SPECTRA OF ALGEBRAS OF ENTIRE FUNCTIONS ON BANACH SPACES

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Abstract. We obtain an explicit description of the spectrum (set of closed maximal ideals) of $H_b(X)$, algebra of analytic functions on a Banach space $X$ which are bounded on bounded subsets. We show that the spectrum of $H_b(X)$ admits a natural linear structure. Some applications to the algebra of uniformly continuous and bounded analytic functions on the unit ball $B \subset X$ are indicated.

Let $A$ be a complex commutative topological algebra. Let us denote by $M(A)$ the spectrum (set of closed maximal ideals = set of continuous characters = set of continuous complex-valued homomorphisms) of $A$. Recall that $A$ is semisimple if the complex homomorphisms from $M(A)$ separate points of $A$. It is well known that every semisimple commutative Fréchet algebra $A$ is isomorphic to some subalgebra of continuous functions on $M(A)$ endowed with a natural topology. More exactly, for every $a \in A$ there exists a function $\hat{a} : M(A) \to \mathbb{C}$ defined by $\hat{a}(\phi) := \phi(a)$. The weakest topology on $M(A)$ such that all functions $\hat{a}, a \in A$, are continuous is called the Gelfand topology. The Gelfand topology coincides with the weak-star topology of the strong dual space $A'$, restricted to $M(A)$. If $A$ is a Banach algebra, $M(A)$ is a weak-star compact subset of the unit ball of $A'$.

If $A$ is a uniform algebra of continuous functions on a metric space $G$, then for any $x \in G$ the point evaluation functional $\delta(x) : f \mapsto f(x)$ belongs to $M(A)$.

The purpose of this paper is to describe the spectrum of the Fréchet algebra $H_b(X)$ of entire analytic functions of bounded type on a Banach space $X$ and to study some related questions of infinite-dimensional holomorphy.

The problem of description of the spectrum of $H_b(X)$ was first studied by Aron, Cole and Gamelin [3, 4]. Using the Aron-Berner extension operation [2, 10], they showed, in particular, that $X''$ belongs to the spectrum of $H_b(X)$. In [5] it is proved that this inclusion is proper if there exists a polynomial on $X$ which is not weakly continuous on bounded sets. This approach was generalized for algebra-valued analytic functions by García et al. in [13]. Some analytic structure on the set of maximal ideals was considered in [5] (for generalization for algebra-valued functions see [17]). In [22] Mujica investigated ideals of analytic functions of bounded type on...
Tsirelson’s space $T$ and showed that each character on $H_b(T)$ is a point evaluation functional. Homomorphisms of $H_b$ were studied by Carando, García and Maestre in \cite{9}. In \cite{11} Alencar et al. considered maximal ideals of algebras of symmetric analytic functions on $\ell_p$.

In this paper we show that every element of the spectrum of $H_b(X)$ can be represented by a sequence of functionals $(u_k)_{k=1}^{\infty}$ such that each $u_k$ belongs to a Banach space $E_k$, where $E_1 = X''$ and $E_n$ coincides with a special subspace of linear functionals on $n$-homogeneous polynomials. It is also shown that the spectrum of $H_b(X)$ contains the linear space of all finite sequences $(u_1, \ldots, u_m, 0, 0, \ldots)$. Finally, some related examples are considered.

For background on analytic functions on infinite-dimensional spaces, we refer the reader to \cite{13} or to \cite{21}. For details on the Aron-Berner extension we refer to \cite{8}.

Recall that for every polynomial $P \in \mathcal{P}(n)X$ there exists a (necessarily unique) symmetric $n$-linear form $A_P$, associated with $P$ such that $A_P(x, \ldots, x) = P(x)$. We will write $A_P(x_1^{k_1}, \ldots, x_n^{k_n})$ instead of $A_P(x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n)$. We will use the fact that $\mathcal{P}(n)X$ is isomorphic to the dual space of the symmetric projective $n$-fold tensor product $\otimes^{n}_s, \pi X$ of $X$.

Let us denote by $A_n(X)$ the closure of the algebra, generated by polynomials from $\mathcal{P}(\leq n)X$ with respect to the uniform topology on bounded subsets. It is clear that $A_1(X) \cap \mathcal{P}(n)X = \mathcal{P}_c(n)X$ and $A_n(X)$ is a Fréchet algebra of entire analytic functions on $X$ for every $n$. The closure of the algebra of all polynomials $\mathcal{P}(X)$ with respect to the uniform topology on bounded subsets is denoted by $H_b(X)$ and is called the algebra of entire functions of bounded type on $X$. It is well known that $H_b(X)$ consists of all entire functions that are bounded on bounded subsets. The closure of the algebra of all polynomials with respect to the uniform topology on the unit ball $B$, $H^{uc}_B(B)$, is the algebra of all analytic functions on $B$ which are uniformly continuous and bounded. We will use the short notation $M_b$ and $M_{uc}$ for the spectra $M(H_b(X))$ and $M(H^{uc}_B(B))$ respectively.

