

ON OPERATORS WHICH COMMUTE WITH ANALYTIC TOEPLITZ OPERATORS MODULO THE FINITE RANK OPERATORS

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ABSTRACT. It is shown that an operator S on the Hardy space $H^2(\mathbb{D}^n)$ (or $H^2(\mathbb{B}_n)$) commutes with all analytic Toeplitz operators modulo the finite rank operators if and only if $S = T_g + F$. Here F is a finite rank operator, and in the case $n = 1$, g is a sum of a rational function and a bounded analytic function, and in the case $n \geq 2$, g is a bounded analytic function.

1. INTRODUCTION

Davidson [Da] studied when an operator S on the classical Hardy space $H^2(\mathbb{D})$ essentially commutes with all analytic Toeplitz operators. He proved that commutators $[S, T_f] = ST_f - T_f S$ are compact for all $f \in H^\infty(\mathbb{D})$ if and only if $S = T_g + K$, where $g \in H^\infty(\mathbb{D}) + C(\mathbb{T})$ and K is compact. Guo [Guo] generalized this result to the Hardy space over the unit ball \mathbb{B}_n .

A natural problem is to characterize operators S such that the commutators $[S, T_f]$ belong to the Schatten-von Neumann class \mathcal{L}^p for all $f \in H^\infty$. Note the inclusion

$$\mathcal{L}^0 \subset \cdots \subset \mathcal{L}^1 \subset \cdots \subset \mathcal{L}^\infty,$$

where \mathcal{L}^0 is the class of all finite rank operators, and \mathcal{L}^∞ is the class of all compact operators. Therefore, Davidson's work completed the case of \mathcal{L}^∞ of the classical Hardy space in the above sequence. Furthermore, on the classical Hardy space $H^2(\mathbb{D})$, Gu [Gu1] showed that for each $f \in H^\infty(\mathbb{D})$, $[S, T_f] = ST_f - T_f S \in \mathcal{L}^0$ if and only if $S = T_g + F$, where g is a sum of a bounded analytic function and a rational function, and F is a finite rank operator. However, the proof in [Gu1] is considerably technical because Gu solved a more general problem from which the above-mentioned result is a consequence. In this note, we will generalize Gu's result to the case of higher dimension, and in the dimension $n = 1$, our proof is different from the proof in [Gu1]. The following is the main result in this note.

Theorem 1.1. *Let S be a bounded linear operator on $H^2(\mathbb{D}^n)$ (or $H^2(\mathbb{B}_n)$). Then $[S, T_f]$ is of finite rank for all $f \in H^\infty$ if and only if $S = T_g + F$. Here F is a finite rank operator, and in the case $n = 1$, g is a sum of a rational function and a bounded analytic function, and in the case $n \geq 2$, g is a bounded analytic function.*

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2. PRELIMINARIES

Let \mathbb{D} be the open unit disk of the complex plane and \mathbb{T} the unit circle. For $n \geq 1$, let \mathbb{D}^n and \mathbb{T}^n be the unit polydisk and n -torus, respectively. The Hardy space $H^2(\mathbb{D}^n)$ is the closure of all polynomials in $L^2(\mathbb{T}^n)$ (with respect to the measure $d\theta_1 \cdots d\theta_n / (2\pi)^n$ on \mathbb{T}^n). Let P be the orthogonal projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{D}^n)$. The Toeplitz operator $T_f : H^2(\mathbb{D}^n) \rightarrow H^2(\mathbb{D}^n)$ with symbol $f \in L^\infty(\mathbb{T}^n)$ is defined by $T_f(h) = P(fh)$ for all $h \in H^2(\mathbb{D}^n)$. The Hankel operator H_f with symbol f is defined as $H_f h = (I - P)(fh)$ for all $h \in H^2(\mathbb{D}^n)$. For $f, g \in L^\infty(\mathbb{T}^n)$, Toeplitz and Hankel operators are connected by the following formula:

$$T_{fg} - T_f T_g = H_{\bar{f}}^* H_g.$$

Let us recall the Hardy space $H^2(\mathbb{B}_n)$ over the unit ball. Let \mathbb{B}_n be the open unit ball in \mathbb{C}^n , and $\partial\mathbb{B}_n$ its boundary. The Hardy space $H^2(\mathbb{B}_n)$ is the closure of all polynomials in $L^2(\partial\mathbb{B}_n)$ (with respect to the unique rotation-invariant probability measure $d\sigma$ on $\partial\mathbb{B}_n$). In the same way, one can define Toeplitz and Hankel operators on the Hardy space $H^2(\mathbb{B}_n)$.

