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GAUGEABILITY FOR FEYNMAN-KAC FUNCTIONALS WITH APPLICATIONS TO SYMMETRIC α-STABLE PROCESSES

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Abstract. For symmetric α-stable processes, an analytic criterion for a measure being gaugeable was obtained by Z.-Q. Chen (2002), M. Takeda (2002) and M. Takeda and T. Uemura (2004). Applying it, we consider the ultracontractivity of Feynman-Kac semigroups and expectations of the number of branches hitting closed sets in branching symmetric α-stable processes.

1. Introduction

Let $\mathbb{M}^\alpha = (\mathbb{P}_x, X_t)$ $(0 < \alpha < 2)$ be a symmetric α-stable process on $\mathbb{R}^d$. Let μ be a smooth measure and A_t^μ the positive continuous additive functional (PCAF) in the Revuz correspondence to μ. Then the measure μ is said to be *gaugeable* on an open set $D \subset \mathbb{R}^d$ if

$$\sup_{x\in D} \mathbb{E}_x\left[\exp(A^\mu_{\tau_D})\right] < \infty, \tag{1.1}$$

where τ_D is the first exit time from D.

For a Brownian motion, Zhao [19] introduced a class of Green-tight measures and Chen [3] generalized the notion of Green-tightness for more general transient Markov processes. Let $\mathbb{M}^D$ be the absorbing process killed upon leaving D and assume that $\mathbb{M}^D$ is transient. Denote by $\mathcal{S}_\infty^D$ the extended class associated with $\mathbb{M}^D$ (see Definition 2.1 below). Chen [3] and Takeda [16] established an analytic condition for $\mu \in \mathcal{S}_\infty^D$ being gaugeable; define

$$\lambda(\mu; D) = \inf\left\{\mathcal{E}^{(\alpha)}(u,u) :\ u \in C_0^\infty(D),\ \int_D u^2(x)\mu(dx) = 1\right\}.$$

Then the gaugeability of μ is equivalent to that of $\lambda(\mu; D) > 1$, which is also equivalent to the subcriticality of Schrödinger operators. Applying these facts, we showed in [17] the differentiability of spectral functions. The objective of this paper is to give two other applications.

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The first is relevant with the ultracontractivity of Schrödinger semigroups; let $p_t^\mu f(x) = \mathbb{E}_x[\exp(A_t^\mu) f(X_t)]$ and denote by $\|p_t^\mu\|_{1,\infty}$ the operator norm of p_t^μ from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$. Then we have

Theorem 1.1. *Suppose that $d > \alpha$. Let $\mu \in \mathcal{S}_\infty^{\mathbb{R}^d}$ with $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x-y|^{\alpha-d} d\mu(x)d\mu(y) < \infty$. Then*

$$\lambda(\mu; \mathbb{R}^d) > 1 \Longleftrightarrow \|p_t^\mu\|_{1,\infty} \leq \frac{c}{t^{d/\alpha}}, \quad t > 0. \tag{1.2}$$

Theorem B.1.1 in [13] says that if 2μ is gaugeable on $\mathbb{R}^d$, the ultracontractivity in the right-hand side of (1.2) holds. In the proof, he used the Schwartz inequality in the Feynman-Kac formula and the duality arguments, which is the reason why the gaugeability of 2μ is required. Our argument is different; a class of Girsanov transforms treated in [5] plays a crucial role.

In the second application, motivated by [10], we consider a branching symmetric α-stable process; let $\mathbb{B}^\alpha = (\bar{X}_t, \bar{\mathbb{P}}_x)$ be the branching α-symmetric stable process with the branching rate k, a smooth measure of $\mathbb{M}^{(\alpha)}$, and the branching mechanism $\{p_n(x)\}_{n\geq 2}$, $\sum_2^\infty p_n(x) = 1$. Set $Q(x) = \sum_{n\geq 2} np_n(x)$, $\mu(dx) = (Q(x)-1)k(dx)$. Denote by Cap the 0-capacity defined by the Dirichlet form generated by $\mathbb{M}^{(\alpha)}$. Then we have

Theorem 1.2. *Assume that $\sup_{x\in\mathbb{R}^d} Q(x) < \infty$. Then for a closed set K with $\mathrm{Cap}(K) > 0$ and $\mu \in \mathcal{S}_\infty^{\mathbb{R}^d\setminus K}$,*

$$\lambda(\mu; \mathbb{R}^d \setminus K) > 1 \Longleftrightarrow \bar{\mathbb{E}}_x[N_K] < \infty. \tag{1.3}$$

Here N_K is the number of branches of $\mathbb{B}^\alpha$ ever hitting K.