According to \cite{3}, every continuous functional $\phi \in H_b(X)'$ can be represented by $\phi = \sum_{k=0}^{\infty} \phi_k$, where $\phi_k = \pi_k(\phi)$ is the restriction of $\phi$ to $\mathcal{P}(k)X$. The infimum of all $r > 0$, $R(\phi)$ such that $\phi$ is continuous with respect to the norm of uniform convergence on the ball $rB$ is called the radius function of $\phi$. It is known \cite{3} that

$$R(\phi) = \limsup_{n \to \infty} \|\phi_n\|^{1/n}.$$  

For every polynomial $P \in \mathcal{P}(mk)X$ we denote by $P_{(m)}(u)$ the polynomial from $\mathcal{P}(k \otimes^{m}\pi s)X$ such that $P_{(m)}(x^{\otimes m}) = P(x)$, where $x^{\otimes m} = x \otimes \cdots \otimes x$. 

Lemma 1. Let $\phi \in H_b(X)'$ such that $\phi(P) = 0$ for every $P \in \mathcal{P}(mX) \cap A_{m-1}(X)$, where $m$ is a fixed positive integer and $\phi_m \neq 0$. Then there is $\psi \in M_b$ such that $\psi_k = 0$ for $k < m$ and $\psi_m = \phi_m$. The radius function $R(\psi) = \|\phi_m\|^{1/m}$.

Proof. Since $\phi_m \neq 0$, there is an element $w \in (\mathcal{S}_{s,n}^m X)'$, $w \neq 0$ such that for any $m$-homogeneous polynomial $P$, $\phi(P) = \phi_m(P) = \bar{P}_m(w)$, where $\bar{P}_m$ is the Aron-Berner extension of the linear functional $P_m$ from $\mathcal{S}_{s,n}^m X$ to $(\mathcal{S}_{s,n}^m X)'$ and $\|w\| = \|\phi_m\|$. For an arbitrary $n$-homogeneous polynomial $Q$ we set

\[
\psi(Q) = \begin{cases} 
\bar{Q}_m(w) & \text{if } n = mk \text{ for some } k \geq 0, \\
0 & \text{otherwise,}
\end{cases}
\]

where $\bar{Q}_m$ is the Aron-Berner extension of the $k$-homogeneous polynomial $Q_m$ from $\mathcal{S}_{s,n}^m X$ to $(\mathcal{S}_{s,n}^m X)'$.

Let $(u_\alpha)$ be a net from $\mathcal{S}_{s,n}^m X$ that converges to $w$ in the weak-star topology of $(\mathcal{S}_{s,n}^m X)'$, where $\alpha$ belongs to an index set $\mathfrak{A}$. We can assume that $u_\alpha$ has a representation $u_\alpha = \sum_{j=1}^{\infty} x_{j,\alpha}^m$ for some $x_{j,\alpha} \in X$. Let us show that $\psi(PQ) = \psi(P)\psi(Q)$ for any homogeneous polynomials $P$ and $Q$. Let us suppose first that $\deg(PQ) = nr + l$ for some integers $r \geq 0$ and $l > 0$. Then $P$ or $Q$ has degree equal to $mk + s$, $k \geq 0$, $m > s > 0$. Thus, by the definition, $\psi(PQ) = 0$ and $\psi(P)\psi(Q) = 0$. Suppose that $\deg(PQ) = nr$ for some integer $r \geq 0$. If $\deg P = mk$ and $\deg Q = mn$ for $k, n \geq 0$, then $\deg(PQ) = m(k+n)$ and $\psi(PQ) = (PQ)_m(w) = \bar{P}_m(w)\bar{Q}_m(w) = \psi(P)\psi(Q)$.

Now let $\deg P = mk + l$ and $\deg Q = mn + r$, $l, r > 0$, $l + r = m$. Write $\nu = 1/(\deg P + \deg Q)! = 1/(m(k+n+1))!$. Let $A_{PQ}$ denote the symmetric multilinear map, associated with $PQ$. Then

\[
A_{PQ}(x_1, \ldots, x_{m(k+n+1)}) = \nu \sum_{\sigma \in S_{m(k+n+1)}} A_P(x_{\sigma(1)}, \ldots, x_{\sigma(mk+l)}) A_Q(x_{\sigma(mk+l+1)}, \ldots, x_{\sigma(m(k+n+1)})],
\]

where $S_{m(k+n+1)}$ is the group of permutations on $\{1, \ldots, m(k+n+1)\}$. Thus for $\alpha_1, \ldots, \alpha_{k+n+1} \in \mathfrak{A}$ we have