It is well known that there exist a lot of inner functions both in the unit ball \mathbb{B}^n and the unit polydisk \mathbb{D}^n [R1, R2]. Inner functions will play an important role in this note. In what follows we will use H^2 to denote the Hardy space $H^2(\mathbb{B}_n)$ or $H^2(\mathbb{D}_n)$.

Lemma 2.1. *Let η be a nonconstant inner function. Then $T_{\bar{\eta}^n} \rightarrow 0$ (SOT) on the Hardy space H^2 . Furthermore, for each compact operator K , we have $T_{\bar{\eta}^n} K \xrightarrow{\|\cdot\|} 0$ as $n \rightarrow \infty$.*

Proof. Let K_λ be the reproducing kernel of H^2 . Take $f = \sum a_i K_{\lambda_i}$ to be a finite linear combination of reproducing kernels. Note that

$$T_{\bar{\eta}^n}(\sum a_i K_{\lambda_i}) = \sum a_i \eta(\overline{\lambda_i})^n K_{\lambda_i}.$$

Hence

$$\|T_{\bar{\eta}^n}(\sum a_i K_{\lambda_i})\| \leq \sum |a_i| |\eta(\lambda_i)|^n \|K_{\lambda_i}\| \rightarrow 0,$$

as $n \rightarrow \infty$. Because the above linear combinations are dense in H^2 , $T_{\bar{\eta}^n} \rightarrow 0$ (SOT). For a rank one operator $f \otimes g$ (here $f \otimes g(h) = \langle h, g \rangle f$), we have

$$T_{\bar{\eta}^n} f \otimes g = (T_{\bar{\eta}^n} f) \otimes g.$$

Note that the set of all finite rank operators is dense in all compact operators, the desired result follows. □

The following lemma may be known by many people. We note it here.

Lemma 2.2. *Let $\{T_\alpha\}$ be a net of operators on H^2 such that $T_\alpha \rightarrow T$ (WOT). If there exists a natural number M such that $\text{rank} T_\alpha \leq M$, then*

$$\text{rank} T \leq M.$$

Proof. If $\text{rank} T \geq M + 1$, then there exist $\{f_i\}_{i=1}^{M+1}$ and $\{g_i\}_{i=1}^{M+1}$ such that

$$\det[\langle T f_i, g_j \rangle]_{1 \leq i, j \leq M+1} \neq 0.$$

Set

$$d_\alpha = [\langle T_\alpha f_i, g_j \rangle]_{1 \leq i, j \leq M+1}.$$

Then $d_\alpha = 0$ since $\text{rank}T_\alpha \leq M$. This leads to a contradiction since

$$d_\alpha \rightarrow \det[\langle Tf_i, g_j \rangle]_{1 \leq i, j \leq M+1} \neq 0.$$

□

Lemma 2.3. *Let S be an operator on H^2 . If $[S, T_f]$ is of finite rank for all $f \in H^\infty$, then there exists a natural number M such that*

$$\text{rank}[S, T_f] \leq M.$$

Proof. Let $\Gamma_n = \{f : \text{rank}[S, T_f] \leq n\}$. By Lemma 2.2, Γ_n is a norm closed subset of H^∞ . Since H^∞ is a Banach space and $H^\infty = \bigcup_n \Gamma_n$, the Baire Category Theorem implies that there exists a natural number N such that Γ_N contains an open subset of H^∞ . This means that the set $\{f - g : f, g \in \Gamma_N\}$ is a neighborhood of the function $f = 0$. Thus for each $h \in H^\infty$, there exists a real number γ and two functions $f, g \in \Gamma_N$ such that $h = \gamma(f - g)$ and hence

$$\text{rank}[S, T_h] \leq \text{rank}[S, T_f] + \text{rank}[S, T_g] \leq 2N.$$

□

The next proposition says that Toeplitz operators on the Hardy space H^2 can be completely characterized by algebraic equations.

Proposition 2.1. *For a bounded linear operator T on H^2 , then T is a Toeplitz operator if and only if $T_\eta^* T T_\eta = T$ for each inner function η .*

Proof. We will prove the proposition in the case of the unit ball, and the same reasoning is valid in the unit polydisk. Set

$$\mathcal{A} = \{\bar{\eta}h : \eta \text{ are inner functions, } h \in H^2\}.$$