Theorem 1.2 is an extension of [10, Theorem 4.1(iv)] to branching processes with jumps. For the proof of Theorem 1.2, we show that for $\mu \in \mathcal{S}_\infty^D$ with $\mathrm{Cap}(\mathbb{R}^d\setminus D) > 0$

$$\sup_{x\in D} \mathbb{E}_x[\exp(A_{\tau_D}^\mu)] < \infty \Longleftrightarrow \sup_{x\in D} \mathbb{E}_x[\exp(A_{\tau_D}^\mu); \tau_D < \infty] < \infty. \tag{1.4}$$

In [6], the gaugeability of μ on D is defined by the right-hand side of (1.4). The equation (1.4) tells us that for a measure in $\mathcal{S}_\infty^D$, two definitions of the gaugeability are equivalent.

2. NOTATIONS AND SOME FACTS

Let $\mathbb{M} = (\Omega, \mathcal{M}, \mathcal{M}_t, \theta_t, \mathbb{P}_x, X_t)$ be a symmetric α-stable process on $\mathbb{R}^d$. Here $\{\mathcal{M}_t\}_{t\geq 0}$ is the minimal (augmented) admissible filtration and θ_t, $t \geq 0$, are the shift operators satisfying $X_s(\theta_t) = X_{s+t}$ identically for $s, t \geq 0$. The Dirichlet form generated by $\mathbb{M}^\alpha$ is given by

$$\begin{aligned} \mathcal{E}^{(\alpha)}(u,v) &= K \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}} dxdy, \\ \mathcal{D}(\mathcal{E}^{(\alpha)}) &= \left\{ u \in L^2(\mathbb{R}^d) : \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus d} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} dxdy < \infty \right\} \end{aligned}$$

($K = \alpha 2^{\alpha-3}\pi^{-\frac{d+2}{2}} \sin(\frac{\alpha\pi}{2})\Gamma(\frac{d+\alpha}{2})\Gamma(\frac{\alpha}{2})$). Every function u in $\mathcal{D}(\mathcal{E}^{(\alpha)})$ admits a quasi-continuous version $\tilde{u}$ ([9, Theorem 2.1.3]). In the sequel we always assume that every function $u \in \mathcal{D}(\mathcal{E}^{(\alpha)})$ is represented by its quasi-continuous version.

Let $D \subset \mathbb{R}^d$ be an open set. Assume that the absorbing process $\mathbb{M}^D$ on D is transient and let $G_D(x,y)$ be the Green function of $\mathbb{M}^D$. Note that $\mathbb{M}^D$ is irreducible due to the strictly positivity of the Levy measure of $\mathbb{M}$. Following [3], we make the following definition.

Definition 2.1. A positive Radon measure μ on $\mathbb{R}^d$ is said to be in the class $\mu \in \mathcal{S}_\infty^D$, if for any $\epsilon > 0$ there exists a compact set $K \subset D$ and $\delta > 0$ such that

$$\sup_{(x,z)\in D\times D\setminus d} \int_{K^c} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)}\mu(dy) \le \epsilon, \tag{2.1}$$

and for all measurable sets $B \subset K$ with $\mu(B) < \delta$,

$$\sup_{(x,z)\in D\times D\setminus d} \int_{B} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)}\mu(dy) \le \epsilon. \tag{2.2}$$

When $D = \mathbb{R}^d$, we remove D in the notations; for example, we simply denote $\mathcal{S}_\infty$ for $\mathcal{S}_\infty^{\mathbb{R}^d}$. By the *3G-inequality*, a measure μ is in $\mathcal{S}_\infty$ if and only if it is Green-tight in the sense of Zhao [19] (cf. [3]). For $\mu \in \mathcal{S}_\infty^D$, let A_t^μ be the positive continuous additive functional of $\mathbb{M}$ in the Revus correspondence to the measures μ. It is known in [3, Proposition 2.2] that for $\mu \in \mathcal{S}_\infty^D$

$$\sup_{x\in D} \mathbb{E}_x[A^\mu_{\tau_D}] = \sup_{x\in D}\int_D G_D(x,y)\mu(dy) < \infty. \tag{2.3}$$

For a measure μ in $\mathcal{S}_\infty^D$, define

$$\lambda(\mu;D) = \inf\left\{\mathcal{E}^{(\alpha)}(u,u):\ u\in C_0^\infty(D),\ \int_D u^2(x)\mu(dx) = 1\right\}. \tag{2.4}$$

By Lemma 3.1 in [15] and Theorem 4.1 below, we see that $\lambda(\mu;D)$ is the principal eigenvalue of the time changed process of $\mathbb{M}^D$ by $A^\mu_{\tau_D\wedge t}$. We abbreviate $\lambda(\mu;\mathbb{R}^d)$ as $\lambda(\mu)$.

