PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 9, September 2006, Pages 2533–2540
S 0002-9939(06)08286-4
Article electronically published on March 23, 2006

ON p-ADIC HERMITIAN EISENSTEIN SERIES

SHOYU NAGAOKA

(Communicated by Wen-Ching Winnie Li)

Dedicated to Professor Yasuo Morita

ABSTRACT. In this paper we generalize the notion of p-adic modular form to the Hermitian modular case and prove a formula that shows a coincidence between certain p-adic Hermitian Eisenstein series and the genus theta series associated with Hermitian matrix with determinant p.

1. INTRODUCTION

In a previous study [8], the present author generalized the notion of p-adic Eisenstein series to the Siegel modular case and found a "curious formula" ([8], (8.6)) that shows a coincidence between p-adic Siegel-Eisenstein series and theta series. Strictly speaking, certain p-adic Siegel-Eisenstein series were shown to coincide with the genus theta series associated with a binary quadratic form of discriminant $-p$.

In another study [5], a more general type of p-adic Siegel-Eisenstein series was investigated. The results of both of these studies indicate that these kinds of p-adic Siegel-Eisenstein series become "true" modular forms, namely, ordinary modular forms of $\Gamma_0(p)$-type.

In this note, we shall extend these findings in a different direction. Namely, we show that a similar phenomenon also occurs in the Hermitian modular case.

2. DEFINITION AND NOTATION

2.1. **Hermitian modular forms.** Let R be a subring of $\mathbb{C}$. We shall denote by $Her_m(R)$ the space of m by m Hermitian matrices over R with respect to the complex conjugate. Let $Her_m^+(R)$ be the subset of $Her_m(R)$ consisting of positive elements. The space

$$\mathbb{H}_n := \left\{ Z \in M_n(\mathbb{C}) \mid \frac{1}{\sqrt{-1}}(Z - {}^t\overline{Z}) \in Her_n^+(\mathbb{C}) \right\}$$

is called the *Hermitian upper-half space* of degree n, where ${}^t\overline{Z}$ denotes the complex conjugate, transpose matrix of Z. The group

$$G_n = U(n,n) := \left\{ M \in M_{2n}(\mathbb{C}) \mid {}^t\overline{M} J_n M = J_n,\ J_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right\}$$

acts on $\mathbb{H}_n$ by the ordinary generalized linear fractional transformation.

Received by the editors December 17, 2004 and, in revised form, April 7, 2005.
2000 *Mathematics Subject Classification.* Primary 11F33; Secondary 11F55.
Key words and phrases. p-adic modular forms, Eisenstein series.

Let $\mathbb{K}$ be an imaginary quadratic field of discriminant $d_{\mathbb{K}}$ with the ring of integers $\mathcal{O}_{\mathbb{K}}$. The *Hermitian modular group* over $\mathbb{K}$ is defined by $\Gamma_n(\mathbb{K}) := G_n \cap M_{2n}(\mathcal{O}_{\mathbb{K}})$. We denote by $M_k(\Gamma_n(\mathbb{K}))$ the complex vector space consisting of Hermitian modular forms of weight k for $\Gamma_n(\mathbb{K})$.

Let $f(Z)$ be a Hermitian modular form in $M_k(\Gamma_n(\mathbb{K}))$. Then, $f(Z)$ has a Fourier expansion of the form

$$f(Z) = \sum_{0 \le H \in \Lambda_n(\mathbb{K})} a_f(H) \exp[2\pi\sqrt{-1}\mathrm{tr}(HZ)], \quad Z \in \mathbb{H}_n,$$

where the index set $\Lambda_n(\mathbb{K})$ is defined by

$$\Lambda_n(\mathbb{K}) := \{\ H = (h_{ij}) \in Her_n(\mathbb{K}) \mid h_{ii} \in \mathbb{Z},\ \sqrt{d_{\mathbb{K}}}h_{ij} \in \mathcal{O}_{\mathbb{K}}\ (i \ne j)\ \}.$$

2.2. **Hermitian Eisenstein series.** Define

$$\Gamma_n(\mathbb{K})_\infty := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n(\mathbb{K}) \;\middle|\; C = O_n \right\}.$$