Then \mathcal{A} is a dense linear subspace of $L^2(\partial\mathbb{B}_n)$ by [R1, Theorem 11.4]. Define a map

$$\Phi : \mathcal{A} \rightarrow \mathbb{C}$$

by $\Phi(\bar{\eta}h) = \langle h, T\eta \rangle$. Then Φ is well defined and linear. In fact, if $\bar{\eta}_1 h_1 = \bar{\eta}_2 h_2$, then we have

$$\begin{aligned} \Phi(\bar{\eta}_1 h_1) &= \langle h_1, T\eta_1 \rangle = \langle h_1, T_{\eta_2}^* T T_{\eta_2} \eta_1 \rangle \\ &= \langle \eta_2 h_1, T\eta_2 \eta_1 \rangle = \langle \eta_1 h_2, T\eta_1 \eta_2 \rangle \\ &= \langle h_2, T_{\eta_1}^* T T_{\eta_1} \eta_2 \rangle = \langle h_2, T\eta_2 \rangle \\ &= \Phi(\bar{\eta}_2 h_2). \end{aligned}$$

So, Φ is well defined. The same reasoning shows that Φ is linear. From the definition of Φ ,

$$|\Phi(\bar{\eta}h)| \leq \|T\| \|h\| = \|T\| \|\bar{\eta}h\|.$$

So, Φ is a bounded linear functional on \mathcal{A} . Since \mathcal{A} is dense in $L^2(\partial\mathbb{B}_n)$, there exists a unique $\phi \in L^2(\partial\mathbb{B}_n)$ such that

$$\Phi(\bar{\eta}h) = \langle \bar{\eta}h, \phi \rangle.$$

Since \mathcal{A} is dense in $L^2(\partial\mathbb{B}_n)$ and

$$|\langle \bar{\eta}h, \phi \rangle| = \left| \int_{\partial\mathbb{B}_n} (\bar{\eta}h)\bar{\phi} d\sigma \right| \leq \|T\| \|\bar{\eta}h\|,$$

we see $\phi \in L^\infty$. From the equalities

$$\langle h, T\eta \rangle = \Phi(\bar{\eta}h) = \langle \bar{\eta}h, \phi \rangle = \langle h, \phi\eta \rangle = \langle h, T_\phi\eta \rangle,$$

and note that the set of all finite linear combinations of inner functions is dense in H^2 [R1, Theorem 11.1], we obtain $T = T_\phi$. \square

By Kronecker’s result about a finite rank Hankel operator, H_g on $H^2(\mathbb{D})$ is of finite rank if and only if g is the sum of a rational function and a bounded analytic function over the disk \mathbb{D} [P]. For $n \geq 2$, if a Hankel operator H_g on $H^2(\mathbb{D}^n)$ has finite rank, then it must be zero (see [Gu2, GuZ]). In the case of unit ball, the following proposition may be known by many people.

Proposition 2.2. *Assume $n \geq 2$ and a Hankel operator H_g is of finite rank. Then $H_g = 0$, that is, $g \in H^\infty$.*

Proof. Setting $M = \ker H_g$, then M is a finitely codimensional multiplier invariant subspace. As a finitely codimensional multiplier invariant subspace, M is generated by finitely many polynomials P_1, \dots, P_m , and the set $\bigcap_{i=1}^m Z(P_i)$ is finite [CG, Corollary 2.2.6], where $Z(P)$ denotes the set of zero points of a polynomial P . From $H_g P_i = 0$, we have $Q_i = g P_i \in H^\infty$, $i = 1, \dots, m$. On $\partial\mathbb{B}_n$, noting

$$g = \frac{Q_1}{P_1} = \dots = \frac{Q_m}{P_m},$$

this implies

$$\frac{Q_1(z)}{P_1(z)} = \dots = \frac{Q_m(z)}{P_m(z)}, \quad z \in \mathbb{B}_n \setminus \bigcup_{i=1}^m Z(P_i).$$

Putting $g(z) = \frac{Q_1(z)}{P_1(z)}$, then from the above equalities, g can be analytically extended to $\mathbb{B}_n \setminus \bigcap_{i=1}^m Z(P_i)$. Since the set $\bigcap_{i=1}^m Z(P_i)$ is finite, g has an extension on \mathbb{B}_n by [Kr]. Furthermore, we have $g \in H^\infty$. The reasoning is as follows: since the set $\bigcap_{i=1}^m Z(P_i)$ is finite, there exist $0 < r < 1$ and $\epsilon > 0$ such that $\sum_{i=1}^m |P_i(z)|^2 > \epsilon$ if $|z| > r$. By

$$|g(z)|^2 = \frac{|Q_1(z)|^2}{|P_1(z)|^2} = \dots = \frac{|Q_m(z)|^2}{|P_m(z)|^2} = \frac{\sum_{i=1}^m |Q_i(z)|^2}{\sum_{i=1}^m |P_i(z)|^2},$$

we see that $|g(z)|$ is bounded on $r < |z| < 1$. This implies $g \in H^\infty$. \square

3. THE PROOF OF THE MAIN THEOREM

Before proving the main theorem we need the following lemma.

