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DEFORMATIONS AND DERIVED EQUIVALENCES

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Abstract. Suppose A and B are block algebras of finite groups over a complete local commutative Noetherian ring whose residue field is a field k of positive characteristic. We prove that a split-endomorphism two-sided tilting complex (as introduced by Rickard) for the derived categories of bounded complexes of finitely generated modules over A, resp. B, preserves the versal deformation rings of bounded complexes of finitely generated modules over kA, resp. kB.

1. Introduction

Suppose k is a field of characteristic $p > 0$, W is a complete local commutative Noetherian ring with residue field k, and G is a profinite group. In [11], Mazur developed a deformation theory of finite-dimensional representations of G over k, using work of Schlessinger [17]. A more explicit approach was later described by de Smit and Lenstra in [8]. In the meantime, deformation theory has become a basic tool in arithmetic algebraic geometry (see e.g. [7], [19, 18], [5] and their references). In [3, 4], the author and Chinburg generalized Mazur's deformation theory to objects of bounded complexes of pseudocompact modules over the completed group algebra $[[kG]]$ (see Section 3 for definitions in case G is finite).

One motivation for [3, 4] arose from the study of universal deformation rings for finite groups. In [2] and [1], Morita equivalences between module categories were a basic tool in determining universal deformation rings of representations associated to blocks with cyclic or Klein four defect groups. Since Morita equivalences cannot be expected to exist in a broad context, one is led to consider other types of equivalences. Conjectures of Broué and others (see e.g. [6, 15] and their references) would establish derived equivalences between various blocks of group algebras of finite groups. For example, Broué conjectured that if a finite group G has an abelian Sylow p-subgroup D, then the principal blocks of WG and of $WN_G(D)$ have equivalent derived categories (in the case W is a complete discrete valuation ring of characteristic zero). In [12], Rickard proved that two block algebras over W are derived equivalent if and only if there are so-called two-sided tilting complexes for the blocks. He further showed that these two-sided tilting complexes can always be

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chosen to have a split-endomorphism property (see e.g. [13, 14]). If one wants to ensure that these tilting complexes are also compatible with the p-local structures of the block algebras, one needs to require additional conditions to obtain so-called splendid tilting complexes (see [13] for principal blocks and [10] for general blocks).

The idea is to use such derived equivalences to simplify the determination of (uni)versal deformation rings of representations or bounded complexes by considering smaller groups. Thus one needs that derived equivalences preserve (uni)versal deformation rings of bounded complexes. In this paper, we consider derived equivalences given by split-endomorphism two-sided tilting complexes. Our main result is Theorem 3.2 which proves that a split-endomorphism two-sided tilting complex for two blocks A and B of finite groups over W induces an isomorphism between the versal deformation ring of a bounded complex of finitely generated kA-modules and the corresponding complex of kB-modules. A key ingredient in the proof is that two-sided tilting complexes behave well when extending scalars from W to a complete local commutative Artinian W-algebra R with residue field k (see [12] and Lemma 2.1).

The paper is organized as follows. In Section 2, we recall the necessary background about split-endomorphism two-sided tilting complexes. In Section 3, we give the basic definitions and results about deformations of objects of bounded complexes of pseudocompact modules over kG in case G is a finite group. We also prove our main result Theorem 3.2. In Section 4, we consider the example of split-endomorphism two-sided tilting complexes for the principal blocks of the alternating groups A_4 and A_5 in characteristic 2.

2. Two-sided tilting complexes and scalar extensions

Suppose k is a field of positive characteristic p and W is a complete local commutative Noetherian ring with residue field k. Define $\hat{\mathcal{C}}$ to be the category of complete local commutative Noetherian W-algebras with residue field k. The morphisms in $\hat{\mathcal{C}}$ are continuous W-algebra homomorphisms which induce the identity on k. Let $\mathcal{C}$ be the full subcategory of Artinian objects in $\hat{\mathcal{C}}$.

