ON OPERATORS WHICH COMMUTE
WITH ANALYTIC TOEPLITZ OPERATORS
MODULO THE FINITE RANK OPERATORS

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Abstract. It is shown that an operator $S$ on the Hardy space $H^2(D^n)$ (or $H^2(B_n)$) commutes with all analytic Toeplitz operators modulo the finite rank operators if and only if $S = T_g + F$. Here $F$ is a finite rank operator, and in the case $n = 1$, $g$ is a sum of a rational function and a bounded analytic function, and in the case $n \geq 2$, $g$ is a bounded analytic function.

1. Introduction

Davidson [Da] studied when an operator $S$ on the classical Hardy space $H^2(D)$ essentially commutes with all analytic Toeplitz operators. He proved that commutators $[S, T_f] = ST_f - T_f S$ are compact for all $f \in H^\infty(D)$ if and only if $S = T_g + K$, where $g \in H^\infty(D) + C(T)$ and $K$ is compact. Gu [Gu] generalized this result to the Hardy space over the unit ball $B_n$.

A natural problem is to characterize operators $S$ such that the commutators $[S, T_f]$ belong to the Schatten-von Neumann class $L^p$ for all $f \in H^\infty$. Note the inclusion

$L^0 \subset \cdots \subset L^1 \subset \cdots \subset L^\infty$,

where $L^0$ is the class of all finite rank operators, and $L^\infty$ is the class of all compact operators. Therefore, Davidson's work completed the case of $L^\infty$ of the classical Hardy space in the above sequence. Furthermore, on the classical Hardy space $H^2(D)$, Gu [Gu] showed that for each $f \in H^\infty(D)$, $[S, T_f] = ST_f - T_f S \in L^0$ if and only if $S = T_g + F$, where $g$ is a sum of a bounded analytic function and a rational function, and $F$ is a finite rank operator. However, the proof in [Gu] is considerably technical because Gu solved a more general problem from which the above-mentioned result is a consequence. In this note, we will generalize Gu's result to the case of higher dimension, and in the dimension $n = 1$, our proof is different from the proof in [Gu]. The following is the main result in this note.

Theorem 1.1. Let $S$ be a bounded linear operator on $H^2(D^n)$ (or $H^2(B_n)$). Then $[S, T_f]$ is of finite rank for all $f \in H^\infty$ if and only if $S = T_g + F$. Here $F$ is a finite rank operator, and in the case $n = 1$, $g$ is a sum of a rational function and a bounded analytic function, and in the case $n \geq 2$, $g$ is a bounded analytic function.
2. Preliminaries

Let $\mathbb{D}$ be the open unit disk of the complex plane and $\mathbb{T}$ the unit circle. For $n \geq 1$, let $\mathbb{D}^n$ and $\mathbb{T}^n$ be the unit polydisk and $n$-torus, respectively. The Hardy space $H^2(\mathbb{D}^n)$ is the closure of all polynomials in $L^2(\mathbb{T}^n)$ (with respect to the measure $d\theta_1 \cdots d\theta_n/(2\pi)^n$ on $\mathbb{T}^n$). Let $P$ be the orthogonal projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{D}^n)$. The Toeplitz operator $T_f : H^2(\mathbb{D}^n) \to H^2(\mathbb{D}^n)$ with symbol $f \in L^\infty(\mathbb{T}^n)$ is defined by $T_f(h) = P(fh)$ for all $h \in H^2(\mathbb{D}^n)$. The Hankel operator $H_f$ with symbol $f$ is defined as $H_fh = (I - P)(fh)$ for all $h \in H^2(\mathbb{D}^n)$. For $f, g \in L^\infty(\mathbb{T}^n)$, Toeplitz and Hankel operators are connected by the following formula:

$$T_fg - T_fT_g = H_f^*H_g.$$

Let us recall the Hardy space $H^2(\mathbb{B}_n)$ over the unit ball. Let $\mathbb{B}_n$ be the open unit ball in $\mathbb{C}^n$, and $\partial\mathbb{B}_n$ its boundary. The Hardy space $H^2(\mathbb{B}_n)$ is the closure of all polynomials in $L^2(\partial\mathbb{B}_n)$ (with respect to the unique rotation-invariant probability measure $da$ on $\partial\mathbb{B}_n$). In the same way, one can define Toeplitz and Hankel operators on the Hardy space $H^2(\mathbb{B}_n)$.

It is well known that there exist a lot of inner functions both in the unit ball $\mathbb{B}^n$ and the unit polydisk $\mathbb{D}^n$. Inner functions will play an important role in this note. In what follows we will use $H^2$ to denote the Hardy space $H^2(\mathbb{B}_n)$ or $H^2(\mathbb{D}_n)$.