Lemma 3.1. *On the Hardy space H^2 , if $[T_g, T_f]$ is of finite rank for all $f \in H^\infty$, then in the case $n = 1$, g is a sum of a rational function and a bounded analytic function, and in the case $n \geq 2$, g is a bounded analytic function.*

Proof. By Lemma 2.3 there exists a natural number M such that

$$\text{rank}[T_g, T_f] \leq M$$

for all $f \in H^\infty$. Below we will give the proof of Lemma 3.1 in the case of unit ball, and the same reasoning is valid in the case of the unit polydisk. For any $h, f \in H^\infty$,

$$\text{rank}(T_{hfg} - T_{hf}T_g) = \text{rank}(T_h(T_gT_f - T_fT_g)) \leq \text{rank}(T_gT_f - T_fT_g) \leq M.$$

Note that $\{\bar{\eta}f : \eta \text{ are inner functions, } f \in H^\infty\}$ is W^* dense in $L^\infty(\partial\mathbb{B}_n)$ [R1, Theorem 11.2]. By Lemma 2.2, $T_{\phi g} - T_\phi T_g$ is of finite rank for all $\phi \in L^\infty(\partial\mathbb{B}_n)$. It follows that $T_{\bar{g}g} - T_{\bar{g}}T_g = H_g^* H_g$ is of finite rank, that is, H_g is of finite rank. In the case $n = 1$, by Kronecker’s result about a finite rank Hankel operator, H_g on

$H^2(\mathbb{D})$ is of finite rank if and only if g is a sum of a rational function and a bounded analytic function over the disk \mathbb{D} [P]. In the case $n \geq 2$, applying Proposition 2.2 gives $g \in H^\infty$. \square

Theorem 3.1. *Given a bounded linear operator S on H^2 , then $[S, T_f]$ is of finite rank for all $f \in H^\infty$ if and only if $S = T_g + F$. Here F is a finite rank operator, and in the case $n = 1$, g is a sum of a rational function and a bounded analytic function, and in the case $n \geq 2$, g is a bounded analytic function.*

Remark. In the case $n = 1$, Gu obtained the above result [Gu1]. However, the present proof is different from Gu's proof.

Proof. First we claim that S has the form $S = T_g + F$, where $g \in L^\infty$, and F is a finite rank operator. For this claim, pick a nonconstant inner function η , and consider the sequence $\{T_{\eta^n}^* S T_{\eta^n}\}$. Note that $\{T_{\eta^n}^* S T_{\eta^n}\}$ is a bounded set, and hence without a loss of generality we may assume that $\{T_{\eta^n}^* S T_{\eta^n}\}$ converges to a operator A in the weak operator topology (if not, we can choose a subnet). It follows that

$$S - A = (\text{WOT}) \lim (S - T_{\eta^n}^* S T_{\eta^n}).$$

Applying Lemma 2.3 shows that there exists a natural number M such that $\text{rank}[S, T_f] \leq M$ for all $f \in H^\infty$ and hence

$$\text{rank}(S - T_{\eta^n}^* S T_{\eta^n}) = \text{rank} T_{\eta^n}^* (T_{\eta^n} S - S T_{\eta^n}) \leq M.$$

By Lemma 2.2, $\text{rank}(S - A) \leq M$, that is, $F = S - A$ is a finite rank operator. Now we prove that A is a Toeplitz operator. For each inner function ζ

$$\begin{aligned} T_\zeta^* A T_\zeta &= (\text{WOT}) \lim T_\zeta^* T_{\eta^n}^* S T_{\eta^n} T_\zeta = (\text{WOT}) \lim T_{\eta^n}^* T_\zeta^* S T_\zeta T_{\eta^n} \\ &= (\text{WOT}) \lim T_{\eta^n}^* T_\zeta^* T_\zeta S T_{\eta^n} + (\text{WOT}) \lim T_\zeta^* T_{\eta^n}^* (S T_\zeta - T_\zeta S) T_{\eta^n}. \end{aligned}$$

Since $S T_\zeta - T_\zeta S$ is of finite rank, the latter is zero by Lemma 2.1. It follows that

$$T_\zeta^* A T_\zeta = (\text{WOT}) \lim T_{\eta^n}^* T_\zeta^* T_\zeta S T_{\eta^n} = A.$$

Proposition 2.1 says that A is a Toeplitz operator, that is, there is a $g \in L^\infty$ such that $A = T_g$. The claim follows. Using Lemma 3.1, we see that in the case $n = 1$, g is a sum of a rational function and a bounded analytic function, and in the case $n \geq 2$, g is a bounded analytic function. The opposite direction is easily proved. \square

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