Let $p_t^{\mu,D}(x,y)$ be the integral kernel of the Feynman-Kac semigroup,

$$\mathbb{E}_x[\exp(A_t^\mu)f(X_t); t<\tau_D] = \int_D p_t^{\mu,D}(x,y)f(y)dy,$$

and $G^{\mu,D}(x,y)$ its Green function, $G^{\mu,D}(x,y) = \int_0^\infty p_t^{\mu,D}(x,y)dt$. We then have

Theorem 2.2 ([3], [16])**.** *Let $\mu \in \mathcal{S}_\infty^D$. Then the following conditions are equivalent:*

(i) $\sup_{x\in D} \mathbb{E}_x[e^{A^\mu_{\tau_D}}] < \infty$;

(ii) $G^{\mu,D}(x,y) < \infty$ *for* $x,y\in D$, $x\ne y$;

(iii) $\lambda(\mu;D) > 1$.

Remark 2.3. The equivalence between (i) and (iii) holds for any $\mu \in \mathcal{S}_\infty$ (see [16, Theorem 2.4]).

3. Proof of Theorem 1.1

Proof of Theorem 1.1*, the* ($\Longleftarrow$) *part.* Let $p(t,x,y)$ be the transition probability density. Then it satisfies the upper estimate,

$$p(t,x,y) \le C\frac{1}{t^{d/\alpha}}\left(\frac{1}{1+\frac{|x-y|}{t^{1/\alpha}}}\right)^{d+\alpha},$$

where C is a positive constant (see [2]). Thus by the same argument as in [1, Theorem 8.1], the integral kernel $p^{\mu}(t,x,y)$ of P_t^{μ} satisfies, for any $\lambda > 1$,

$$\begin{aligned} p^{\mu}(t,x,y) &\leq C' \frac{1}{t^{d/\alpha}} \left(\frac{1}{1+\frac{|x-y|}{t^{1/\alpha}}} \right)^{\frac{d+\alpha}{\lambda}} \\ &\leq C' \frac{t^{(\frac{d}{\alpha}+1)\frac{1}{\lambda}-\frac{d}{\alpha}}}{|x-y|^{\frac{d+\alpha}{\lambda}}}, \quad 0 < t \leq 1, \end{aligned}$$

where C' is a positive constant depending on λ. In particlar, for any x, y with $x \neq y$, $\sup_{0<t\leq 1} p^{\mu}(t,x,y) < \infty$. Moreover, the assumption implies that $p^{\mu}(t,x,y) \leq \frac{c}{t^{d/\alpha}}$. Therefore we have

$$G^{\mu}(x,y) \leq \int_0^1 p^{\mu}(t,x,y)dt + c\int_1^{\infty} \frac{1}{t^{d/\alpha}}dt < \infty.$$

Now it follows from Theorem 2.2 that $\lambda(\mu) > 1$. □

To prove the converse, we need some results. Let $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$ be the *extended Dirichlet space*, that is, the family of measurable function u on $\mathbb{R}^d$ such that $|u| < \infty$ a.e. and there exists an $\mathcal{E}^{(\alpha)}$-Cauchy sequence $\{u_n\}$ of functions in $\mathcal{D}(\mathcal{E}^{(\alpha)})$ such that $\lim_{n\to\infty} u_n = u$ a.e. (see [9]). We denote $G\mu(x) = \int_{\mathbb{R}^d} G(x,y)d\mu(y)$.

Lemma 3.1. *If $\mu \in \mathcal{S}_{\infty}$ satisfies $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x,y)d\mu(x)d\mu(y) < \infty$, then $G\mu$ belongs to $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$.*

Proof. First note that for $\mu \in \mathcal{S}_{\infty}$

$$\int_{\mathbb{R}^d} u^2 d\mu \leq \|G\mu\|_{\infty}\mathcal{E}^{(\alpha)}(u,u), \quad u \in \mathcal{D}_e(\mathcal{E}^{(\alpha)}) \tag{3.1}$$