**Lemma 2.1.** Let $\eta$ be a nonconstant inner function. Then $T_\eta^n \to 0$ ($\text{SOT}$) on the Hardy space $H^2$. Furthermore, for each compact operator $K$, we have $T_\eta^nK \to 0$ as $n \to \infty$.

**Proof.** Let $K_\lambda$ be the reproducing kernel of $H^2$. Take $f = \sum a_iK_\lambda_i$ to be a finite linear combination of reproducing kernels. Note that

$$T_\eta^n(\sum a_iK_\lambda_i) = \sum a_i\eta(\lambda_i)K_\lambda_i.$$

Hence

$$\|T_\eta^n(\sum a_iK_\lambda_i)\| \leq \sum |a_i| |\eta(\lambda_i)|^n \|K_\lambda_i\| \to 0,$$

as $n \to \infty$. Because the above linear combinations are dense in $H^2$, $T_\eta^n \to 0$ ($\text{SOT}$). For a rank one operator $f \otimes g$ (here $f \otimes g(h) = \langle h, gf \rangle$), we have

$$T_\eta^n f \otimes g = (T_\eta^n f) \otimes g.$$

Note that the set of all finite rank operators is dense in all compact operators, the desired result follows. $\square$

The following lemma may be known by many people. We note it here.

**Lemma 2.2.** Let $\{T_\alpha\}$ be a net of operators on $H^2$ such that $T_\alpha \to T$ ($\text{WOT}$). If there exists a natural number $M$ such that $\text{rank}T_\alpha \leq M$, then $\text{rank}T \leq M$.

**Proof.** If $\text{rank}T \geq M + 1$, then there exist $\{f_i\}_{i=1}^{M+1}$ and $\{g_i\}_{i=1}^{M+1}$ such that

$$\det[\langle Tf_i, g_j\rangle]_{1 \leq i,j \leq M+1} \neq 0.$$

Set

$$d_\alpha = [\langle T_\alpha f_i, g_j\rangle]_{1 \leq i,j \leq M+1}.$$
Then $d_\alpha = 0$ since $\text{rank } T_\alpha \leq M$. This leads to a contradiction since
\[ d_\alpha \rightarrow \det([T_{f_i}, g_j])_{1 \leq i, j \leq M+1} \neq 0. \]
\[ \square \]

**Lemma 2.3.** Let $S$ be an operator on $H^2$. If $[S, T_f]$ is of finite rank for all $f \in H^\infty$, then there exists a natural number $M$ such that
\[ \text{rank } [S, T_f] \leq M. \]

**Proof.** Let $\Gamma_n = \{ f : \text{rank } [S, T_f] \leq n \}$. By Lemma 2.2, $\Gamma_n$ is a norm closed subset of $H^\infty$. Since $H^\infty$ is a Banach space and $H^\infty = \bigcup_n \Gamma_n$, the Baire Category Theorem implies that there exists a natural number $N$ such that $\Gamma_N$ contains an open subset of $H^\infty$. This means that the set $\{ f - g : f, g \in \Gamma_N \}$ is a neighborhood of the function $f = 0$. Thus for each $h \in H^\infty$, there exists a real number $\gamma$ and two functions $f, g \in \Gamma_N$ such that $h = \gamma(f - g)$ and hence
\[ \text{rank } [S, T_h] \leq \text{rank } [S, T_f] + \text{rank } [S, T_g] \leq 2N. \]
\[ \square \]

The next proposition says that Toeplitz operators on the Hardy space $H^2$ can be completely characterized by algebraic equations.

**Proposition 2.1.** For a bounded linear operator $T$ on $H^2$, then $T$ is a Toeplitz operator if and only if $T_\eta^*TT_\eta = T$ for each inner function $\eta$.

**Proof.** We will prove the proposition in the case of the unit ball, and the same reasoning is valid in the unit polydisk. Set
\[ \mathcal{A} = \{ \bar{\eta}h : \eta \text{ are inner functions, } h \in H^2 \}. \]

Then $\mathcal{A}$ is a dense linear subspace of $L^2(\partial\mathbb{B}_n)$ by [R1, Theorem 11.4]. Define a map
\[ \Phi : \mathcal{A} \rightarrow \mathbb{C} \]
by $\Phi(\bar{\eta}h) = \langle h, T\eta \rangle$. Then $\Phi$ is well defined and linear. In fact, if $\bar{\eta}_1 h_1 = \bar{\eta}_2 h_2$, then we have
\[ \Phi(\bar{\eta}_1 h_1) = \langle h_1, T\eta_1 \rangle = \langle h_1, T_{\eta_2}^* TT_{\eta_2} \eta_1 \rangle = \langle \eta_2 h_1, T\eta_1 \eta_2 \rangle = \langle h_2, T_{\eta_1}^* TT_{\eta_1} \eta_2 \rangle = \langle h_2, T\eta_2 \rangle = \Phi(\bar{\eta}_2 h_2) . \]