For an integer k such that $k > 2n$ and $w_{\mathbb{K}} | k$ ($w_{\mathbb{K}}$ is the order of the unit group of $\mathbb{K}$), define a series given by

$$G_k^{(n)}(Z) := \sum_{\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_n(\mathbb{K})_\infty \backslash \Gamma_n(\mathbb{K})} \det(CZ + D)^{-k}, \quad Z \in \mathbb{H}_n.$$

This series belongs to $M_k(\Gamma_n(\mathbb{K}))$ and is called the *Hermitian Eisenstein series* of weight k for $\Gamma_n(\mathbb{K})$. It was shown by Braun [3] that all of the Fourier coefficients of $G_k^{(n)}$ are rational.

2.3. **Genus theta series.** Let $S \in Her_m^+(\mathcal{O}_{\mathbb{K}})$. Define

$$\Theta^{(n)}(S; Z) := \sum_{X \in M_{m,n}(\mathcal{O}_{\mathbb{K}})} \exp[\pi\sqrt{-1}\mathrm{tr}(S[X]Z)], \quad Z \in \mathbb{H}_n,$$

where $S[X] := {}^t\overline{X}SX$. Let $\{S_1, \dots, S_h\}$ be a set of representatives of unimodular equivalence classes of the genus containing S. The *genus theta series* associated with S is defined by

$$\mathrm{genus}\,\Theta^{(n)}(S)(Z) := \left(\sum_{i=1}^{h} \frac{\Theta^{(n)}(S_i; Z)}{E(S_i)}\right) \Big/ \left(\sum_{i=1}^{h} \frac{1}{E(S_i)}\right),$$

where $E(S_i)$ is the order of the unit group of S_i.

2.4. p**-adic Hermitian Eisenstein series.** Put $\omega = (d_{\mathbb{K}} + \sqrt{d_{\mathbb{K}}})/2$ and define the matrices $\dot{Z} = (\dot{z}_{ij})$ and $\ddot{Z} = (\ddot{z}_{ij})$ by

$$\dot{Z} := \frac{\omega\, {}^tZ - \bar{\omega} Z}{\omega - \bar{\omega}}, \quad \ddot{Z} := \frac{Z - {}^tZ}{\omega - \bar{\omega}}.$$

Thus, any Hermitian modular form $f(Z)$ can be considered as a function of $n(n-1)/2$ complex variables $\ddot{z}_{ij} (i < j)$ in $\ddot{Z}$ and $n(n+1)/2$ complex variables $\dot{z}(i \le j)$ in $\dot{Z}$. If we put

$$\dot{q}_{ij} := \exp(2\pi\sqrt{-1}\dot{z}_{ij})\ (i \le j), \qquad \ddot{q}_{ij} := \exp(2\pi\sqrt{-1}\ddot{z}_{ij})\ (i < j),$$

then

$$q^H := \exp[2\pi\sqrt{-1}\mathrm{tr}(HZ)] = \prod_{i \le j} \dot{q}_{ij}^{\xi_{ij}} \prod_{i<j} \ddot{q}_{ij}^{\eta_{ij}},$$

where $H = (h_{ij})$ and

$$\xi_{ii} = h_{ii},\ \xi_{ij} = h_{ij} + \bar{h}_{ij}\ (i < j),\ \eta_{ij} = \bar{\omega} h_{ij} + \omega \bar{h}_{ij} (i < j)$$

are rational integers. Based on the semi-positivity of H, we may regard a Hermitian modular form $f(Z)$ as an element of a formal power series ring

$$\mathbb{C}[\dot{q}_{ij}^{\pm}, \ddot{q}_{ij}^{\pm}][[\dot{q}_{11}, \dots, \dot{q}_{nn}]].$$

Let $\{k_m\}_{m=1}^{\infty}$ be a sequence of increasing natural numbers. If the corresponding sequence of Hermitian Eisenstein series $\{G_{k_m}^{(n)}\} \subset \mathbb{Q}[\dot{q}_{ij}^{\pm}, \ddot{q}_{ij}^{\pm}]\ [[\dot{q}_{11}, \dots, \dot{q}_{nn}]]$ converges p-adically to an element of $\mathbb{Q}_p[\dot{q}_{ij}^{\pm}, \ddot{q}_{ij}^{\pm}][[\dot{q}_{11}, \dots, \dot{q}_{nn}]]$, we shall call the limit $\lim_{m\to\infty} G_{k_m}^{(n)}$ a *p-adic Hermitian Eisenstein series* ([10] and [8], (5.3)).