\[
\psi(PQ) = \psi(\bar{P}_Q)_m(w) = \lim_{\alpha_1, \ldots, \alpha_{k+n+1} \in \mathfrak{A}} \bar{A}_{PQ_m}(u_{\alpha_1}, \ldots, u_{\alpha_{k+n+1}})
\]

\[
= \lim_{\alpha_1, \ldots, \alpha_{k+n+1} \in \mathfrak{A}} \bar{A}_{PQ_m} \left( \sum_{j=1}^{\infty} x_{j,\alpha_1}^m \cdots, \sum_{j=1}^{\infty} x_{j,\alpha_{k+n+1}}^m \right)
\]

\[
= \nu \sum_{\sigma \in S_{m(k+n+1)}} \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k+n+1)} \sum_{j_1, \ldots, j_{k+n+1} = 1}^{\infty} A_P(x_{j_{\sigma(1)}}, \alpha_{\sigma(1)}) \cdots, x_{j_{\sigma(k+n+1)}}, \alpha_{\sigma(k+n+1)}).
\]

Fix some $\sigma \in S_{m(k+n+1)}$ and fix all $x_{j_{\sigma(i)}}, \alpha_{\sigma(i)}$, for $i \leq k$ and for $i > k + 1$. Then

\[
\sum_{j_1, \ldots, j_{k+n+1} = 1}^{\infty} \lim_{\alpha_{\sigma(k+n+1)} \in \mathfrak{A}} \sum_{j_{\sigma(k+n+1)} = 1}^{\infty} A_P(x_{j_{\sigma(1)}}, \alpha_{\sigma(1)}) \cdots, x_{j_{\sigma(k+n+1)}}, \alpha_{\sigma(k+n+1)}) \times A_Q(x_{j_{\sigma(k+n+1)}}, \alpha_{\sigma(k+n+1)}).
\]
because for fixed $x_{k(i), \alpha(i)}$, $i \leq k$,
\[
P_\sigma(y) := \sum_{j_1, \ldots, j_k, j_{k+2}, \ldots, j_{k+n+1} = 1}^\infty A_P(x_{j_1(\sigma(1)), \alpha(i_1)}, \ldots, x_{j_k(\sigma(k)), \alpha(i_k)}, y^l)
\]
is an $l$-homogeneous polynomial and for fixed $x_{k(i), \alpha(i)}$, $i > k + 1$,
\[
Q_\sigma(y) := \sum_{j_1, \ldots, j_k, j_{k+2}, \ldots, j_{k+n+1} = 1}^\infty A_Q(y^r, x_{j_{k+2}(\sigma(k+2)), \alpha(i_{k+2})}, \ldots, x_{j_{k+n+1}(\sigma(k+n+1)), \alpha(i_{k+n+1})})
\]
is an $r$-homogeneous polynomial. Thus $P_\sigma Q_\sigma \in A_{m-1}(X)$. Hence
\[
\lim_{\alpha}(P_\sigma Q_\sigma)_{(m)}(u_{\alpha}) = \psi(P_\sigma Q_\sigma) = 0
\]
for every fixed $\sigma$. Thus $\psi(PQ) = 0$. On the other hand, $\psi(P)\psi(Q) = 0$ by the definition of $\psi$. So $\psi(PQ) = \psi(P)\psi(Q)$.

Thus we have defined the multiplicative function $\psi$ on homogeneous polynomials. We can extend it by linearity and distributivity to a linear multiplicative functional on the algebra of all continuous polynomials $P(X)$. If $\psi_n$ is the restriction of $\psi$ to $P(nX)$, then $\|\psi_n\| = \|w\|^{n/m}$ if $n/m$ is a positive integer and $\|\psi_n\| = 0$ otherwise. Hence $\psi_n = \sum_{n=0}^{\infty} \psi_n$ is a continuous linear multiplicative functional on $H_b(X)$ by [3] 2.4. Theorem and the radius function of $\psi$ can be computed by
\[
R(\psi) = \limsup_{n \to \infty} \|\psi_n\|^{1/n} = \limsup_{n \to \infty} \|w\|^{n/m} = \|w\|^{1/m} = \|\phi_m\|^{1/m}
\]
as required. \hfill \Box

For each fixed $x \in X$, the translation operator $T_x$ is defined on $H_b(X)$ by
\[
(T_xf)(y) = f(y + x), \quad f \in H_b(X).
\]
It is not complicated to check that $T_x f \in H_b(X)$ and for fixed $\phi \in H_b(X)'$ the function $x \mapsto \phi(T_x f)$, $x \in X$, belongs to $H_b(X)$ (see [3]).