Suppose G (resp. H) is a finite group, and let A (resp. B) be a block of WG (resp. WH). Then A and B are both W-algebras that are projective as W-modules. Moreover, A and B are symmetric W-algebras in the sense that A (resp. B) is isomorphic to its W-linear dual $\check{A} = \mathrm{Hom}_W(A, W)$ (resp. $\check{B} = \mathrm{Hom}_W(B, W)$) as an A-A-bimodule (resp. B-B-bimodule). For $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, define RA (resp. RB) to be the block algebra in RG (resp. RH) corresponding to A (resp. B), i.e. $RA = R \otimes_W A$ (resp. $RB = R \otimes_W B$).

If S is a ring, S-mod denotes the category of finitely generated left S-modules. Let $C^b(S\text{-mod})$ be the category of bounded complexes in S-mod, let $K^b(S\text{-mod})$ be the homotopy category of $C^b(S\text{-mod})$, and let $D^b(S\text{-mod})$ be the derived category of $K^b(S\text{-mod})$.

Rickard proved in [12] (see also [13, §2] and [14, §9.2]) that the derived categories $D^b(A\text{-mod})$ and $D^b(B\text{-mod})$ are equivalent as triangulated categories if and only if there is a bounded complex $P^\bullet$ of finitely generated A-B-bimodules and a bounded complex $Q^\bullet$ of finitely generated B-A-bimodules such that

$$(2.1)\qquad \begin{aligned} P^\bullet \otimes^{\mathbf{L}}_B Q^\bullet &\cong A \qquad \text{in } D^b((A \otimes_W A^{op})\text{-mod}), \text{ and} \\ Q^\bullet \otimes^{\mathbf{L}}_A P^\bullet &\cong B \qquad \text{in } D^b((B \otimes_W B^{op})\text{-mod}). \end{aligned}$$

If $P^\bullet$ and $Q^\bullet$ exist, then the functors

$$P^\bullet \otimes^{\mathbf{L}}_B - : D^b(B\text{-mod}) \to D^b(A\text{-mod}) \quad \text{and} \tag{2.2}$$
$$Q^\bullet \otimes^{\mathbf{L}}_A - : D^b(A\text{-mod}) \to D^b(B\text{-mod})$$

are equivalences of derived categories, and $Q^\bullet$ is isomorphic to $\mathbf{R}\mathrm{Hom}_A(P^\bullet, A)$ in the derived category of B-A-bimodules. The complexes $P^\bullet$ and $Q^\bullet$ are called *two-sided tilting complexes*. Moreover, Rickard showed (see [13, §2] and [14, §9.2.2]) that we may assume that all terms of $P^\bullet$ are projective as left A-modules and as right B-modules and that all but one of the terms are actually projective as A-B-bimodules. Since A is a symmetric W-algebra, the functors $\mathrm{Hom}_A(-, A)$ and $\mathrm{Hom}_W(-, W)$ are naturally isomorphic, and so we may take $Q^\bullet$ to be the linear dual $\check{P}^\bullet = \mathrm{Hom}_W(P^\bullet, W)$ of $P^\bullet$. In this situation, (2.1) is equivalent to

$$A \cong P^\bullet \otimes_B \check{P}^\bullet \cong \mathrm{Hom}_B(P^\bullet, P^\bullet) \quad \text{in } K^b((A \otimes_W A^{op})\text{-mod}), \text{ and} \tag{2.3}$$
$$B \cong \check{P}^\bullet \otimes_A P^\bullet \cong \mathrm{Hom}_A(P^\bullet, P^\bullet) \quad \text{in } K^b((B \otimes_W B^{op})\text{-mod}).$$

Rickard calls a bounded complex $P^\bullet$ of finitely generated A-B-bimodules a *split-endomorphism two-sided tilting complex*, if all terms of $P^\bullet$ are projective as left and as right modules and (2.3) is satisfied.