3. Main result

In the remainder of this note, we will restrict ourselves to the case in which

$$\boxed{\mathbb{K} = \mathbb{Q}(\sqrt{-1}) : \text{the Gaussian field}}.$$

In this case, we have $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\sqrt{-1}]$ is the ring of Gaussian integers, $d_{\mathbb{K}} = -4$, and $w_{\mathbb{K}} = 4$. An integral Hermitian matrix $S \in Her_m(\mathcal{O}_{\mathbb{K}})$ is called *even* if $S[\mathbf{g}] \in 2\mathbb{Z}$ for all $\mathbf{g} \in M_{m,1}(\mathcal{O}_{\mathbb{K}})$.

A main result of this note is as follows.

Theorem 3.1. *Let p be a prime number such that $p \equiv 3 \pmod 4$, and let $S \in Her_2^+(\mathcal{O}_{\mathbb{K}})$ be an even integral matrix such that $\det(S) = p$ (actually such a matrix S exists for each prime p satisfying $p \equiv 3 \pmod 4$). If we define a sequence*

$$k_m = k_m(p) := 2 + (p-1)p^{m-1},$$

then the corresponding sequence $G_{k_m}^{(n)}$ defines a p-adic Hermitian Eisenstein series. Moreover, we have

$$\lim_{m\to\infty} G_{k_m}^{(n)} = genus\,\Theta^{(n)}(S). \tag{3.1}$$

Namely, the p-adic Hermitian Eisenstein series $\lim_{m\to\infty} G_{k_m}^{(n)}$ becomes an ordinary Hermitian modular form of weight 2 and level p.

4. Proof of the main theorem

We shall prove (3.1) by showing the coincidence of each Fourier coefficient of both sides. Let

$$\lim_{m\to\infty} G_{k_m}^{(n)} = \sum \tilde{a}^{(n)}(H) q^H$$

and

$$\text{genus}\,\Theta^{(n)}(S) = \sum b^{(n)}(H) q^H$$

be the Fourier expansions. Our goal is to prove the following identity:

$$\tilde{a}^{(n)}(H) = b^{(n)}(H). \tag{4.1}$$

However, before proving this identity, we must investigate the coefficients $\tilde{a}^{(n)}(H)$ and $b^{(n)}(H)$. The coefficient $\tilde{a}^{(n)}(H)$ is the p-adic limit of $a_{k_m}^{(n)}(H) (m \to \infty)$. On the other hand, by the definition of the genus theta series, the coefficient $b^{(n)}(H)$ is given by

$$\left(\sum_{i=1}^{h}\frac{A(S_i,2H)}{E(S_i)}\right)\Big/\left(\sum_{i=1}^{h}\frac{1}{E(S_i)}\right),$$

where

$$A(S_i,2H) := \sharp\{X\in M_{2,n}(\mathcal{O}_{\mathbb{K}}) \mid {}^t\overline{X}S_iX=2H\ \}.$$

First, we shall prove the following lemma.

