frac{C_0}{\delta^{2m}} \log \frac{1}{\delta^m}.$$

We note that for this particular case, where $\mathcal{A} = H^\infty(D)$, a better estimate in the matricial corona theorem due to Trent [13] gives

$$\|M_G\| \leq \frac{C_0}{\delta^{m+1}} \log \frac{1}{\delta^m}.$$

(2) If $\Omega \subset \mathbb{C}$ is a finitely connected domain, then the corona theorem holds for $H^\infty(\Omega)$ by Stout [9] and Forelli [4]. (See Fisher [3] for an exposition of these results.) Our result gives a matricial version.

(3) For $\mathcal{A} = \mathcal{D}^2(D) \cap H^\infty(D)$ (where $\mathcal{D}^2(D)$ denotes Dirichlet space), by results of Nicolau [6], $\mathcal{A}$ satisfies a "finite" corona theorem. Thus our results give a "finite" matricial version corona theorem for $\mathcal{D}^2(D) \cap H^\infty(D)$.

(4) For $\mathcal{A} = \mathcal{M}(\mathcal{D}^2(D))$, multipliers on $\mathcal{D}^2(D)$, the ∞-corona theorem is due to Trent [12]. Our results give a matricial corona theorem for $\mathcal{M}(D^2(D))$.

(5) For $1 \leq p < \infty$, let

$$\begin{aligned}
\mathcal{H}^p(D^n) = \{ f : D^n \to l^2 \ \mid\ & f \text{ is analytic, } l^2\text{-valued in } D^n \text{ and} \\
|f|_p^p \overset{\text{def}}{=} \sup_{r \nearrow 1} (& \int_{T^n} \| f(r\, e^{it_1}, r\, e^{it_2}, \ldots, r\, e^{it_n}) \|_2^p \, d\sigma(t_1) \cdots d\sigma(t_n) < \infty) \}.
\end{aligned}$$

In [14], Trent has proven the $\mathcal{H}^p$-corona theorem.

Theorem ($\mathbf{H^p}$-CT). *Let* $F \in \mathcal{H}^\infty(D^n)$ *satisfy* $0 < \epsilon^2 \leq F(z)F(z)^* \leq 1$ *for all* $z \in D^n$*. Then the Toeplitz operator,* $T_F : \mathcal{H}^p(D^n) \to H^p(D^n)$*, is onto for each* $1 \leq p < \infty$*.*

[Note: The case that $p = \infty$ is an unsolved problem for the polydisk.]

A modification of the proof of **MCT** leads to an H^p-matricial corona theorem. That is,

Theorem ($\mathbf{H^p}$-MCT). *Let* F *be a* $m \times \infty$ *matrix of elements of* $H^\infty(D^n)$*. Assume that*

$$0 < \epsilon^2 I_m \leq F(z)F(z)^* \leq I_m.$$

Then for $1 \leq p < \infty$*, the Toeplitz operator* $T_F : \mathcal{H}^p(D^n) \to \bigoplus_1^m H^p(D^n)$*, is onto.*

Proof. We will outline the modifications of the proof of the **MCT** necessary for this proof. First

$$\|T_F\| = \sup_{z \in D^n} \|F(z)\|_{B(l^2, \mathbb{C}^m)} \leq 1$$

by hypothesis. For

$$H(z) = f_1(z)\, Q_{f_2}^{(1)}(z) \cdots Q_{f_m}^{(m-1)}(z)$$

with $z \in D^n$, we have

$$\epsilon^{2m} \leq H(z)H(z)^* \leq 1.$$

This follows since

$$Q_{f_j}^{(r)}(z) Q_{f_j}^{(r)}(z)^* \leq (f_j(z)f_j(z)^*)I \leq I.$$

Thus $T_H : \mathcal{H}^p(D^n) \to H^p(D^n)$ is onto. Let $\underline{h} = (h_1, \ldots, h_m) \in \bigoplus_1^m H^p(D^n)$. Choose $u_k \in H^p(D^n)$, so that $T_H(u_k) = H\, u_k = h_k$ for $k = 1, \ldots, m$. Let

$$\underline{g}^T = [Q_2^{(1)} \cdots Q_m^{(m-1)} u_1, \ldots, (-1)^{m-1} Q_1^{(1)} \cdots Q_{m-1}^{(m-1)} u_m].$$