The following lemma follows from [12, Thm. 2.1].

Lemma 2.1. *Suppose A, B are block algebras as above, $R \in \mathrm{Ob}(\mathcal{C})$ is Artinian, and $P^\bullet$ is a split-endomorphism two-sided tilting complex in $D^b((A \otimes_W B^{op})\text{-mod})$. Then*

$$P^\bullet_R = R \otimes_W P^\bullet$$

is a split-endomorphism two-sided tilting complex in $D^b((RA \otimes_R RB^{op})\text{-mod})$.

3. Two-sided tilting complexes and deformations

Keeping the notation of the previous section, we want to use Lemma 2.1 to prove that split-endomorphism two-sided tilting complexes preserve versal deformation rings. We first remind the reader of the basic definitions of quasi-lifts and deformations of complexes that were introduced in [3, 4].

Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$. Note that since G is assumed to be a finite group, the abstract group algebra RG is the same as the completed group algebra $[[RG]]$. An RG-module M is said to be *pseudocompact*, if it is the projective limit of RG-modules of finite length having the discrete topology. In particular, every finitely generated RG-module is pseudocompact. Let $C^-(\mathrm{PCMod}(RG))$ be the abelian category of bounded above complexes of pseudocompact RG-modules, let $K^-(\mathrm{PCMod}(RG))$ be the corresponding homotopy category, and let $D^-(\mathrm{PCMod}(RG))$ be the corresponding derived category. We say that a complex $M^\bullet$ in $K^-(\mathrm{PCMod}(RG))$ has *finite pseudocompact R-tor dimension*, if there exists an integer N such that for all pseudocompact R-modules S, and for all integers $i < N$, $\mathrm{H}^i(S \hat{\otimes}^{\mathbf{L}}_R M^\bullet) = 0$. Here $\hat{\otimes}_R$ denotes the completed tensor product in the category of pseudocompact R-modules. Note that if M is finitely generated as pseudocompact R-module, then the functors $M \otimes_R -$ and $M \hat{\otimes}_R -$ are naturally isomorphic.

Definition 3.1. Let $V^\bullet$ be a complex in $D^-(\mathrm{PCMod}(kG))$ which has only finitely many non-zero cohomology groups, all of which have finite k-dimension. A *quasi-lift* of $V^\bullet$ over an object R of $\hat{\mathcal{C}}$ is a pair $(M^\bullet, \phi)$ consisting of a complex $M^\bullet$ in $D^-(\mathrm{PCMod}(RG))$ which has finite pseudocompact R-tor dimension together with an isomorphism $\phi : k\hat{\otimes}^{\mathbf{L}}_R M^\bullet \to V^\bullet$ in $D^-(\mathrm{PCMod}(kG))$. Two quasi-lifts $(M^\bullet, \phi)$ and $(M'^\bullet, \phi')$ are *isomorphic* if there exists an isomorphism $M^\bullet \to M'^\bullet$ in $D^-(\mathrm{PCMod}(RG))$ which carries ϕ to ϕ'. A *deformation* of $V^\bullet$ over R is an isomorphism class of quasi-lifts of $V^\bullet$.

The *deformation functor* $\hat{F}_{V^\bullet} : \hat{\mathcal{C}} \to \mathrm{Sets}$ associated to $V^\bullet$ is defined as follows. It sends an object R of $\hat{\mathcal{C}}$ to the set $\hat{F}_{V^\bullet}(R)$ of all deformations of $V^\bullet$ over R, and it sends a morphism $\alpha : R \to R'$ in $\hat{\mathcal{C}}$ to the set map $\hat{F}_{V^\bullet}(R) \to \hat{F}_{V^\bullet}(R')$ induced by $M^\bullet \mapsto R'\hat{\otimes}^{\mathbf{L}}_{R,\alpha} M^\bullet$. Let $F_{V^\bullet}$ be the restriction of $\hat{F}_{V^\bullet}$ to the subcategory $\mathcal{C}$ of Artinian objects in $\hat{\mathcal{C}}$.