Lemma 4.1. *If $r := rank(H) > 2$, then*

$$\tilde{a}^{(n)}(H)=b^{(n)}(H)=0. \tag{4.2}$$

Proof. Since $\mathrm{rank}(H) > \mathrm{rank}(S_i) = 2$, the numbers $A(S_i, 2H)$ vanish. This implies $b^{(n)}(H) = 0$. Next, we prove that $\tilde{a}^{(n)}(H) = \lim_{m\to\infty} a_{k_m}(H)= 0$. For this purpose, we set

$$\mathbb{B}_k=\begin{cases}\dfrac{B_k}{k} & \text{if } k \text{ is even,}\\[2ex] \dfrac{B_{k,\chi_{-4}}}{k} & \text{if } k \text{ is odd,}\end{cases}$$

where B_k(resp. $B_{k,\chi}$) is the k-th Bernoulli number (resp. the k-th generalized Bernoulli number). If we pursue an argument similar to that of Boecherer [1] in the Hermitian modular case, then we see that the coefficient $a_{k_m}(H)$ can be written as

$$a_{k_m}^{(n)}(H)=\prod_{i=0}^{r-1}\mathbb{B}_{k_m-i}^{-1}\cdot(\text{rational integer}).$$

(Refer also to [11], Proposition 4.7 ($Case\ SU$).)

If $i \equiv 2 \pmod{p-1}$, then

$$\mathrm{ord}_p\left(\mathbb{B}_{k_m-i}^{-1}\right)=\mathrm{ord}_p\left(\frac{k_m-i}{B_{k_m-i}}\right)\geq 1$$

(based on the classical theorem of von Staudt and Clausen). In particular, if $i = 2$, then

$$\mathrm{ord}_p\left(\mathbb{B}_{k_m-2}^{-1}\right)=\mathrm{ord}_p\left(\frac{p^{m-1}(p-1)}{B_{p^{m-1}(p-1)}}\right)=m.$$

If $i \not\equiv 2 \pmod{p-1}$, then there exists a p-adic number $d_i \in \mathbb{Q}_p$ satisfying

$$\lim_{m\to\infty}\frac{k_m-i}{B_{k_m-i,\chi_{-4}}}=d_i.$$

Therefore, we have

$$\mathrm{ord}_p\left(\mathbb{B}_{k_m-i}\right)=\mathrm{ord}_p\left(\frac{k_m-i}{B_{k_m-i,\chi_{-4}}}\right)=\mathrm{ord}_p(d_i)\quad(\text{for sufficiently large } m).$$

Combining these results, we have $\lim_{m\to\infty}\prod_{i=0}^{r-1}\mathbb{B}_{k_m-i}^{-1}=0$. This implies

$$\lim_{m\to\infty}a_{k_m}^{(n)}(H)=0,$$

and completes the proof. ∎

From this result, for the purpose of proving (3.1), it suffices to prove that the identity (4.1) holds for the case $n = 2$. Namely, our goal is reduced to the proof of the identity

$$\tilde{a}^{(2)}(H) = b^{(2)}(H) \tag{4.3}$$

for any $H \in \Lambda_2(\mathbb{K})$.

Lemma 4.2. *If rank$(H) = 2$, then*

$$\tilde{a}^{(2)}(H) = b^{(2)}(H).$$

Proof. We shall first calculate $\tilde{a}^{(2)}(H)$. For this purpose, we introduce an explicit formula for $a_k^{(2)}(H)$ given by Krieg ([6], p. 678, Theorem and p. 679, Corollary):

$$a_k^{(2)}(H) = \frac{4k(k-1)}{B_k \cdot B_{k-1,\chi_{-4}}} \sum_{0<d|\varepsilon(H)} d^{k-1} G_{\chi_{-4}}\left(k-2, \frac{\det(2H)}{d^2}\right),$$

where

$$G_{\chi_{-4}}(s,N) := \frac{1}{1+|\chi_{-4}(N)|}(\sigma_{s,\chi_{-4}}(N) - \sigma^*_{s,\chi_{-4}}(N)), \quad (s,N) \in \mathbb{Z}^2,$$
$$\sigma_{s,\chi_{-4}}(N) := \sum_{0<d|N} \chi_{-4}(d) d^s, \quad \sigma^*_{s,\chi_{-4}}(N) := \sum_{0<d|N} \chi_{-4}(N/d) d^s,$$
$$\varepsilon(H) := \max\{l \in \mathbb{N} \mid l^{-1}H \in \Lambda_2(\mathbb{K})\}.$$