(cf. [14]). Then by applying (3.1) to $\mu_K(\cdot) = \mu(K \cap \cdot)$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi d\mu_K &\leq (\mu(K))^{1/2} \left(\int_{\mathbb{R}^d} \varphi^2 d\mu_K \right)^{1/2} \\ &\leq (\mu(K))^{1/2} \|G\mu_K\|_{\infty}^{1/2} \mathcal{E}^{(\alpha)}(\varphi,\varphi)^{1/2}. \end{aligned}$$

Hence equation (2.3) says that the measure μ_K is of finite energy integral, and thus

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi d\mu_K &\leq \mathcal{E}^{(\alpha)}(G\mu_K, G\mu_K)^{1/2} \mathcal{E}^{(\alpha)}(\varphi,\varphi)^{1/2} \\ &\leq \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x,y)d\mu_K(x)d\mu_K(y) \right)^{1/2} \mathcal{E}^{(\alpha)}(\varphi,\varphi)^{1/2}. \end{aligned}$$

By letting K increase to $\mathbb{R}^d$, we find that μ is of finite energy integral, and thus $G\mu$ is in $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$. □

Assume that $\lambda(\mu) > 1$ and set

$$h(x) = \mathbb{E}_x[e^{A_{\infty}^{\mu}}].$$

Then by Theorem 2.2, $1 \leq h(x) \leq \sup_x \mathbb{E}_x[e^{A_{\infty}^{\mu}}] < \infty$.

Lemma 3.2. *Assume that $\lambda(\mu) > 1$. Then it holds that*

$$h(x) = G(h\mu)(x) + 1.$$

Proof. Define $M_t := \mathbb{E}[\exp(A^\mu_\infty)|\mathcal{M}_t]$. Then by the Markov property

$$\begin{aligned} h(X_t) &= \mathbb{E}_{X_t}[\exp(A^\mu_\infty)] = \mathbb{E}_x[\exp(A^\mu_\infty(\theta_t))|\mathcal{M}_t] \\ &= \mathbb{E}_x[\exp(A^\mu_\infty - A^\mu_t)|\mathcal{M}_t] = \exp(-A^\mu_t)M_t, \end{aligned}$$

and thus

$$\begin{aligned} \mathbb{E}_x\left[\int_0^t h(X_s)dA^\mu_s\right] &= \mathbb{E}_x\left[\int_0^t \exp(-A^\mu_s)M_s dA^\mu_s\right] \\ &= \mathbb{E}_x[M_0] - \mathbb{E}_x[\exp(-A^\mu_t)M_t] - \mathbb{E}_x\left[\int_0^t \exp(-A^\mu_s)dM_s\right] \\ &= h(x) - \mathbb{E}_x[h(X_t)]. \end{aligned} \tag{3.2}$$

Noting that

$$\lim_{t\to\infty} h(X_t) = \lim_{t\to\infty} \exp(-A^\mu_t)M_t = \exp(-A^\mu_\infty)\exp(A^\mu_\infty) = 1,$$

we have the lemma by letting t to ∞ in (3.2). □

Suppose that $\mu \in \mathcal{S}_\infty$ satisfies $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x,y)d\mu(x)d\mu(y) < \infty$. Then it follows from Lemma 3.1 that $G(h\mu) \in \mathcal{D}_e(\mathcal{E}^{(\alpha)})$. Thus we have the Fukushima decomposition for $G(h\mu)$ ([9, Theorem 5.2.2]):

$$G(h\mu)(X_t) - G(h\mu)(X_0) = M^{[G(h\mu)]}_t + N^{[G(h\mu)]}_t.$$

Since the left-hand side of the above equals $h(X_t) - h(X_0)$ by Lemma 3.3, we may denote $M^{[h]}_t = M^{[G(h\mu)]}_t$. Furthermore, it follows from [9, Lemma 5.4.1] that $N^{[G(h\mu)]}_t = -\int_0^t h(X_s)dA^\mu_s$. Therefore we have

$$h(X_t) - h(X_0) = M^{[h]}_t - \int_0^t h(X_s)dA^\mu_s. \tag{3.3}$$

Define a martingale by

$$M_t = \int_0^t \frac{1}{h(X_{s-})}dM^h_s$$

and denote by L_t the unique solution of the Doleans-Dade equation:

$$Z_t = 1 + \int_0^t Z_{s-}dM_s. \tag{3.4}$$

Then we see from the Doleans-Dade formula that L_t is expressed by

$$\begin{aligned} L_t &= \exp\left(M_t - \frac{1}{2}\langle M^c\rangle_t\right)\prod_{0<s\le t}(1+\Delta M_s)\exp(-\Delta M_s) \\ &= \exp\left(M_t - \frac{1}{2}\langle M^c\rangle_t\right)\prod_{0<s\le t}\frac{h(X_s)}{h(X_{s-})}\exp\left(1 - \frac{h(X_s)}{h(X_{s-})}\right). \end{aligned}$$