It was proved in [4, Thm. 2.14] that $F_{V^\bullet}$ has a pro-representable hull $R(G, V^\bullet) \in \mathrm{Ob}(\hat{\mathcal{C}})$ in the sense of [17, Def. 2.7], and that $\hat{F}_{V^\bullet}$ is continuous. The ring $R(G, V^\bullet)$ is called the *versal deformation ring of* $V^\bullet$. Moreover, if $\mathrm{Hom}_{D^-(kG)}(V^\bullet, V^\bullet) = k$, then $\hat{F}_{V^\bullet}$ is represented by $R(G, V^\bullet)$, in which case $R(G, V^\bullet)$ is called the *universal deformation ring of* $V^\bullet$.

Theorem 3.2. *Suppose A, B are block algebras as above, and $Q^\bullet$ is a split-endomorphism two-sided tilting complex in $D^b((B \otimes_W A^{op})$-mod). Let $V^\bullet$ be a bounded complex of finitely generated kA-modules, and let $V'^\bullet = (k \otimes_W Q^\bullet) \otimes_{kA} V^\bullet$. Then $R(G, V^\bullet)$ and $R(H, V'^\bullet)$ are isomorphic.*

Proof. Suppose $R \in \mathrm{Ob}(\mathcal{C})$ is Artinian, and let $(M^\bullet, \phi)$ be a quasi-lift of $V^\bullet$ over R.

Claim 1. The quasi-lift $(M^\bullet, \phi)$ is isomorphic to a quasi-lift $(N^\bullet, \psi)$ of $V^\bullet$ over R, where $N^\bullet$ and ψ are in $D^b(RA\text{-mod})$, such that the terms of $N^\bullet$ are abstractly free as R-modules.

Proof of Claim 1. By [4, Cor. 3.6], we may assume without loss of generality that $M^\bullet$ is a bounded-above complex of finitely generated abstractly free RG-modules. Then $k\hat{\otimes}^{\mathbf{L}}_R M^\bullet = k \otimes_R M^\bullet$ is also a bounded above complex of finitely generated abstractly free kG-modules, and we may assume that ϕ is given by a morphism in $C^-(\mathrm{PCMod}(kG))$. Let e be the idempotent corresponding to the block A. Then $M^\bullet = eM^\bullet \oplus (1-e)M^\bullet$ and $\phi = e\phi \oplus (1-e)\phi$. Since $M^\bullet$ has finite pseudocompact R-tor dimension and since the functor H^n is additive for all n, $eM^\bullet$ and $(1-e)M^\bullet$ also have finite pseudocompact R-tor dimension. Hence $(eM^\bullet, e\phi)$ is a quasi-lift of $eV^\bullet = V^\bullet$, and $((1-e)M^\bullet, (1-e)\phi)$ is a quasi-lift of the zero complex $(1-e)V^\bullet$. It follows that the quasi-lift $(M^\bullet, \phi)$ is isomorphic to the quasi-lift $(eM^\bullet, e\phi)$ of $V^\bullet$. Since $eM^\bullet$ has finite pseudocompact R-tor dimension and since the terms of $eM^\bullet$ are finitely generated abstractly free R-modules, we can truncate $eM^\bullet$ to obtain a quasi-lift $(N^\bullet, \psi)$ of $V^\bullet$ which is isomorphic to $(eM^\bullet, e\phi)$ such that $N^\bullet$ is a bounded complex of finitely generated RA-modules, all of which are abstractly free as R-modules. This completes the proof of Claim 1.