From this formula, we obtain

$$\begin{aligned}\tilde{a}^{(2)}(H) &= \frac{8}{B_{2,\chi_{-p}^2} \cdot B_{1,\chi_{-p}^2\chi_{-4}}} \sum_{\substack{0<d|\varepsilon(H)\\(d,p)=1}} d\, \tilde{G}_{\chi_{-4}}\left(\frac{\det(2H)}{d^2}\right) \\ &= \frac{48}{p-1} \sum_{\substack{0<d|\varepsilon(H)\\(d,p)=1}} d\, \tilde{G}_{\chi_{-4}}\left(\frac{\det(2H)}{d^2}\right),\end{aligned}$$

where

$$\tilde{G}_{\chi_{-4}}(N) := \frac{1}{1+|\chi_{-4}(N)|}(\tilde{\sigma}_{\chi_{-4}}(N) - \tilde{\sigma}^*_{\chi_{-4}}(N)), \quad N \in \mathbb{Z},$$
$$\tilde{\sigma}_{\chi_{-4}}(N) := \sum_{\substack{0<d|N\\(d,p)=1}} \chi_{-4}(d), \quad \tilde{\sigma}^*_{\chi_{-4}}(N) := \sum_{\substack{0<d|N\\(d,p)=1}} \chi_{-4}(N/d).$$

Next, we shall calculate $b^{(2)}(H)$. By the Siegel formula in the case of Hermitian forms (cf. Braun [2], Otremba [9]), we have

$$b^{(2)}(H) = \frac{1}{2} \prod_{q \le \infty} \alpha_q(S, 2H),$$

where

$$\alpha_q(S,2H) = \lim_{a\to\infty} q^{-4a} A_{q^a}(S,2H),$$
$$A_{q^a}(S,2H) := \sharp\{\, X \in M_2(\mathcal{O}_{\mathbb{K}}) \bmod q^a \mid {}^t\overline{X}SX \equiv 2H \bmod q^a \,\}$$

and

$$\alpha_\infty(S,2H)=\det(S)^{-2}\cdot\prod_{j=1}^{2}\frac{(2\pi)^j}{\sqrt{|d_{\mathbb{K}}|}^j\cdot(j-1)!}=\frac{\pi^3}{p^2}.$$

In order to calculate $\alpha_q(S,2H)$ more explicitly, we introduce the following notation: let $f(H)$ be the integer defined by $\det(2H)=\varepsilon(H)^2\cdot f(H)$ and put

$$\det(2H)=\prod_{q:\text{prime}}q^{e_q},\quad \varepsilon(H)=\prod_{q:\text{prime}}q^{\varepsilon_q},\quad f(H)=\prod_{q:\text{prime}}q^{f_q}.$$

We proceed the calculation of $\alpha_q(S,2H)$ by dividing the following three cases:
(i) q is a prime number such that $q\neq 2,\ p$,
(ii) $q=p$,
(iii) $q=2$ (ramified prime).
For case (i), we have

$$\alpha_q(S,2H)=(1-q^{-2})(1-\chi_{-4}(q)q^{-1})\sum_{l=0}^{\varepsilon_q}q^l\left(\sum_{m=0}^{e_q-2l}\chi_{-4}(q)^m\right).$$

This follows from the main theorem in [7] (p. 235) (unramified case).

For case (ii), we obtain

$$\alpha_q(S,2H)=\alpha_p(S,2H)=\begin{cases}\dfrac{(p+1)^2}{p} & \text{if } f_p \text{ is odd},\\ 0 & \text{otherwise.}\end{cases}$$

This formula is derived from a formula by Hironaka [4] by setting $m=n=2$, $\lambda=(\varepsilon_p+f_p,\varepsilon_p)$, and $\mu=(1,0)$ in the main formula in Theorem 4 ([4], p. 61).

For case (iii), we have

$$\begin{aligned}&\alpha_2(S,2H)\\ &=2^2\cdot(1-2^{-2})\sum_{l=0}^{\varepsilon_2}2^l\{1-\chi_{-4}(\det(2H)/2^{\varepsilon_2})(1-\chi_{-4}(\det(2H)/2^{2l})^2)\}.\end{aligned}$$

As in case (i), this formula can be obtained from the main theorem in [7] (p. 235) (ramified case).