Here M^c_t is the continuous part of M_t and $\Delta M_s = M_s - M_{s-}$. By Itô's formula applied to the semi-martingale $h(X_t)$ with the function $\log x$, we see that L_t has the following expression:

$$L_t = \frac{h(X_t)}{h(X_0)}\exp(A^\mu_t). \tag{3.5}$$

Denote by $\mathbb{M}^L = (\Omega, \mathbb{P}^L_x, X_t)$ the transformed process of $\mathbb{M}^\alpha$ by L_t, $d\mathbb{P}^L_x(\omega) = L_t(\omega)d\mathbb{P}_x(\omega)$. Lemma 3.1 says that $u := \log(G(h\mu)+1) \in \mathcal{D}_e(\mathcal{E}^{(\alpha)})$ and $h = \exp(u)$.

Thus the transform by L_t belongs to the class of Girsanov transforms considered in [5]. In particular, the Dirichlet form generated by $\mathbb{M}^L$ is identified as follows.

Theorem 3.3 ([5]). *The transformed process* $\mathbb{M}^L$ *is an* h^2dx*-symmetric Hunt process and its Dirichlet form* $(\mathcal{E}^h, \mathcal{F}^h)$ *on* $L^2(\mathbb{R}^d, h^2dx)$ *is written as*

$$\begin{cases} \mathcal{E}^h(u,v) = K \displaystyle\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus d} \frac{(u(x)-u(y))(v(x)-v(y))h(x)h(y)}{|x-y|^{d+\alpha}}dxdy, \\ \mathcal{D}(\mathcal{E}^h) = \mathcal{D}(\mathcal{E}^{(\alpha)}). \end{cases}$$

Proof of Theorem 1.1, *the* ($\Longrightarrow$) *part.* Noting that $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ is equivalent to $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$, we have the Sobolev inequality

$$\left(\int_{\mathbb{R}^d} |f|^{\frac{2d}{d-\alpha}} h^2 dx\right)^{\frac{d-\alpha}{2d}} \le c\sqrt{\mathcal{E}^h(f,f)}, \quad f \in \mathcal{F}^h,$$

and so we have by [7, Theorem 2.4.2] that

$$\|p_t^h f\|_\infty \le \frac{c}{t^{\frac{d}{\alpha}}} \int_{\mathbb{R}^d} |f| h^2 dx, \quad t > 0.$$

Here p_t^h is the semigroup of $\mathbb{M}^L$. We see from (3.5) that

$$p_t^h f(x) = \frac{1}{h(x)} p_t^\mu (hf)(x).$$

Hence

$$\begin{aligned} \|p_t^\mu f\|_\infty &= \|h p_t^h \left(\frac{|f|}{h}\right)\|_\infty \le \frac{c\|h\|_\infty}{t^{\frac{d}{\alpha}}} \int_{\mathbb{R}^d} \frac{|f|}{h} h^2 dx \\ &\le \frac{c\|h\|_\infty^2}{t^{\frac{d}{\alpha}}} \int_{\mathbb{R}^d} |f| dx, \end{aligned}$$

which implies

$$\|p_t^\mu\|_{1,\infty} \le \frac{c'}{t^{\frac{d}{\alpha}}}, \quad t > 0.$$

□

Example 3.4. Let σ_r be the surface measure of the sphere $\partial B_r = \{|x| = r\}$. Since the symmetric α-stable process hits the sphere ∂B_r if $1 < \alpha$, the measure σ_r is in $\mathcal{S}_\infty$. Combining Theorem 1.1 with [18, Example 4.1], we find that

$$\begin{aligned} \|p_t^{\sigma_r}\|_{1,\infty} \le \frac{c}{t^{\frac{d}{\alpha}}}, \quad t > 0 \quad &\Longleftrightarrow \quad \lambda(\sigma_r) > 1 \\ &\Longleftrightarrow \quad \left\{ \frac{\sqrt{\pi}\,\Gamma\left(\frac{d+\alpha}{2} - 1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha-1}{2}\right)\Gamma\left(\frac{d-\alpha}{2}\right)} \right\}^{\frac{1}{\alpha-1}} > r. \end{aligned}$$

Here $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$.