Then

$$F\, \underline{g} = \underline{h}.$$

Moreover, $\underline{g}$ can be chosen so that

$$\begin{aligned} \|\underline{g}\|_p^p &\leq \sum_{j=1}^m \|u_j\|_p^p \\ &\leq \|\underline{h}\|_p^p \\ &\leq m\, \|[T_H\, |_{(\ker\ T_H)^\perp}]^{-1}\|^p\, \|h\|_p^p. \end{aligned}$$

Finally,

$$\|g\|_p \leq m^{\frac{1}{p}} C_1(\epsilon^m, p)\, \|h\|_p.$$

Here $C_1(\delta, p)$ denotes

$$\|(T_F\, |_{(\ker\ T_F)^\perp})^{-1}\|_{B(H^p(D^n), \mathcal{H}^p(D^n))}.$$

The estimate for $C_1(\delta, p)$ in [14] is

$$C_1(\delta, p) \leq \frac{C_0\, p^{2n-1}}{\delta^{3n+1}}.$$

So

$$C_m(\delta, p) \leq \frac{m^{\frac{1}{p}} C_0\, p^{2n-1}}{\delta^{(3n+1)m}}$$

gives the matricial estimate. □

Appendix

Let $\mathcal{A}$ be an algebra of functions defined on Ω. The basic exterior algebra proof of the linear algebra results we need concerning the Koszul complex can be found in, for example, Birkhoff-Mac Lane [1, Problem 4, page 566].

We will sketch the basic idea. Note that although our operators defined below are (of course) "bases free", it is only with respect to a particular choice of basis that the entries of the corresponding matrices belong to $\mathcal{A}$.

For our notation, $l^2_{(n)}$ will denote the exterior product of l^2 with itself n-times, i.e. $l^2_{(n)} = l^2 \wedge \cdots \wedge l^2$ (n times). For $n = 0$, $l^2_{(0)} = \mathbb{C}$. Let $\{e_j\}_{j=1}^\infty$ denote the standard basis in l^2. If $\mathcal{I}_n$ denotes increasing n-tuples of positive integers and if $(i_1, \ldots, i_n) \in \mathcal{I}_n$, we let $\pi_n = \{i_1, \ldots, i_n\}$, and abusing notation, we write $\pi_n \in \mathcal{I}_n$.

Define $e_{\pi_n} = e_{i_1} \wedge \cdots \wedge e_{i_n}$. Then $\{e_{\pi_n}\}_{\pi_n \in \mathcal{I}_n}$ denotes the standard basis for $l^2_{(n)}$.

For $f_j \sim \{v_n\}_{n=1}^\infty$ and $v_n \in \mathcal{A}$, we assume that $\epsilon^2 \leq f_j(\mathbf{z}) f_j(\mathbf{z})^* \leq 1$ for all $\mathbf{z} \in \Omega$. Fix $\mathbf{z} \in \Omega$. For $n = 0, 1, \ldots$ define

$$Q_j^{(n)*}(\mathbf{z}) : \, l^2_{(n)} \to l^2_{(n+1)}$$

by

$$Q_j^{(n)*}(\mathbf{z})(w_n) = \overline{f_j(\mathbf{z})} \wedge w_n, \text{ where } w_n \in l^2_{(n)}.$$

Clearly (1) holds and since

$$Q_j^{(n+1)*}(\mathbf{z}) Q_k^{(n)*}(\mathbf{z})(w_n) = \overline{f_j(\mathbf{z})} \wedge \overline{f_k(\mathbf{z})} \wedge w_n = -Q_k^{(n+1)*}(\mathbf{z}) Q_j^{(n)*}(\mathbf{z})(w_n)$$

we see that (4) holds.

Now

$$Q_j^{(n)*}(\mathbf{z})(e_{\pi_n}) = \sum_{p=1}^\infty \overline{v_p(\mathbf{z})}\, e_p \wedge e_{\pi_n},$$

so with respect to the standard basis, entries of $Q_j^{(n)*}(\mathbf{z})$ are 0 or else $\pm\overline{v_n(\mathbf{z})}$ for some n. Thus $Q^{(n)}(\cdot)$ has entries belonging to $\mathcal{A}$ with respect to the standard basis.