For fixed $\phi, \theta \in H_b(X)'$ the convolution product $\phi \ast \theta$ in $H_b(X)$ is defined by
\[
(\phi \ast \theta)(f) = \phi(\theta(T_x f)), \quad f \in H_b(X).
\]

Let $\phi, \theta \in M_b$. According to [3] 4.7. Corollary, there exist nets $(x_{\alpha})$, $(y_{\beta}) \subset X$ such that
\[
(2) \quad \phi(P) = \lim_{\alpha} P(x_{\alpha}), \quad \theta(P) = \lim_{\beta} P(y_{\beta})
\]
for every polynomial $P$. We will write the condition (2) by $x_{\alpha} \xrightarrow{\mathcal{E}} \phi$ and $y_{\beta} \xrightarrow{\mathcal{E}} \theta$. Thus for every polynomial $P$ we have: $(\phi \ast \theta)(P) = \lim_{\beta} \lim_{\alpha} P(x_{\alpha} + y_{\beta})$. Note that $M_b$ is a semigroup with respect to the convolution product and $\phi \ast \theta \neq \theta \ast \phi$ in general (see [3] Remark 3.5). We denote $\phi_1 \ast \cdots \ast \phi_n$ briefly by $\phi_n \ast \cdots \ast \phi_1$.

Let $I_k$ be the minimal closed ideal in $H_b(X)$, generated by all $m$-homogeneous polynomials, $0 < m \leq k$. Evidently, $I_k$ is a proper ideal (contains no unit) so it is contained in a closed maximal ideal (see [21] p. 228). Let
\[
\Phi_k := \{ \phi \in M_b : \ker \phi \supset I_k \}.
\]
We set $\Phi_0 := M_b$. The functional $\delta(0)$, that is, point evaluation at zero, belongs to $\Phi_k$ for every $k > 0$. 

Corollary 2. If $A_m(X) \neq A_{m-1}(X)$ for some $m > 1$, then there exists $\psi \in \Phi_{m-1}$ such that $\psi \notin \Phi_m$.

Proof. Let $P \in \mathcal{P}(mX)$ and $P \notin A_{m-1}(X)$. Since $A_{m-1}(X)$ is a closed subspace of $H_b(X)$, by the Hahn-Banach Theorem there exists a linear functional $\phi \in H_b(X)'$ such that $\phi(Q) = 0$ for every $Q \in A_{m-1}(X)$ and $\phi(P) \neq 0$. So $\phi_k \equiv 0$ for $k < m$ and $\phi_m(P) \neq 0$. By Lemma 4 there exists $\psi \in M_b$ such that $\psi_k = \phi_k$ for $k = 1, \ldots, m$. Thus $\psi \in \Phi_{m-1}$, but $\psi \notin \Phi_m$. □

Note that $A_1(c_0) = A_n(c_0)$ for every $n$, but $A_k(\ell_p) = A_m(\ell_p)$ for $k \neq m$ if and only if $k < p$ and $m < p$. Moreover, if $X$ admits a polynomial which is not weakly sequentially continuous, then the chain of algebras $\{A_k(X)\}$ does not stabilize and if $X$ contains $\ell_1$, then $A_k(X) \neq A_m(X)$ for $k \neq m$ [19] [12].

Lemma 3. If $\phi, \psi \in M_b$ and $\psi \in \Phi_{k-1}$, then $\phi \ast \psi(P) = \phi(P) + \psi(P)$ for every $P \in \mathcal{P}(kX)$.

Proof. Given $x_\alpha$ and $y_\beta$ be nets in $X$ such that $x_\alpha \overset{\mathcal{E}_\beta}{\to} \phi$ and $y_\beta \overset{\mathcal{E}_\alpha}{\to} \psi$. For any fixed $y_\beta$ and $0 < n < k$, $A_P(x_{k-n}^n, y_\beta^m)$ is a $(k - n)$-homogeneous polynomial. Thus

$$\phi(A_P(x_{k-n}^n, y_\beta^m)) = \lim_{\alpha} A_P(x_\alpha, y_\beta^n) = 0.$$

Therefore,

$$\phi \ast \psi(P) = \lim_{\beta, \alpha} P(x_\alpha + y_\beta) = \sum_{n+m=k} \lim_{\beta, \alpha} A_P(x_\alpha^n, y_\beta^m) = \sum_{n+m=k} \lim_{\beta, \alpha} A_P(x_\alpha^n, y_\beta^m) = \phi(P) + \psi(P).$$

□

Lemma 4. If $P \in \mathcal{P}(kX)$, $\phi_j \in \Phi_{j-1}$, then for every $m > k$, $\sum_{j=1}^{m} \phi_j(P) = \sum_{j=1}^{k} \phi_j(P)$.