Now we assume that $\mathrm{ord}_p(\det(2H))$ is odd, that is, the number f_p is odd. In this case, by the above formulas, we have

$$\begin{aligned}b^{(2)}(H)=&\frac{1}{2}\cdot\prod_{q\leq\infty}\alpha_q(S,2H)\\ =&2\cdot\frac{1}{\zeta(2)L(1;\chi_{-4})}\cdot\frac{1}{(1-p^{-2})(1-\chi_{-4}(p)p^{-1})}\cdot\frac{(p+1)^2}{p}\cdot\frac{\pi^3}{p^2}\\ &\cdot\sum_{\substack{0<d|\varepsilon(H)\\(d,p)=1}}dF\left(\frac{\det(2H)}{d^2}\right)\\ =&\frac{48}{p-1}\sum_{\substack{0<d|\varepsilon(H)\\(d,p)=1}}dF\left(\frac{\det(2H)}{d^2}\right),\end{aligned}$$

where

$$F(N) := \prod_{\substack{q:\text{prime}\\ q\neq p,2}} \left(\sum_{m=0}^{\beta_q} \chi_{-4}(q)^m\right) \cdot \frac{1-\chi_{-4}(N')}{1+|\chi_{-4}(N)|}, \quad N = \prod q^{\beta_q} = 2^{\beta_2}\cdot N'.$$

It is easy to see that $G_{\chi_{-4}}(s,N)$ has the following expression:

$$G_{\chi_{-4}}(s,N) = \frac{1}{1+|\chi_{-4}(N)|} \prod_{q\neq 2} \left(\sum_{m=0}^{\beta_q} \chi_{-4}(q)^m q^{ms}\right)(1-\chi_{-4}(N')2^{\beta_2 s}).$$

Hence, we obtain

$$F(N) = \tilde{G}_{\chi_{-4}}(N).$$

Therefore, if $\mathrm{ord}_p(\det(2H))$ is odd, then $\tilde{a}^{(2)}(H) = b^{(2)}(H)$. On the other hand, if $\mathrm{ord}_p(\det(2H))$ is even, we have $\tilde{a}^{(2)}(H) = b^{(2)}(H) = 0$. This completes the proof of Lemma 4.2. □

Lemma 4.3. *If* $\mathrm{rank}(H) = 1$, *then*

$$\tilde{a}^{(2)}(H) = b^{(2)}(H).$$

Proof. In the case where $\mathrm{rank}(H) = 1$, we have

$$a_k^{(2)}(H) = a_k^{(1)}(\varepsilon(H)) = -\frac{2k}{B_k}\sigma_{k-1}(\varepsilon(H)).$$

Thus, we obtain

$$\tilde{a}^{(2)}(H) = -\frac{4}{B_{2,\chi_{-p}^2}}\sigma_{1,\chi_{-p}^2}(\varepsilon(H)) = -\frac{4}{(1-p)B_2}\sum_{\substack{0<d|\varepsilon(H)\\ (d,p)=1}} d = \frac{24}{p-1}\sum_{\substack{0<d|\varepsilon(H)\\ (d,p)=1}} d.$$

Next, we shall calculate $b^{(2)}(H)$ for the case in which $\mathrm{rank}(H) = 1$. In this case, we have $2H \sim \left(\begin{smallmatrix} 2\varepsilon(H) & 0\\ 0 & 0\end{smallmatrix}\right)$ (unimodular equivalence). As in the case of $\mathrm{rank}(H) = 2$, we can apply the Siegel formula and obtain

$$b^{(2)}(H) = \prod_{q\leq\infty} \alpha_q(S, 2\varepsilon(H)),$$

where

$$\alpha_q(S,2\varepsilon(H)) = \lim_{a\to\infty} q^{-3a} A_{q^a}(S,2\varepsilon(H)),$$
$$A_{q^a}(S,2\varepsilon(H)) := \sharp\{\, X\in M_{2,1}(\mathcal{O}_{\mathbb{K}}) \bmod q^a \mid {}^t\overline{X}SX \equiv 2\varepsilon(H) \bmod q^a \,\}$$

and

$$\alpha_\infty(S,2\varepsilon(H)) = \det(S)^{-1}\cdot 2\varepsilon(H)\cdot\pi^2 = \frac{2\varepsilon(H)\pi^2}{p}.$$

The value $\alpha_q(S,2\varepsilon(H))$ is given as follows.