Proof. Fix $\mathbf{z} \in \Omega$ and let $a = \overline{f_j(\mathbf{z})}$, for fixed j, and $Q_n = Q_j^{(n)}(\mathbf{z})$. Then $Q_n^*(w_n) = a \wedge w_n$. Choose an orthonormal basis $\{u_n\}_{n=1}^\infty$ of l^2 with $u_1 = \frac{a}{\|a\|}$. (Note that $\|a\|^2 \geq \epsilon^2$.) Then it follows that for $\pi_n \in \mathcal{I}_n$ and $u_{\pi_n} = u_{i_1} \wedge \cdots \wedge u_{i_n}$, we have that $\{u_{\pi_n}\}_{\pi_n \in \mathcal{I}_n}$ is an orthonormal basis for $l^2_{(n)}$.

Thus

$$\begin{aligned} Q_n(w_{n+1}) &= \sum_{\pi_n \in I_n} \langle Q_n(w_{n+1}), u_{\pi_n} \rangle u_{\pi_n} \\ &= \sum_{\pi_n \in \mathcal{I}_n} \langle w_{n+1}, a \wedge u_{\pi_n} \rangle u_{\pi_n} \\ &= \|a\| \sum_{\pi_n \in \mathcal{I}_n} \langle w_{n+1}, u_1 \wedge u_{\pi_n} \rangle u_{\pi_n}. \end{aligned} \tag{6}$$

We wish to show that for $n = 0, 1, \ldots,$

$$Q_n^* Q_n + Q_{n+1} Q_{n+1}^* = \|a\|^2 I_{l^2_{(n+1)}}. \tag{7}$$

For $n = 0$, $\frac{Q_0^* Q_0}{\|a\|^2}$ is the rank one projection of l^2 onto a. So given (7), $\frac{Q_1 Q_1^*}{\|a\|^2}$ is a projection. But then $\frac{Q_1^* Q_1}{\|a\|^2}$ must be a projection. Applying (7) again, we see that $\frac{Q_2 Q_2^*}{\|a\|^2}$ is a projection. Repeating this procedure, we conclude that $\frac{Q_n Q_n^*}{\|a\|^2}$ is the projection onto the range of Q_n. Also, given (7), it follows that Ker $Q_n =$ ran Q_{n+1}.

To prove (7) it suffices to check that for $w_{n+1} \in l^2_{(n+1)}$,

$$\|Q_n(w_{n+1})\|^2 + \|Q_{n+1}^*(w_{n+1})\|^2 = \|a\|^2 \|w_{n+1}\|^2. \tag{8}$$

Denote w_{n+1} by w. Then from (8), we see that

$$\begin{aligned}\|Q_n(w)\|^2 &= \|a\|^2 \sum_{\substack{\pi_n \in \mathcal{I}_n \\ 1 \notin \pi_n}} |\langle w, u_{1,\pi_n}\rangle|^2 \\ &= \|a\|^2 \sum_{\substack{\pi_{n+1} \in \mathcal{I}_{n+1} \\ 1 \in \pi_{n+1}}} |\langle w, u_{\pi_{n+1}}\rangle|^2.\end{aligned}$$

Also, since

$$\begin{aligned}Q_{n+1}^*(w) = a \wedge \underline{w} &= \|a\|\, u_1 \wedge \sum_{\pi_{n+1} \in \mathcal{I}_{n+1}} \langle w, u_{\pi_{n+1}}\rangle\, u_{\pi_{n+1}} \\ &= \|a\| \sum_{\substack{\pi_{n+1} \in \mathcal{I}_{n+1} \\ 1 \notin \pi_{n+1}}} \langle w, u_{\pi_{n+1}}\rangle u_1 \wedge u_{\pi_{n+1}},\end{aligned}$$

we compute that

$$\|Q_{n+1}^*(w)\|^2 = \|a\|^2 \sum_{\substack{\pi_{n+1} \in \mathcal{I}_{n+1} \\ 1 \notin \pi_{n+1}}} |\langle w, u_{\pi_{n+1}}\rangle|^2.$$

So (7) holds.

Let $a = f_j(\mathbf{z})$ and $b = f_k(\mathbf{z})$ with $j \neq k$. Then

$$b = \langle b, a\rangle \frac{a}{\|a\|^2} + b_\perp, \text{ where } \quad b_\perp \perp a.$$

We claim that $Q_b^{(n)*} Q_a^{(n)} + Q_a^{(n+1)} Q_b^{(n+1)*} = \langle a, b\rangle I$.