4. Proof of Theorem 1.2

Let K be a closed set with $\mathrm{Cap}(K)>0$ and denote $D=\mathbb{R}^d\setminus K$. Let $(\mathcal{E}_D^{(\alpha)}, \mathcal{D}(\mathcal{E}_D^{(\alpha)}))$ be the Dirichlet form generated by the absorbing process of $\mathbb{M}^{(\alpha)}$ on D (cf. Section 4.4 in [9]). Let $\mathcal{D}_e(\mathcal{E}_D^{(\alpha)})$ be the extended Dirichlet space of $(\mathcal{E}_D^{(\alpha)}, \mathcal{D}(\mathcal{E}_D^{(\alpha)}))$. Because of the remark just below Definition 3.1 in [3] and Proposition 2.3 in [3], we have the next theorem in the same manner as in [17, Theorem 3.3].

Theorem 4.1. *Let $\mu \in \mathcal{S}_\infty^D$. Then the embedding of $(\mathcal{E}_D^{(\alpha)}, \mathcal{D}_e(\mathcal{E}_D^{(\alpha)}))$ to $L^2(D;\mu)$ is compact.*

Moreover, by the same argument as in [17, Section 4], we have

Lemma 4.2. *If $\lambda(\mu;D)=1$, then there exists a positive continuous bounded function u_0 on D such that*

$$u_0(x)=\mathbb{E}_x\left[\int_0^{\tau_D} u_0(X_t)\,dA_t^\mu\right] \tag{4.1}$$

and so equation (4.1) says that the function u_0 is $p_t^{\mu,D}$-excessive, $p_t^{\mu,D}u_0(x)\le u_0(x)$ for every $t>0$ and $x\in D$.

Set

$$h_D(x)=\mathbb{E}_x[\exp(A_{\tau_D}^\mu);\tau_D<\infty].$$

Lemma 4.3. *If $\sup_{x\in D}h_D(x)<\infty$, then h_D is a $p_t^{\mu,D}$-excessive function. Moreover, it satisfies*

$$h_D(x)=\mathbb{P}_x[\tau_D<\infty]+\mathbb{E}_x\left[\int_0^{\tau_D} h_D(X_t)\,dA_t^\mu\right]. \tag{4.2}$$

Proof. The $p_t^{\mu,D}$-excessiveness follows from the Markov property

$$\begin{aligned} p_t^{\mu,D}h_D(x) &= \mathbb{E}_x[\exp(A_t^\mu)\mathbb{E}_{X_t}[\exp(A_{\tau_D}^\mu);\tau_D<\infty];t<\tau_D] \\ &= \mathbb{E}_x[\mathbb{E}_x[\exp(A_t^\mu+A_{\tau_D}^\mu(\theta_t));\tau_D(\theta_t)<\infty, t<\tau_D|\mathcal{M}_t] \\ &= \mathbb{E}_x[\exp(A_{\tau_D}^\mu);t<\tau_D<\infty]\le h_D(x). \end{aligned}$$

For the proof of (4.2), note that

$$\begin{aligned} \mathbb{E}_x\left[\int_0^{\tau_D} h_D(X_t)\,dA_t^\mu\right] &= \mathbb{E}_x\left[\int_0^{\tau_D} \mathbb{E}_{X_t}[\exp(A_{\tau_D}^\mu);\tau_D<\infty]\,dA_t^\mu\right] \\ &= \mathbb{E}_x\left[\int_0^{\tau_D} \mathbb{E}_x[\exp(A_{\tau_D}^\mu(\theta_t));\tau_D(\theta_t)<\infty|\mathcal{M}_t]\,dA_t^\mu\right] \\ &= \mathbb{E}_x\left[\int_0^{\tau_D} \exp(-A_t^\mu)\mathbb{E}_x[\exp(A_{\tau_D}^\mu);t<\tau_D<\infty|\mathcal{M}_t]\,dA_t^\mu\right]. \end{aligned} \tag{4.3}$$

Set

$$\begin{aligned} X_t &= \exp(-A_t^\mu)\exp(A_{\tau_D}^\mu)I_{\{t<\tau_D<\infty\}}, \\ Y_t &= \exp(-A_t^\mu)\mathbb{E}_x[\exp(A_{\tau_D}^\mu)|\mathcal{M}_t]I_{\{t<\tau_D<\infty\}}. \end{aligned}$$

Then applying Exercise (1.13) of Chap. V in [11], we have

$$\mathbb{E}_x\left[\int_0^\infty X_t\,dA_t^\mu\right]=\mathbb{E}_x\left[\int_0^\infty Y_t\,dA_t^\mu\right].$$

Namely, the right-hand side of (4.3) equals

$$\begin{aligned} &\mathbb{E}_x\left[\exp(A^\mu_{\tau_D})\int_0^{\tau_D}\exp(-A^\mu_t)dA^\mu_t\right] \\ = \quad &\mathbb{E}_x\left[\exp(A^\mu_{\tau_D});\tau_D<\infty\right]-\mathbb{P}_x\left[\tau_D<\infty\right], \end{aligned}$$

which implies (4.2). □

Lemma 4.4. *If* $\sup_{x\in D}h_D(x)<\infty$, *then* $\lambda(\mu;D)\geq 1$.