Proof. Since $\phi_j \in \Phi_{j-1}$, $\phi_j(P) = 0$ for every $j > k$. □

Given a sequence $(\phi_n)_{n=1}^{\infty} \subset M_b$, $\phi_n \in \Phi_{n-1}$, the infinite convolution $\star \phi_n$ denotes a linear multiplicative functional on the algebra of all polynomials $\mathcal{P}(X)$ such that $\star \phi_n(P) = \star (P)$ if $P \in \mathcal{P}(kX)$ for an arbitrary $k$. This multiplicative functional uniquely determines a functional in $M_b$ (which we denote by the same symbol $\star \phi_n$) if it is continuous.

The point evaluation operator $\delta$ maps $X$ into $M_b$ by $x \mapsto \delta(x)$, $\delta(x)(f) = f(x)$. The operator $\tilde{\delta}$ is the extension of $\delta$ onto $X''$, i.e. $\tilde{\delta}(x'')(f) = \tilde{f}(x'')$ for every $x'' \in X''$.

Theorem 5. There exists a sequence of dual Banach spaces $(E_n)_{n=1}^{\infty}$ and a sequence of maps $\delta^{(n)}: E_n \to M_b$ such that $E_1 = X''$, $E_n = \mathcal{P}(nX)' \cap I_{n-1}$, $\delta^{(1)} = \delta$ and such that an arbitrary complex homomorphism $\phi \in M_b$ has a representation

$$\phi = \lim_{n=1}^{\infty} \delta^{(n)}(u_n)$$

for some $u_n \in E_n$, $n = 1, 2, \ldots$. 

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Proof. Put $E_1 = X''$. Then $\delta(1_X) = \delta(x'') \in M_b$ for every $x'' \in X''$. Suppose that spaces $E_k$ and maps $\delta(k)$ are constructed for $k < n$. Denote by $E_n$ the set 
\{\pi_n(\phi) : \phi \in \Phi_{n-1}\},
where $\pi_n(\phi)$ is the restriction of $\phi$ onto the subspace $P_n(X)$. In other words, $E_n$ consists of linear continuous functionals on $P_n(X)$ that vanish on all polynomials in $P_n(X) \cap A_{n-1}$. If $A_n = A_{n-1}$, then $E_n = 0$. Otherwise, by Corollary \ref{corollary_exist}, there are nonzero points in $E_n$.

By Lemma \ref{Lemma3} for $P \in P_n(X)$ and $\phi, \psi \in \Phi_{n-1} \subset M_b$, $\pi_n(\phi \psi)(P) = \phi(\psi(P)) + \psi(\phi(P)) = \pi_n(\phi(P)) + \pi_n(\psi(P))$. Hence $\pi_n(\phi) = \pi_n(\phi) + \pi_n(\psi)$. For an arbitrary complex number $a$, $a\phi \in H_b(X)$ and $\pi_k(a\phi) = a\pi_k(\phi)$. So $a\phi$ vanishes on all homogeneous polynomials of degree less than $n$. By Lemma \ref{Lemma4} there exists $\psi \in M_b$ such that $\psi_k = a\psi_k$ for $1 \leq k \leq n$. Thus $\psi \in \Phi_{n-1}$ and $a\phi_n = \psi_n \in E_n$. Hence $E_n$ is a linear space and polynomials from $P_n(X)$ are acting on $E_n$ as linear functionals.

Put $W_n = P_n(X)/(I_{n-1} \cap P_n(X))$. Then $W_n$ is a Banach space of linear functionals on $E_n$ and the functionals from $W_n$ separate points of $E_n$. Let us define a norm on $E_n$, $|| \cdot ||_n$ as the supremum of values of a vector from $E_n$ on the unit ball of $W_n$. Therefore $W_n' = (P_n(X)/(I_{n-1} \cap P_n(X)))' = P_n(X)' \cap I_{n-1}' \supset E_n'$. On the other hand, if $u \in P_n(X)' \cap I_{n-1}'$, then by Lemma \ref{Lemma1} $u = \pi_n(\phi)$ for some $\phi \in M_b$ and so $u \in E_n$. Thus $E_n = W_n$.

For given $w \in E_n$ let us define $\delta(n)(w)(Q) = \psi(Q)$ on homogeneous polynomials $Q$ by formula (4) and extend it to the unique complex homomorphism on $H_b(X)$ as in Lemma \ref{Lemma1}. So $\delta(n)$ maps $E_n$ into $M_b$. For any $\phi \in M_b$ put $u_1 := \phi_1 \in X'' = E_1$, $u_2 := \phi_2 - \pi_2(\delta(1)(u_1))$. It is clear that $u_2 \in E_2$. Suppose that we have defined $u_k \in E_k$, $k < n$. Set 
\begin{equation} u_n := \phi_n - \pi_n \left( \delta(k)(u_k) \right). \end{equation}