Claim 2. Suppose $R' \in \mathrm{Ob}(\mathcal{C})$ is Artinian, and $\alpha : R \to R'$ is a morphism in $\mathcal{C}$. Let $\pi : R \to k$, resp. $\pi' : R' \to k$, be the canonical surjection, and let $Q^\bullet_R = R \otimes_W Q^\bullet$ and $Q^\bullet_{R'} = R' \otimes_W Q^\bullet$. Then for each complex $X^\bullet$ in $D^b(RA\text{-mod})$, there is an isomorphism

$$h^{X^\bullet}_{R,R',\alpha} : (Q^\bullet_R \otimes_{RA} X^\bullet) \otimes_{R,\alpha} R' \to Q^\bullet_{R'} \otimes_{R'A} (X^\bullet \otimes_{R,\alpha} R') \tag{3.1}$$

in the category of bounded complexes of left $R'B$-modules such that

$$h^{X^\bullet\otimes_{R,\alpha}R'}_{R',k,\pi'}\circ(h^{X^\bullet}_{R,R',\alpha}\otimes k)=h^{X^\bullet}_{R,k,\pi} \tag{3.2}$$

when we identify

$$((Q^\bullet_R\otimes_{RA}X^\bullet)\otimes_{R,\alpha}R')\otimes_{R'}k=(Q^\bullet_R\otimes_{RA}X^\bullet)\otimes_R k$$

and

$$(X^\bullet\otimes_{R,\alpha}R')\otimes_{R'}k=X^\bullet\otimes_R k.$$

Proof of Claim 2. Suppose first that Q is a finitely generated B-A-bimodule which is projective as right A-module, and suppose X is a finitely generated left RA-module. Then the map

$$\begin{array}{rcl} h^{Q,X}_{R,R',\alpha}:((R\otimes_W Q)\otimes_{RA}X)\otimes_{R,\alpha}R' & \to & (R'\otimes_W Q)\otimes_{R'A}(X\otimes_{R,\alpha}R') \\ ((r\otimes q)\otimes x)\otimes r' & \mapsto & (\alpha(r)\otimes q)\otimes(x\otimes r') \end{array}$$

is an $R'B$-module isomorphism which is natural in both variables Q and X. It follows from the definition of the tensor product complex $Q^\bullet_R\otimes_{RA}X^\bullet$ that there is an isomorphism in $C^b(R'B\text{-mod})$ as stated in (3.1). Since α induces the identity on k, one easily checks (3.2), which proves Claim 2.

It follows from Lemma 2.1 that $Q^\bullet_R=R\otimes_W Q^\bullet$ is a split-endomorphism two-sided tilting complex in $D^b((RB\otimes_W RA^{op})\text{-mod})$. Let $(N^\bullet,\psi)$ be as in Claim 1, and define $N'^\bullet=Q^\bullet_R\otimes_{RA}N^\bullet$. Since the terms of $Q^\bullet_R$ are finitely generated projective right RA-modules and the terms of $N^\bullet$ are finitely generated abstractly free R-modules, it follows that the terms of $N'^\bullet=Q^\bullet_R\otimes_{RA}N^\bullet$ are finitely generated projective, and hence abstractly free, R-modules. Because $N'^\bullet$ is a bounded complex, this means that $N'^\bullet$ has finite pseudocompact R-tor dimension. Moreover,

$$N'^\bullet\hat{\otimes}^{\mathbf{L}}_R k=N'^\bullet\otimes_R k=(Q^\bullet_R\otimes_{RA}N^\bullet)\otimes_R k\xrightarrow{h^{N^\bullet}_{R,k,\pi}}Q^\bullet_k\otimes_{kA}(N^\bullet\otimes_R k)\xrightarrow{Q^\bullet_k\otimes\psi}V'^\bullet$$