Proof. Suppose that $\lambda(\mu;D)<1$. Then there is $0<\theta<1$ such that $\lambda(\theta\mu;D)=1$. It follows from Lemma 4.3 that h-tranformation $p_t(x,y):=\frac{1}{h_D(x)}p^{\mu,D}_t(x,y)h_D(y)$ is a transition probability density. Denote by $\mathbb{M}$ the Markov processes generated by $p_t(x,y)$. The kernel $p^\theta_t(x,y):=\frac{1}{h_D(x)}p^{\theta\mu,D}_t(x,y)h_D(y)$ is also a transition probability density. In fact, $p^\theta_t(x,y)$ is the transition density of the subprocess of $\mathbb{M}$ by the multiplicative functional $\exp(-(1-\theta)A^\mu_t)$;

$$\frac{1}{h_D(x)}p^{\theta\mu,D}_t(h_Df)(x)=\frac{1}{h_D(x)}\mathbb{E}_x[\exp(-(1-\theta)A^\mu_t)\exp(A^\mu_t)h_D(X_t)f(X_t);t<\tau_D].$$

Hence $p^\theta_t(x,y)$ defines a transient Markov process, which is contradictory to Theorem 2.2. □

Lemma 4.5. *If* $\sup_{x\in D}h_D(x)<\infty$, *then* $\lambda(\mu;D)>1$.

Proof. Suppose that $\lambda(\mu;D)=1$. Let u_0 be the function in Lemma 4.2. Then Theorem 2.2 says that $\frac{1}{u_0(x)}p^{\mu,D}_t(x,y)u_0(y)$ is a transition probability density generating a recurrent Markov process. Moreover, since

$$\frac{1}{u_0(x)}\mathbb{E}_x\left[\exp(A^\mu_t)u_0(X_t)\left(\frac{h_D}{u_0}\right)(X_t);t<\tau_D\right]\leq\frac{h_D}{u_0}(x),$$

that is, h_D/u_0 is excessive with respect to $\frac{1}{u_0(x)}p^{\mu,D}_t(x,y)u_0(y)$, there exists a positive constant c such that $h_D=cu_0$. However, this is impossible by Lemma 4.2 and Lemma 4.3 because $\mathbb{P}_x[\tau_D<\infty]>0$ for all $x\in D$. Therefore, Lemma 4.4 leads us to this lemma. □

By combining Lemma 4.5 and Theorem 2.2, we have

Theorem 4.6. *Let* $\mu\in\mathcal{S}^D_\infty$. *Then*

$$\begin{aligned} \lambda(\mu;D)>1 \quad &\Longleftrightarrow \quad \sup_{x\in D}\mathbb{E}_x[\exp(A^\mu_{\tau_D})]<\infty \\ &\Longleftrightarrow \quad \sup_{x\in D}\mathbb{E}_x[\exp(A^\mu_{\tau_D});\tau_D<\infty]<\infty. \end{aligned}$$

Proof of Theorem 1.2. Let $\bar{X}^D_t$ be the process that stops when $\bar{X}_t$ reaches K. Then, N_K is the number of the offsprings of $\bar{X}^D_t$ that reach K. Let N_t be the number of the offspring of $\bar{X}^D_t$ at t and N^D_t the number of the offspring of $\bar{X}^D_t$ that stay in D at t. Then by the strong Markov property, $m(t,x):=\bar{\mathbb{E}}_x[N_t]$ and $m^D(t,x):=\bar{\mathbb{E}}_x[N^D_t]$ satisfy

$$m(t,x)=\mathbb{E}_x[\int_0^{\sigma_K\wedge t}e^{-A^k_s}m(t-s,X_s)Q(X_s)dA^k_s]+\mathbb{E}_x[e^{-A^k_{\sigma_K\wedge t}}]$$

and

$$m^D(t,x) = \mathbb{E}_x\Big[\int_0^{\sigma_K\wedge t} e^{-A^k_s} m^D(t-s,X_s)Q(X_s)dA^k_s\Big] + \mathbb{E}_x[e^{-A^k_t}; t<\sigma_K],$$