If b is replaced by $b = \langle b, a\rangle \frac{a}{\|a\|^2}$, this follows from (7). So we need only check that, if $c \perp a$ in l^2, then

$$Q_c^{(n)*} Q_a^{(n)} + Q_a^{(n+1)} Q_c^{(n+1)*} = 0.$$

This easily follows by choosing an orthonormal basis for l^2, $\{u_j\}_{j=1}^\infty$ with $u_1 = \frac{a}{\|a\|}$ and $u_2 = \frac{c}{\|c\|}$.

For $k = 2, \ldots, m$, let $a_k{}' = P^\perp_{sp\{a_1, \ldots, a_{k-1}\}}(a_k)$. Then, using (7) and (4) repeatedly, we see that for $1 \le r \le m$ and $\mathbf{z} \in \Omega$ fixed,

$$a_1 Q_{a_2}^{(1)} \ldots Q_{a_r}^{(r-1)} Q_{a_r}^{(r-1)*} \ldots a_1^* = \|a_1\|^2 \prod_{j=2}^r \|a_j'\|^2.$$

So when $r = m$, (5) holds. That is,

$$a_1 Q_{a_2}^{(1)} \dots Q_{a_m}^{(m-1)} Q_{a_m}^{(m-1)*} \dots a_1^* = \|a_1\|^2 \prod_{j=2}^{m} \|a_j'\|^2 = \det(F(\mathbf{z})F(\mathbf{z})^*).$$

This completes the proof of the linear algebra material. □

References

[1] G. Birkhoff and S. Mac Lane, *Algebra*, MacMillan, Toronto, 1971.

[2] L. Carleson, *Interpolation by bounded analytic functions and the corona problem*, Annals of Math. **76** (1962), 547-559. MR0141789 (25:5186)

[3] S. Fisher, *Function Theory on Planar Domains, a Second Course in Complex Analysis*, John Wiley and Sons, New York, 1983. MR0694693 (85d:30001)

[4] F. Forelli, *Bounded holomorphic functions and projections*, Illinois J. Math. **10** (1966), 367-380. MR0193534 (33:1754)

[5] P. A. Fuhrmann, *On the corona theorem and its applications to spectral problems in Hilbert space*, Trans. Amer. Math. Soc. **132** (1968), 55-66. MR0222701 (36:5751)

[6] A. Nicolau, *The corona property for bounded analytic functions in some Besov spaces*, Proc. Amer. Math. Soc. **110** (1990), 135-140. MR1017007 (90m:46090)

[7] N. K. Nikolski, *Treatise on the Shift Operator*, Springer-Verlag, New York, 1985. MR0827223 (87i:47042)

[8] M. Rosenblum, *A corona theorem for countably many functions*, Integral Equa. Oper. Theory **3** (1980), 125-137. MR0570865 (81e:46034)

[9] E. L. Stout, *Bounded holomorphic functions on finite Riemann surfaces*, Trans. Amer. Math. Soc. **120** (1965), 255-285. MR0183882 (32:1358)

[10] V. A. Tolokonnikov, *Estimates in Carleson's corona theorem and finitely generated ideals in the algebra $H^\infty(D)$*, Functional Anal. I Prilozhen **14** (1980), 85-86 (in Russian). MR0595742 (82a:46058)

[11] S. R. Treil, *Angles between coinvariant subspaces and an operator-valued corona problem, a question of Szökefalvi-Nagy*, Soviet Math. Dokl. **38** (1989), 394-399. MR0981054 (90b:47057)

[12] T. T. Trent, *A corona theorem for multipliers on Dirichlet space*, Integral Equa. Oper. Theory **49** (2004), 123-139. MR2057771

[13] ———, *A new estimate for the vector-valued corona problem*, J. Func. Anal. **189** (2002), 267-282. MR1887635 (2002m:30067)

[14] ———, *An H^p-corona theorem on the bidisk for infinitely many functions*, submitted.

[15] X. Zhang, *A matrix version of corona theorem for algebras of functions on reproducing kernel Hilbert spaces*, Ph.D. dissertation, The University of Alabama, Tuscaloosa, AL, August 2004.

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