Let us show that $u_n \in E_n$. It is enough to check that for every $P \in P_n(X)$ such that $P = P_kP_m$, $\deg P_k = k \neq 0$, $\deg P_m = n \neq 0$ implies $u_n(P) = 0$. Note that for all $n$-homogeneous polynomials $P_n$,
\begin{equation} \phi_n - \pi_n \left( \delta(k)(u_k) \right)(P_n) = \phi_n - n^{-1}\delta(k)(u_k)(P_n). \end{equation}

From the multiplicativity of $\phi$ and Lemma \ref{Lemma1} it follows that
\begin{align*}
\phi_n(P_kP_m) - \pi_n \delta(j)(u_j)(P_kP_m) &= \phi_k(P_k)\phi_m(P_m) \\
&\quad - \left( \delta(k)(u_k)(P_k) \right) \left( \delta(k)(u_k)(P_m) \right) \\
&\quad - \left( \delta(j)(u_j)(P_k) \right) \left( \delta(j)(u_j)(P_m) \right) \\
&\quad - \left( \delta(j)(u_j)(P_k) \right) \left( \delta(j)(u_j)(P_m) \right) = 0.
\end{align*}

The last equality holds because by the induction assumption, $u_k \in E_k$, $u_m \in E_m$ and hence, by Lemma \ref{Lemma3},
\begin{equation} u_k(P_k) + \delta(j)(u_j)(P_k) = \delta(j)(u_j)(P_k) \end{equation}
Proposition 7. Let \( u_k \in E_k \), by Lemma 3

\[
\phi \left( \sum_{j=1}^{m} \delta(j)(u_j)(f_n) \right) = f(0) + \sum_{n=1}^{\infty} \left( \sum_{j=1}^{n} \delta(j)(u_j)(f_n) \right),
\]

where \( f = \sum f_n \) is the Taylor series expansion of \( f \). Hence \( \phi \) is well defined on \( \mathcal{P}(X) \). On the other hand, applying (4) and (5) we obtain

\[
\phi(u_n(P)) = \phi_n(P) - \sum_{j=1}^{n} \delta(j)(u_j)(P_n) = u_n(P) + \sum_{j=1}^{n} \delta(j)(u_j)(P_n) = 0
\]

for arbitrary \( P_n \in \mathcal{P}(^nX) \). Thus \( \phi = \sum_{j=1}^{\infty} \delta(j)(u_j) \) on \( \mathcal{P}(X) \). Hence \( \phi = \sum_{j=1}^{\infty} \delta(j)(u_j) \) on \( H_b(X) \).

Let us denote by \( E^\infty \) the space of all finite sequences \( (u_1, \ldots, u_m, 0, \ldots) \), \( u_k \in E_k \). According to Theorem 5 every finite sequence \( u = (u_1, \ldots, u_m, 0, \ldots) \) defines a character \( \phi_u = \sum_{k=1}^{m} \delta(k)(u_k) \in M_b \). Thus \( E^\infty \subset M_b \) and for every \( u, v \in E^\infty \), \( \phi_{u+v} \in M_b \). Moreover, from the density of polynomials in \( H_b(X) \) it follows that \( E^\infty \) is dense in \( M_b \) with respect to the Gelfand topology. So we have proved the following theorem.

**Theorem 6.** \( M_b \) contains the dense linear subspace of all finite subsequences \( (u_1, \ldots, u_m, 0, \ldots) \), \( u_k \in E^k \).

According to [3], [7], the operation of sum on \( X \) may be discontinuous with respect to the Gelfand topology, induced from \( M_b \). Hence, in general, \( E^\infty \) is not a topological vector space. Thus, the density of \( E^\infty \) in \( M_b \) does not imply that \( M_b \) is a linear space.

We need to have some properties of the radius function, proved by Aron, Cole and Gamelin in [3].

**Proposition 7.**

1. For each \( r > 0 \), the set of \( \phi \in M_b \) satisfying \( R(\phi) \leq r \) coincides with the spectrum of \( H^{\infty}_{uc}(rB) \). In particular, \( M_{uc} = \{ \phi \in M_b : R(\phi) \leq 1 \} \).
2. For every \( \phi, \psi \in H_b(x)' \), \( R(\phi \ast \psi) \leq R(\phi) + R(\psi) \).

