

BOUNDARY STRUCTURE OF HYPERBOLIC 3-MANIFOLDS ADMITTING ANNULAR FILLINGS AT LARGE DISTANCE

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ABSTRACT. We show that if a hyperbolic 3-manifold M with ∂M a union of tori admits two annular Dehn fillings at distance $\Delta \geq 3$, then M is bounded by at most three tori.

An annulus or torus embedded in a 3-manifold is *essential* if it is incompressible, boundary-incompressible and is not boundary-parallel. A 3-manifold is *annular* (resp. *toroidal*) if it contains an essential annulus (resp. torus). Otherwise, it is *anannular* (resp. *atoroidal*). Also, a 3-manifold is *irreducible* if any embedded 2-sphere bounds an embedded 3-ball, and is *boundary-irreducible* if its boundary is incompressible. Thurston [9] has shown that a compact, orientable 3-manifold M with non-empty boundary is irreducible, boundary-irreducible, atoroidal and anannular if and only if it is *hyperbolic*, in the sense that M with its boundary tori removed admits complete finite volume hyperbolic structure with totally geodesic boundary.

Let M be a compact, connected, orientable 3-manifold with torus boundary component T_0 . The Dehn filling of M with slope γ is the manifold $M(\gamma)$ obtained by attaching a solid torus V_γ to M along their boundary so that a meridian of V_γ is identified with a curve of slope γ on T_0 . For two slopes γ_1, γ_2 on T_0 , $\Delta(\gamma_1, \gamma_2)$ denotes their minimal geometric intersection number and is called a *distance* between the slopes.

Suppose that given a hyperbolic 3-manifold M , there are two slopes γ_1, γ_2 on T_0 such that both $M(\gamma_1)$ and $M(\gamma_2)$ are annular. Gordon [2] showed that $\Delta(\gamma_1, \gamma_2) \leq 5$, and furthermore, together with Wu [5], showed that there are only three specific manifolds M realizing $\Delta(\gamma_1, \gamma_2) = 4$ and 5. These manifolds are the exteriors of the Whitehead link, the Whitehead sister link and the 2-bridge link corresponding to $3/10$ in the 3-sphere S^3 . Also, he [3, Theorem 5.3] constructed a hyperbolic k -component link in $S^2 \times S^1$ for any $k \geq 4$ whose exterior realizes $\Delta(\gamma_1, \gamma_2) = 2$, and he asked [3, Question 5.3] what is the maximal value for $\Delta(\gamma_1, \gamma_2)$ if M is bounded by at least four tori. In this paper, we give an answer to this question.

Theorem 1. *Let M be a hyperbolic 3-manifold with ∂M a union of tori. Suppose that there are two slopes γ_1 and γ_2 on a specified boundary torus of M such that*

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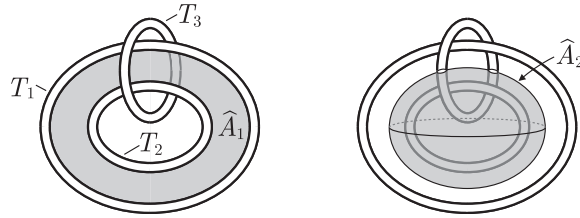


FIGURE 1.

both $M(\gamma_1)$ and $M(\gamma_2)$ are annular. If $\Delta(\gamma_1, \gamma_2) \geq 3$, then ∂M is a union of at most three tori.

We remark that there is a hyperbolic 3-manifold with three boundary tori, called the *magic manifold*, which has two annular Dehn fillings at distance 3. See [4, Lemma 7.6].

We now go on to the proof of Theorem 1. As mentioned before, if $\Delta(\gamma_1, \gamma_2) \geq 4$, then M is bounded by only two tori. So, we assume $\Delta = 3$ and assume for contradiction that ∂M is a union of at least four tori.

Lemma 2. $M(\gamma_i)$ is homeomorphic to $P \times S^1$, where P is a pair of pants.

Proof. Put $X = M(\gamma_i)$. Then X is irreducible by [8, 10], and atoroidal by [7]. Since ∂M is a union of at least four tori, X has at least three torus boundary components T_1, T_2, T_3 .

By assumption X is annular. If A is an essential annulus with boundary components on T_1 and T_2 , then the frontier of $N(A \cup T_1 \cup T_2)$ is a torus T . Since X is connected, irreducible and atoroidal, T must be parallel to T_3 , hence $X = N(A \cup T_1 \cup T_2)$ and the result follows. The case that both boundary components of A are on the same torus is similar. We omit the details. \square

Let T_1, T_2, T_3 be the three torus boundary components of $M(\gamma_i)$. Note that $P \times S^1$ contains an essential annulus connecting two distinct boundary tori and an essential annulus meeting only one boundary torus.