is an isomorphism in $D^b(kB\text{-mod})$. This implies that $(N'^\bullet,(Q^\bullet_k\otimes\psi)\circ h^{N^\bullet}_{R,k,\pi})$ is a quasi-lift of $V'^\bullet$ over R. We therefore obtain for all $R\in\mathrm{Ob}(\mathcal{C})$ a bijection τ_R from the set of deformations of $V^\bullet$ over R onto the set of deformations of $V'^\bullet$ over R. Using (3.1) and (3.2) from Claim 2, it follows that the τ_R are natural with respect to homomorphisms $\alpha:R\to R'$ in $\mathcal{C}$. Since the deformation functors $\hat{F}_{V^\bullet}$ and $\hat{F}_{V'^\bullet}$ are continuous, this implies that they are naturally isomorphic. Hence the versal deformation rings $R(G,V^\bullet)$ and $R(H,V'^\bullet)$ are isomorphic. This completes the proof of Theorem 3.2.

4. The principal blocks of A_4 and A_5 in characteristic 2

In this section, we want to look at the following example. Let k be an algebraically closed field of characteristic 2, and let W be the ring of infinite Witt vectors over k. Let A_4, resp. A_5, denote the alternating group on 4, resp. 5, letters. Then WA_4 is its own principal block, and we denote the principal block of WA_5 by B. Rickard showed in [13, §3] (see also [16, Example 1]) that there is a split-endomorphism two-sided tilting complex $Q^\bullet_k$ for kB and kA_4. In fact, Rickard showed that this complex is a so-called splendid tilting complex, and can thus be lifted to a splendid tilting complex $Q^\bullet$ for B and WA_4 [13, Thm. 5.2]. We now describe the tilting complex $Q^\bullet_k$, as given in [13, §3]. We identify A_4 with the normalizer of a Sylow 2-subgroup of A_5.

The group algebra kA_4 has three simple modules: the trivial simple module S_0, and two other one-dimensional modules S_1 and S_2. The Loewy structures of the projective indecomposable kA_4-modules are as follows:

$$P_0 = \begin{array}{ccc} & S_0 & \\ S_1 & & S_2 \\ & S_0 & \end{array}, \qquad P_1 = \begin{array}{ccc} & S_1 & \\ S_0 & & S_2 \\ & S_1 & \end{array}, \qquad P_2 = \begin{array}{ccc} & S_2 & \\ S_1 & & S_0 \\ & S_2 & \end{array}.$$

The principal block kB of kA_5 also has three simple modules: the trivial simple module T_0, and two two-dimensional modules T_1 and T_2. When considering T_1 and T_2 as kA_4-modules, then T_1 corresponds to the uniserial module $\begin{array}{c} S_1 \\ S_2 \end{array}$, and T_2 corresponds to the uniserial module $\begin{array}{c} S_2 \\ S_1 \end{array}$. The Loewy structures of the projective indecomposable kB-modules are as follows:

$$Q_0 = \begin{array}{ccc} & T_0 & \\ T_1 & & T_2 \\ T_0 & & T_0 \\ T_2 & & T_1 \\ & T_0 & \end{array}, \qquad Q_1 = \begin{array}{c} T_1 \\ T_0 \\ T_2 \\ T_0 \\ T_1 \end{array}, \qquad Q_2 = \begin{array}{c} T_2 \\ T_0 \\ T_1 \\ T_0 \\ T_2 \end{array}.$$

The split-endomorphism two-sided tilting complex $Q_k^\bullet$ has exactly two non-zero terms, one in degree -1 and one in degree 0:

$$Q_k^\bullet = \cdots \to 0 \to X^{-1} \xrightarrow{d^{-1}} X^0 \to 0 \to \cdots ,$$

where X^0 is kB, considered as a kB-kA_4-bimodule, X^{-1} is the projective cover, as a bimodule, of the augmentation ideal of kB, and the differential d^{-1} is the natural surjection onto the augmentation ideal. It follows from [13, Lemma 3.1] and its proof that, as kB-kA_4-bimodules,

$$\begin{aligned} \mathrm{top}(X^0) &\cong (T_0 \otimes_k S_0^*) \oplus (T_1 \otimes_k S_1^*) \oplus (T_2 \otimes_k S_2^*) , \\ X^{-1} &\cong (Q_1 \otimes_k P_1^*) \oplus (Q_2 \otimes_k P_2^*) , \end{aligned} \tag{4.1}$$

where $M^* = \mathrm{Hom}_k(M, k)$ and $\mathrm{top}(X^0)$ denotes the largest semisimple quotient module of X^0.