respectively. Moreover, functions

$$v(t,x) := \mathbb{E}_x[\exp(A^\mu_{\sigma_K\wedge t})], \quad v^D(t,x) := \mathbb{E}_x[\exp(A^\mu_t); t<\sigma_K]$$

also satisfy the equations above, respectively. Since the bounded solution of each equation above is unique by [8, Lemma 4.4.2, Lemma 4.3.1], we see that

$$\bar{\mathbb{E}}_x[N_t] = \mathbb{E}_x[\exp(A^\mu_{\sigma_K\wedge t})], \quad \bar{\mathbb{E}}_x[N^D_t] = \mathbb{E}_x[\exp(A^\mu_t); t<\sigma_K]$$

and

$$\begin{aligned}\bar{\mathbb{E}}_x[N_K] &= \lim_{t\to\infty}\left(\bar{\mathbb{E}}_x[N_t] - \bar{\mathbb{E}}_x[N^D_t]\right)\\ &= \mathbb{E}_x[\exp(A^\mu_{\sigma_K}); \sigma_K<\infty].\end{aligned}$$

Now Theorem 4.6 leads us to Theorem 1.2. □

Lemma 4.7. *Let K be a closed set with* $\mathrm{Cap}(K)>0$. *Then for $\mu\in\mathcal{S}^D_\infty$, $\lambda(\mu;D)>\lambda(\mu)$.*

Proof. Let F be the topological support of μ and denote by $(\check{\mathcal{E}}^{(\alpha)}, \mathcal{D}(\check{\mathcal{E}}^{(\alpha)}))$ the Dirichlet form on $L^2(F,\mu)$ generated by the time changed process by A^μ_t. By (3.1) the restriction of $u\in\mathcal{D}_e(\mathcal{E}^{(\alpha)})$ to F, $u|_F$, belongs to $\mathcal{D}(\check{\mathcal{E}}^{(\alpha)})$ ([9, Lemma 6.2.2]), and

$$\mathcal{E}^{(\alpha)}(u,u) = \check{\mathcal{E}}^{(\alpha)}(u|_F, u|_F).$$

As a result of Theorem 4.1, the embedding of $(\mathcal{E}^{(\alpha)}_D, \mathcal{D}_e(\mathcal{E}^{(\alpha)}_D))$ to $L^2(D;\mu)$ is compact. Hence there exists the function which attains the infimum in the definition of $\lambda(\mu;D)$. Consequently, $\lambda(\mu;D)\geq\lambda(\mu)$. Indeed, suppose that $\lambda(\mu;D)=\lambda(\mu)$ and let u_0 and u_D be functions attaining $\lambda(\mu)$ and $\lambda(\mu;D)$, respectively. Then $u_0|_F = u_D|_F$ μ-a.e. because $u_0|_F$ and $u_D|_F$ are normalized principal eigenfunctions of the time changed process by A^μ_t and it is irreducible.

Since for any $\varphi\in\mathcal{D}_e(\mathcal{E}^{(\alpha)})$

$$\mathcal{E}^{(\alpha)}(u_0,\varphi) = \int_{\mathbb{R}^d} u_0\varphi d\mu, \quad \mathcal{E}^{(\alpha)}(u_D,\varphi) = \int_{\mathbb{R}^d} u_D\varphi d\mu,$$

we have $\mathcal{E}^{(\alpha)}(u_0-u_D,\varphi)=0$ for any $\varphi\in\mathcal{D}_e(\mathcal{E}^{(\alpha)})$, Therefore $u_0=u_D$, which contradicts that $\mathrm{Cap}(K)>0$. □

Corollary 4.8. *Let K be a closed set with* $\mathrm{Cap}(K)>0$. *If $\lambda(\mu)\geq 1$ and $\mu\in\mathcal{S}^D_\infty$, then $\bar{\mathbb{E}}_x[N_K]<\infty$.*

Example 4.9. Suppose that $d=1$ and $1<\alpha<2$. Let $k=\delta_a$, $a\neq 0$, the Dirac measure at a and $p_2(x)=1$. Then it is known in [12] that

$$\lambda(\delta_a;\mathbb{R}\setminus\{0\}) = -\frac{\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}{2|a|^{\alpha-1}}.$$

Hence

$$\sup_{x\in\mathbb{R}\setminus\{0\}}\bar{\mathbb{E}}[N_{\{0\}}]<\infty \Longleftrightarrow 0<|a|<\left(-\frac{\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}{2}\right)^{1/(\alpha-1)}.$$

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