If q is a prime number satisfying $q\neq 2,\ p$ (which corresponds to case (i) in the proof of Lemma 4.2), then

$$\alpha_q(S,2\varepsilon(H)) = (1-q^{-2})\sum_{l=0}^{\varepsilon_q} q^{-l}.$$

If $q = p$ (case (ii)), then

$$\alpha_p(S, 2\varepsilon(H)) = \frac{1+p}{p^{1+\varepsilon_p}}.$$

If $q = 2$ (case (iii)), then

$$\alpha_2(S, 2\varepsilon(H)) = 2 \cdot (1 - 2^{-2}) \sum_{l=0}^{\varepsilon_2} 2^{-l}.$$

Combining these formulas, we have

$$\begin{aligned} b^{(2)}(H) &= \prod_{q \le \infty} \alpha_q(S, 2\varepsilon(H)) \\ &= \frac{1}{\zeta(2) \cdot (1 - p^{-2})} \cdot \frac{1+p}{p^{1+\varepsilon_p}} \cdot 2 \prod_{q \ne p} \left(\sum_{l=0}^{\varepsilon_q} q^{-l} \right) \cdot \frac{2\varepsilon(H)\pi^2}{p} \\ &= \frac{24}{p-1} \sum_{\substack{0<d|\varepsilon(H) \\ (d,p)=1}} d \\ &= \tilde{a}^{(2)}(H). \end{aligned}$$

This completes the proof of Lemma 4.3. □

Since $\tilde{a}^{(2)}(O_2) = b^{(2)}(O_2) = 1$, we conclude the proof of the desired formula (4.3).

References

1. S. Boecherer, *Über die Fourierkoeffizienten der Siegelschen Eisensteinreihen.* Manuscripta Math. **45**(1984), 273–288. MR0734842 (86b:11037)
2. H. Braun, *Zur Theorie der hermiteschen Formen*, Abh. Math. Sem. Univ. Hamburg **14**(1941), 61–150. MR0004856 (3:70i)
3. H. Braun, *Hermitian modular functions I*, Ann. of Math. **50**(1949), 827–855. MR0032699 (11:333a)
4. Y. Hironaka, *Local zeta functions on Hermitian forms and its application to local densities*, J. Number Theory **71**(1998), 40–64. MR1631034 (99e:11045)
5. H. Katsurada and S. Nagaoka, *On some p-adic properties of Siegel-Eisenstein series*, J. Number Theory **104**(2004), 100–117. MR2021628 (2004i:11042)
6. A. Krieg, *The Maass spaces on the Hermitian half-space of degree 2*, Math. Ann. **289**(1991), 663–681. MR1103042 (93d:11051)
7. S. Nagaoka, *An explicit formula for Siegel series*, Abh. Math. Sem. Univ. Hamburg **59**(1989), 235–262. MR1049898 (91i:11053)
8. S. Nagaoka, *A remark on Serre's example of p-adic Eisenstein series*, Math. Z. **235**(2000), 227–250. MR1795506 (2001m:11068)
9. G. Otremba, *Zur Theorie der hermiteschen Formen in imaginär-quadratischen Zahlkörpern*, J. Reine Angew. Math. **249**(1971), 1–19. MR0318060 (47:6609)
10. J.-P. Serre, *Formes modulaires et fonction zêta p-adiques*, Lecture Notes in Math., Vol. 350, Springer, Berlin 1973, pp.191–268. MR0404145 (53:7949a)
11. G. Shimura, *On Eisenstein series*, Duke Math. J. **50**(1983), 417–476. MR0705034 (84k:10019)

Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan
E-mail address: `nagaoka@math.kindai.ac.jp`