**Example 8.** 1. Let \( X = c_0 \) or Tsirelson’s space. Then \( E_k = \{0\} \) for \( k > 1 \) [4, 22].
2. Let \( X = \ell_1 \) and \( \phi \in H_b(\ell_1)' \), \( ||\phi|| = 1 \). According to [3], \( \phi \in M_b(\ell_1) \) if and only if for every \( m = 1, 2, \ldots \) there exists a symmetric measure on \( \beta(\mathbb{N}^m) \), \( \nu_m \) and a constant \( c > 0 \) such that \( ||\nu_m|| \leq c^m \) and for each \( P_m \in \mathcal{P}(^m\ell_1) \),

\[
\phi(P_m) = \int_{\beta(\mathbb{N}^m)} \hat{P}_m d\nu_m,
\]
where \( \hat{P}_m \) is just \( P_m \) regarded as a vector from \( \ell^\infty(\mathbb{N}^m) \). By Theorem 5 \( \phi \in M_b(\ell_1) \) if and only if there is a sequence of symmetric measures \((\mu_m)\) which are orthogonal to \( \beta(\mathbb{N}^j) \times \beta(\mathbb{N}^k) \subset \beta(\mathbb{N}^m) \), for \( m > 1, k + j = m, k, j > 0 \), and functionals

\[
u_m(P_m) = \int_{\beta(\mathbb{N}^m)} \hat{P}_m d\mu_m
\]
determine \( \phi \) by formula (3).

3. (Cf. [1] Example 3.1.) Let \( X = \ell_p \) for some integer \( p, 1 < p < \infty \). For every \( n, \) put

\[
v_n = \frac{1}{n^{1/p}} (e_1 + \cdots + e_n),
\]
where \((e_k)\) is the standard basis in \( \ell_p \). Since \( \|v_n\| = 1 \), \( R(\delta(v_n)) = 1 \) and so \( \delta(v_n) \in M_{uc} \subset M_b \). By compactness of \( M_{uc} \) there is an accumulation point \( \phi \in M_{uc} \) of the sequence \((\delta(v_n))\). If \( 0 < k < p \), then by Pitt’s Theorem (see [16, Theorem 5.1]) every polynomial \( P \in \mathcal{P}(k\ell_p) \) is weakly continuous on bounded sets. Since \( v_n \) is weakly null in \( \ell_p \), \( \phi(P) = 0 \). On the other hand, \( \phi(Q) = 1 \) for the polynomial

\[
Q(x) = \sum_{n=1}^{\infty} x_n^p.
\]
Thus \( \phi \in \Phi_{p-1} \) and \( \phi \neq 0 \). In other words, if \( \phi = \sum_{k=1}^\infty \delta^{(k)}(u_k) \) is the representation of \( \phi \) by Theorem 5, then \( u_k = 0 \) for \( k < p \) and \( v_p \neq 0 \).

**Proposition 9.** Let \( \phi \in M_b \) and let \( \phi = \sum_{k=1}^{\infty} \delta^{(k)}(u_k), u_k \in E_k \), be its representation. Then

\[
\limsup_{k \to \infty} \| u_k \|_k^{1/k} \leq R(\phi) \leq \sum_{k=1}^{\infty} \| u_k \|_k^{1/k}.
\]

**Proof.** The first inequality holds because \( \|u_k\|_k \leq \|\phi_k\| \) and by the definition of the radius function. The second inequality follows from Proposition 7 and the following calculation:

\[
R(\delta^{(k)}(u_k)) = \limsup_{m \to \infty} \| \pi_{km}(\delta^{(k)}(u_k)) \|_1^{1/km} = \| \delta^{(k)}(u_k) \|_m^{1/km} = \| u_k \|_k^{1/k}.
\]

Let \( F \) be an analytic map from \( \ell_1 \) to \( \ell_1 \) defined by

\[
F(x) = F \left( \sum_{n=1}^{\infty} x_n e_n \right) = \sum_{n=1}^{\infty} x_n^n e_n.
\]
We denote by \( F(\ell_1) \) the range of \( F \) and by \( F(B_{\ell_1}) \) the range of \( F \) restricted to the unit ball \( B_{\ell_1} \subset \ell_1 \).

Given a sequence of Banach spaces \((E_n, \| \cdot \|_n)_{n=1}^\infty \) and \( 0 < \rho \leq \infty \) the Kôthe sequence space \( \lambda^1(K_\rho; (E_n)) \) (where \( K_\rho = \{ (r^n)_{n=1}^\infty: 0 < r < \rho \} \) is the Fréchet space

\[
\left\{ (x_n)_{n=1}^\infty \in \prod_{n=1}^\infty E_n : p_r((x_n)_{n=1}^\infty) = \sum_{n=1}^\infty \|x_n\| r^n < \infty \forall r, 0 < r < \rho \right\},
\]
endowed with the topology given by the seminorms \( \{ p_r \}_{0 < r < \rho} \). By Cauchy-Hadamard’s formula,

\[
\lambda^1(K_\rho; (E_n)) = \left\{ (x_n)_{n=1}^\infty \in \prod_{n=1}^\infty E_n : \limsup_{n \to \infty} \|x_n\|_n^{1/n} \leq \frac{1}{\rho} \right\}.
\]
Corollary 10. \(1\) \(M_b\) contains every sequence \(u = (u_k)_{k=1}^{\infty}, u_k \in E_k,\) such that the sequence \((||(u_k)||_{k=1}^{\infty})\) is in \(F(\ell_1)\).
\(2\) \(M_{uc}\) contains every sequence \(u = (u_k)_{k=1}^{\infty}, u_k \in E_k,\) such that the sequence \((||(u_k)||_{k=1}^{\infty})\) is in \(F(B_{\ell_1})\).
\(3\) Every complex homomorphism \(\phi \in M_b\) is contained in a Köthe sequence space \(\lambda^1(K_p; (E_n))\) for \(p = 1/R(\phi)\).
\(4\) \(M_{uc}\) is contained in \(\lambda^1(K_1; (E_n))\).