So, $\Phi$ is well defined. The same reasoning shows that $\Phi$ is linear. From the definition of $\Phi$,
\[ |\Phi(\bar{\eta}h)| \leq \|T\|\|h\| = \|T\|\|\bar{\eta}h\|. \]
So, $\Phi$ is a bounded linear functional on $\mathcal{A}$. Since $\mathcal{A}$ is dense in $L^2(\partial\mathbb{B}_n)$, there exists a unique $\phi \in L^2(\partial\mathbb{B}_n)$ such that
\[ \Phi(\bar{\eta}h) = \langle \bar{\eta}h, \phi \rangle . \]

Since $\mathcal{A}$ is dense in $L^2(\partial\mathbb{B}_n)$ and
\[ |\langle \bar{\eta}h, \phi \rangle| = \left| \int_{\partial\mathbb{B}_n} (\bar{\eta}h)\bar{\phi} \, d\sigma \right| \leq \|T\|\|\bar{\eta}h\|, \]
we see $\phi \in L^\infty$. From the equalities
\[ \langle h, T\eta \rangle = \Phi(\bar{\eta}h) = \langle \bar{\eta}h, \phi \rangle = \langle h, \phi\eta \rangle = \langle h, T_\phi\eta \rangle , \]
and note that the set of all finite linear combinations of inner functions is dense in $H^2$ [R1 Theorem 11.1], we obtain $T = T_g$. □

By Kronecker’s result about a finite rank Hankel operator, $H_g$ on $H^2(\mathbb{D})$ is of finite rank if and only if $g$ is the sum of a rational function and a bounded analytic function over the disk $\mathbb{D}$ [P]. For $n \geq 2$, if a Hankel operator $H_g$ on $H^2(\mathbb{D}^n)$ has finite rank, then it must be zero (see Gu2, GuZ). In the case of unit ball, the following proposition may be known by many people.

**Proposition 2.2.** Assume $n \geq 2$ and a Hankel operator $H_g$ is of finite rank. Then $H_g = 0$, that is, $g \in H^\infty$.

**Proof.** Setting $M = \ker H_g$, then $M$ is a finitely codimensional multiplier invariant subspace. As a finitely codimensional multiplier invariant subspace, $M$ is generated by finitely many polynomials $P_1, \ldots, P_m$, and the set $\bigcap_{i=1}^m Z(P_i)$ is finite [CC Corollary 2.2.6], where $Z(P)$ denotes the set of zero points of a polynomial $P$. From $H_gP_i = 0$, we have $Q_i = gP_i \in H^\infty$, $i = 1, \ldots, m$. On $\partial \mathbb{B}_n$, noting
\[
g = \frac{Q_1}{P_1} = \cdots = \frac{Q_m}{P_m},
\]
this implies
\[
Q_1(z)P_i(z) = \cdots = Q_m(z)P_i(z), \quad z \in \mathbb{B}_n \setminus \bigcup_{i=1}^m Z(P_i).
\]
Putting $g(z) = \frac{Q_1(z)}{P_1(z)}$, then from the above equalities, $g$ can be analytically extended to $\mathbb{B}_n \setminus \bigcap_{i=1}^m Z(P_i)$. Since the set $\bigcap_{i=1}^m Z(P_i)$ is finite, $g$ has an extension on $\mathbb{B}_n$ by [Kr]. Furthermore, we have $g \in H^\infty$. The reasoning is as follows: since the set $\bigcap_{i=1}^m Z(P_i)$ is finite, there exist $0 < r < 1$ and $\epsilon > 0$ such that $\sum_{i=1}^m |P_i(z)|^2 > \epsilon$ if $|z| > r$. By
\[
|g(z)|^2 = \frac{|Q_1(z)|^2}{|P_1(z)|^2} = \cdots = \frac{|Q_m(z)|^2}{|P_m(z)|^2} = \frac{\sum_{i=1}^m |Q_i(z)|^2}{\sum_{i=1}^m |P_i(z)|^2},
\]
we see that $|g(z)|$ is bounded on $r < |z| < 1$. This implies $g \in H^\infty$. □

3. THE PROOF OF THE MAIN THEOREM

Before proving the main theorem we need the following lemma.