Let \hat{A}_1 be an essential annulus in $M(\gamma_1)$ which has one boundary circle in each of T_1 and T_2 . Let \hat{A}_2 be an essential annulus in $M(\gamma_2)$ whose boundary is contained in T_3 . Note that $\partial \hat{A}_1 \cap \partial \hat{A}_2 = \emptyset$. Also, notice that \hat{A}_1 is non-separating, while \hat{A}_2 is separating. See Figure 1.

We may assume that \hat{A}_1 meets the core of V_{γ_1} transversely. Then $\hat{A}_1 \cap V_{\gamma_1}$ is a disjoint union of meridian disks u_1, u_2, \dots, u_{n_1} of V_{γ_1} , numbered successively along V_{γ_1} , and n_1 is chosen to be minimal among all essential annuli. Similarly, \hat{A}_2 intersects V_{γ_2} in meridian disks v_1, v_2, \dots, v_{n_2} of V_{γ_2} (n_2 is even). Let $A_i = \hat{A}_i \cap M$ ($i = 1, 2$) and assume that A_1 and A_2 intersect transversely and minimally.

In the usual way [1, 2], the arc components of $A_1 \cap A_2$ define labelled graphs G_1 on \hat{A}_1 and G_2 on \hat{A}_2 . The vertices of G_i are u_1, u_2, \dots, u_{n_1} (or v_1, v_2, \dots, v_{n_2}), each having a sign according to whether the core of the attached solid torus passes \hat{A}_i from the positive side or the negative side at the vertex. An edge of G_i is an arc component of $A_1 \cap A_2$. Since $\partial \hat{A}_1 \cap \partial \hat{A}_2 = \emptyset$, each edge of G_i connects two (possibly the same) vertices of G_i . If an edge endpoint lies in $\partial u_x \cap \partial v_y$, then the

point is labelled y at u_x and labelled x at v_y . An edge is called an x -edge if one endpoint is labelled x .

An edge in G_i is *positive* if it connects the vertices of the same sign. Otherwise, it is *negative*. The parity rule [1] says that an edge is positive in one graph if and only if it is negative in the other graph.

Assume that G_i has at least two labels. For a label x , $G_i^+(x)$ denotes a subgraph of G_i consisting of all vertices and all positive x -edges of G_i . A disk face in $G_i^+(x)$ is called an x -face. A cycle σ in G_i consisting of positive edges is a *Scharlemann cycle* if it bounds a disk face of G_i , and all the edges in σ have the same pair of labels $\{x, x+1\}$ at their endpoints. It is a well-known fact that any x -face contains a Scharlemann cycle [6, Proposition 5.1].

Lemma 3. *Neither G_1 nor G_2 contains a Scharlemann cycle.*

Proof. By [10, Lemma 5.4(1)], G_2 cannot contain a Scharlemann cycle.

Assume that G_1 contains a Scharlemann cycle with label pair $\{x, x+1\}$ and let f be the disk face bounded by the Scharlemann cycle. Let H be the annulus in ∂V_{γ_2} cobounded by ∂v_x and ∂v_{x+1} such that $H \cap \partial f \neq \emptyset$. Then surgering $(\widehat{A}_2 - \text{Int}(v_x \cup v_{x+1})) \cup H$ along f gives an annulus A in $M(\gamma_2)$ which intersects the core of V_{γ_2} less than n_2 times. Since $\partial A = \partial \widehat{A}_2$, A is incompressible. Thus A is boundary-parallel in $M(\gamma_2)$. Then one side of \widehat{A}_2 cannot contain a component of ∂M . This is a contradiction. See Figure 1. \square

If $n_1 = 1$, then the edges of G_1 are mutually parallel. Then G_1 would contain Scharlemann cycles. Thus $n_1 \geq 2$. Since \widehat{A}_2 is separating, $n_2 \geq 2$.

Lemma 4. *Let $\{i, j\} = \{1, 2\}$. Then any vertex of G_i has at most n_j negative edge endpoints.*

Proof. Assume, for example, that a vertex u_x has more than n_2 negative edge endpoints. Suppose that $G_2^+(x)$ has V vertices, E edges, and F disk faces. Then $V = n_2$, $E > n_2$ and $F \geq E - V + \chi(\widehat{A}_2) = E - V > 0$. Thus G_2 contains an x -face and hence it contains a Scharlemann cycle, contradicting Lemma 3. \square

Let K_i be the number of negative edge endpoints in G_i . Then by Lemma 4 we have $K_i \leq n_1 n_2$. Since there is a total of $3n_1 n_2$ edge endpoints on G_1 , by the parity rule we have $K_1 = 3n_1 n_2 - K_2$. Therefore

$$K_1 = 3n_1 n_2 - K_2 \geq 3n_1 n_2 - n_1 n_2 = 2n_1 n_2.$$

This gives a contradiction and completes the proof of Theorem 1.

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