We now want to use Theorem 3.2 to obtain a connection between $R(A_4, V^\bullet)$ and $R(A_5, V'^\bullet)$ when $V'^\bullet = Q_k^\bullet \otimes_{kA_4} V^\bullet$, in case $V^\bullet$ is a one-term complex with a simple kA_4-module in degree 0. Since, as left kB-modules, $X^0 \cong Q_0 \oplus Q_1^2 \oplus Q_2^2$, it follows that as right kA_4-modules $X^0 \cong P_0^* \oplus P_1^{*5} \oplus P_2^{*5}$. Note that X^0 induces a stable equivalence between kB and kA_4. Thus it follows from [9, Thm. 3.4] (see also [16, Lemma 5]) that if V is a simple kA_4-module, then $X^0 \otimes_{kA_4} V$ is an indecomposable kB-module. Moreover, it follows e.g. from the proof of [16, Lemma 2] that, as kB-kA_4-bimodules,

$$\mathrm{top}(X^0) \;\cong\; \bigoplus_{i=0}^{2} \mathrm{top}(X^0 \otimes_{kA_4} S_i) \otimes_k S_i^*,$$

which implies by (4.1) that $\mathrm{top}(X^0 \otimes_{kA_4} S_i) \cong T_i$ for $i \in \{0, 1, 2\}$. Thus, considering possible indecomposable kB-modules with a given top and dimension, it follows that, as kB-modules,

$$X^0 \otimes_{kA_4} S_0 \cong T_0\,, \qquad X^0 \otimes_{kA_4} S_1 \cong \begin{matrix} T_1 \\ T_0 \\ T_2 \end{matrix}\,, \qquad X^0 \otimes_{kA_4} S_2 \cong \begin{matrix} T_2 \\ T_0 \\ T_1 \end{matrix}\,.$$

Moreover, as kB-modules,

$$X^{-1} \otimes_{kA_4} S_0 \cong 0\,, \qquad X^{-1} \otimes_{kA_4} S_1 \cong Q_1\,, \qquad X^{-1} \otimes_{kA_4} S_2 \cong Q_2\,.$$

Thus, we obtain that if $V_0^\bullet$ is the one-term complex S_0, then $V_0'^\bullet$ is the one-term complex T_0. On the other hand, if $V_1^\bullet$ is the one-term complex S_1, then $V_1'^\bullet$ is the complex

$$V_1'^\bullet = \cdots 0 \to 0 \to Q_1 \xrightarrow{\pi_1} \begin{matrix} T_1 \\ T_0 \\ T_2 \end{matrix} \to 0 \to \cdots\,,$$

where π_1 is the natural surjection map. Thus $V_1'^\bullet$ is isomorphic to the one-term complex $\begin{matrix} T_0 \\ T_1 \end{matrix}$, this time concentrated in degree (-1). Similarly, if $V_2^\bullet$ is the one-term complex S_2, then $V_2'^\bullet$ is isomorphic to the one-term complex $\begin{matrix} T_0 \\ T_2 \end{matrix}$, concentrated in degree (-1). We conclude from Theorem 3.2 that $R(A_4, S_0) \cong R(A_5, T_0)$, $R(A_4, S_1) \cong R(A_5, \begin{matrix} T_0 \\ T_1 \end{matrix})$ and $R(A_4, S_2) \cong R(A_5, \begin{matrix} T_0 \\ T_2 \end{matrix})$. This coincides with the results found in [1].

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