Proof. Since \(F^{-1}((||u_k||)_{k=1}^{\infty}) \in \ell_1, \sum_{k=1}^{\infty} ||u_k||^{1/k} \leq \infty\) and by Proposition \(9\), \(R(\phi_u) < \infty\). Thus \(\phi_u \in M_b\). Moreover, if \(F^{-1}((||u_k||)_{k=1}^{\infty}) \in F(B_{\ell_1})\), then \(R(\phi_u) \leq 1\) and \(\phi_u \in M_{uc}\).

Suppose that \(\phi_u \in M_b\) for some \(u = (u_k)_{k=1}^{\infty}\). Then \(R(\phi_u) < \infty\) and by Proposition \(9\), \(\limsup_{k \to \infty} ||u_k||^{1/k} \leq R(\phi_u)\). Hence \(\phi_u \in \lambda^1(K_1/R(\phi_u); (E_n))\). In particular, if \(R(\phi_u) \leq 1\), then \(\phi_u \in \lambda^1(K_1; (E_n))\). \(\square\)

Dixon \([14]\) has given an example of an algebra of polynomials of infinitely many variables which admits discontinuous scalar-valued homomorphisms. In \([13]\) a construction is given of a discontinuous scalar-valued homomorphism of an algebra of polynomials on an arbitrary infinite-dimensional Banach space. The next corollary shows that the restriction of a discontinuous complex homomorphism on \(A_n(X) \cap \mathcal{P}(X)\) can be continuous for every \(n\). Note that the problem of existence of discontinuous complex homomorphisms on \(H_b(X)\) for an infinite-dimensional Banach space \(X\) is still open and equivalent to the famous Michael Problem \([20, 21, p. 240]\).

Corollary 11. If the sequence of algebras \(A_n(X)\) does not stabilize, then there is a discontinuous complex homomorphism \(\zeta\) on \(\mathcal{P}(X)\) such that the restriction of \(\zeta\) on \(A_n(X) \cap \mathcal{P}(X)\) is a continuous complex homomorphism for every \(n\).

Proof. By Corollary \(2\) and Theorem \(5\) there exists an infinite sequence \((u_k)_{k=1}^{\infty}, u_k \in E_k, u_k \neq 0\). Since each \(E_k\) is a linear space, we can choose \(u_k\) such that \(\limsup_{k \to \infty} ||u_k||^{1/k} = \infty\). Put \(\zeta = \bigoplus_{k=1}^{\infty} \delta^{(k)}(u_k)\). Evidently,
\[\zeta(f) = \bigoplus_{k=1}^{n} \delta^{(k)}(u_k)(f)\]
for every \(f \in A_n(X)\). So \(\zeta\) is well defined and continuous on \(A_n(X) \cap \mathcal{P}(X)\). If \(\zeta\) is continuous on \(\mathcal{P}(X)\), then it can be extended to a continuous complex homomorphism on \(H_b(X)\). But this contradicts Proposition \(4\). \(\square\)

In \([11]\) Deghoul, using Borsuk’s theorem, shows that there is a “exceptional” character \(\phi\) on \(H_b(\ell_2)\) such that \(\phi\) vanishes on odd degree homogeneous polynomials and is different from the evaluation at 0. The next proposition delivers the existence of exceptional characters on \(H_b(X)\) for a large number of \(X\).

Proposition 12. Suppose that \(A_m(X) \neq A_k(X)\) for some \(m > 1\) and all \(k < m\). Then there exists a nontrivial character \(\psi_0 \in M_b\) such that \(\psi_0(P) = 0\) for every homogeneous polynomial \(P, \deg P \neq nm, n = 1, \ldots, \infty\).
Proof. By Corollary 2 there exists a nontrivial character \( \psi \in M_b \) which vanishes on all \( k \)-homogeneous polynomials for \( k < m \). From Theorem 5 it follows that \( E_m \) contains a nonzero vector \( u_m \). Put \( \psi_0 = \delta^{(m)}(u_m) \). Then \( \psi_0 \) vanishes on all homogeneous polynomials excepting \( nm \)-homogeneous polynomials, \( n = 1, 2, \ldots \).

\[ \square \]

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