**Lemma 3.1.** On the Hardy space $H^2$, if $[T_g, T_f]$ is of finite rank for all $f \in H^\infty$, then in the case $n = 1$, $g$ is a sum of a rational function and a bounded analytic function, and in the case $n \geq 2$, $g$ is a bounded analytic function.

**Proof.** By Lemma 2.3 there exists a natural number $M$ such that
\[
\text{rank } [T_g, T_f] \leq M
\]
for all $f \in H^\infty$. Below we will give the proof of Lemma 3.1 in the case of unit ball, and the same reasoning is valid in the case of the unit polydisk. For any $h, f \in H^\infty$,
\[
\text{rank}(T_{hf} - T_{h}T_g) = \text{rank}(T_h(T_gT_f - T_fT_g)) \leq \text{rank}(T_gT_f - T_fT_g) \leq M.
\]
Note that $\{\eta : \eta \text{ are inner functions}, f \in H^\infty\}$ is $W^*$ dense in $L^\infty(\partial \mathbb{B}_n)$ [R1 Theorem 11.2]. By Lemma 2.2, $T_{g} - T_{\bar{g}}T_{g}$ is of finite rank for all $\phi \in L^\infty(\partial \mathbb{B}_n)$. It follows that $T_{g} - T_{\bar{g}}T_{g} = H_{g}H_{g}$ is of finite rank, that is, $H_{g}$ is of finite rank. In the case $n = 1$, by Kronecker’s result about a finite rank Hankel operator, $H_{g}$ on
\(H^2(\mathbb{D})\) is of finite rank if and only if \(g\) is a sum of a rational function and a bounded analytic function over the disk \(\mathbb{D}\) [P]. In the case \(n \geq 2\), applying Proposition 2.2 gives \(g \in H^\infty\).

**Theorem 3.1.** Given a bounded linear operator \(S\) on \(H^2\), then \([S,T_f]\) is of finite rank for all \(f \in H^\infty\) if and only if \(S = T_g + F\). Here \(F\) is a finite rank operator, and in the case \(n = 1\), \(g\) is a sum of a rational function and a bounded analytic function, and in the case \(n \geq 2\), \(g\) is a bounded analytic function.

**Remark.** In the case \(n = 1\), Gu obtained the above result [Gu1]. However, the present proof is different from Gu’s proof.

**Proof.** First we claim that \(S\) has the form \(S = T_g + F\), where \(g \in L^\infty\), and \(F\) is a finite rank operator. For this claim, pick a nonconstant inner function \(\eta\), and consider the sequence \(\{T^n_\eta ST^n_\eta\}\). Note that \(\{T^n_\eta ST^n_\eta\}\) is a bounded set, and hence without a loss of generality we may assume that \(\{T^n_\eta ST^n_\eta\}\) converges to a operator \(A\) in the weak operator topology (if not, we can choose a subnet). It follows that

\[
S - A = (WOT) \lim(S - T^n_\eta ST^n_\eta).
\]

Applying Lemma 2.3 shows that there exists a natural number \(M\) such that

\[
\text{rank}(S - T^n_\eta ST^n_\eta) = \text{rank}T^n_\eta (S - ST^n_\eta) \leq M.
\]

By Lemma 2.2, \(\text{rank}(S - A) \leq M\), that is, \(F = S - A\) is a finite rank operator. Now we prove that \(A\) is a Toeplitz operator. For each inner function \(\zeta\)

\[
T^n_\zeta AT^n_\zeta = (WOT) \lim T^n_\zeta T^n_\eta ST^n_\zeta T^n_\zeta = (WOT) \lim T^n_\eta T^n_\zeta ST^n_\eta
\]

\[
= (WOT) \lim T^n_\eta T^n_\zeta T^n_\eta ST^n_\eta + (WOT) \lim T^n_\zeta T^n_\eta (ST^n_\eta - T^n_\zeta S)T^n_\eta.
\]

Since \(ST^n_\eta - T^n_\zeta S\) is of finite rank, the latter is zero by Lemma 2.1. It follows that

\[
T^n_\zeta AT^n_\zeta = (WOT) \lim T^n_\eta T^n_\zeta T^n_\eta ST^n_\eta = A.
\]

Proposition 2.1 says that \(A\) is a Toeplitz operator, that is, there is a \(g \in L^\infty\) such that \(A = T_g\). The claim follows. Using Lemma 3.1, we see that in the case \(n = 1\), \(g\) is a sum of a rational function and a bounded analytic function, and in the case \(n \geq 2\), \(g\) is a bounded analytic function. The opposite direction is easily proved. \(\square\)

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