RIBBON-MOVES FOR 2-KNOTS WITH 1-HANDLES ATTACHED AND KHOVANOV-JACOBSSON NUMBERS

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Abstract. We prove that a crossing change along a double point circle on a 2-knot is realized by ribbon-moves for a knotted torus obtained from the 2-knot by attaching a 1-handle. It follows that any 2-knots for which the crossing change is an unknotting operation, such as ribbon 2-knots and twist-spun knots, have trivial Khovanov-Jacobsson number.

A surface-knot or -link is a closed surface embedded in 4-space $\mathbb{R}^4$ locally flatly. Throughout this note, we always assume that all surface-knots are oriented. A ribbon-move (cf. [10]) is a local operation for (a diagram of) a surface-knot as shown in Figure 1. We say that surface-knots $F$ and $F'$ are ribbon-move equivalent, denoted by $F \sim F'$, if $F'$ is obtained from $F$ by a finite sequence of ribbon-moves.

The ribbon-move is a special case of the crossing change: Assume that a surface-knot $F$ has a double point circle $L$ in a diagram such that (i) $L$ has no self-intersection, and (ii) at every triple point on $L$, the sheet transverse to $L$ is either top or bottom (not middle). The condition (i) means that $L$ does not go through the same triple point twice. When $L$ satisfies these conditions, we can perform a crossing change along $L$ by exchanging the roles of over- and under-sheets as indicated in Figure 2 (cf. [10]). See [4] for details on diagrams of surface-knots.

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For a 2-knot \( K \) (a knotted sphere in \( \mathbb{R}^4 \)), a crossing change is not necessarily realized by ribbon-moves; indeed, a ribbon-move does not change the Farber-Levine pairing of \( K \), but a crossing change might (cf. \cite{10}). On the other hand, when we consider the \( T^2 \)-knot (knotted torus in \( \mathbb{R}^4 \)) \( K + h \) obtained from \( K \) by attaching a 1-handle \( h \) on \( K \), we obtain the following. 

**Theorem 1.** Let \( K \) and \( K' \) be 2-knots such that \( K' \) is obtained from \( K \) by a crossing change. Then for any 1-handles \( h \) and \( h' \) on \( K \) and \( K' \), respectively, the \( T^2 \)-knot \( K + h \) is ribbon-move equivalent to \( K' + h' \).

**Proof.** Along the double point circle \( L \) for which we perform the crossing change, there is a neighborhood \( N \) identified with \((B^3, t) \times S^1\), where \((B^3, t)\) is a tangle with two strings as shown on the left of Figure 3. In the figure, the orientations of tangles are induced from that of \( K \), and all bands are attached in an orientation-compatible manner. For an interval \( I \) in \( S^1 \), we take a 1-handle \( h_1 = b_1 \times I \) on \( K \), where \( b_1 \) is a band as indicated in the figure.

We observe that \( K + h_1 \) is ambient isotopic to \((K' \cup T) + h_2 \) (cf. \cite{12}), where \( T = m \times S^1 \) is a \( T^2 \)-knot linking with \( K' \), and the 1-handle \( h_2 = b_2 \times I \) connects between \( K' \) and \( T \). See the center of Figure 3.

Consider a 1-handle \( h_3 = b_3 \times I \) on \( K' \cup T \). Since both \( h_2 \) and \( h_3 \) connect between \( K' \) and \( T \), the \( T^2 \)-knot \((K' \cup T) + h_2 \) is ribbon-move equivalent to \((K' \cup T) + h_3 \).

Finally we see that \((K' \cup T) + h_3 \) is ambient isotopic to \( K' + h_4 \), where \( h_4 = b_4 \times I \) is the 1-handle on \( K' \) as shown on the right of the figure. Thus we obtain

\[
K + h \sim (K' \cup T) + h_2 \sim (K' \cup T) + h_3 \sim K' + h_4 \sim K' + h'.
\]

This completes the proof. \( \square \)

We say that the crossing change is an *unknotting operation* for a surface-knot \( F \) if the trivial surface-knot is obtained from \( F \) by a finite sequence of crossing changes. It is still unknown whether the crossing change is an unknottting operation for any surface-knot.
Khovanov [8] introduced a categorification of the Jones polynomial, that is, a chain complex for a given classical knot diagram such that its graded Euler characteristic is the Jones polynomial. Khovanov [9] and Jacobsson [5] proved that it defines an invariant for cobordisms (relative to boundary diagrams). Specifically, a cobordism between two knot diagrams gives rise to a chain map (we call it a Khovanov-Jacobsson homomorphism) between corresponding chain complexes that is invariant under equivalence of cobordisms of diagrams. See also [2]. In particular, a diagram of a $\mathbb{T}^2$-knot is a cobordism between empty diagrams, giving rise to a homomorphism $\mathbb{Z} \to \mathbb{Z}$ defined up to sign, a multiplication by a constant. We call this constant the Khovanov-Jacobsson number.

**Theorem 2.** Let $K$ be a 2-knot for which a crossing change is an unknotting operation. Then for any 1-handle $h$ on $K$, the $\mathbb{T}^2$-knot $K + h$ has trivial Khovanov-Jacobsson number.

**Proof.** Let $K_0$ be the trivial 2-knot and $h_0$ the trivial 1-handle on $K_0$. By assumption and Theorem 1, the $\mathbb{T}^2$-knot $K + h$ is ribbon-move equivalent to $K_0 + h_0$, which is the trivial $\mathbb{T}^2$-knot.

Consider two movies as shown in Figure 4. It is seen from the definitions [2, 5] that the corresponding Khovanov-Jacobsson homomorphisms $H^*(\bigcirc) \to H^*(\bigcirc\bigcirc)$ are the same for these movies. This implies that a ribbon-move does not change the Khovanov-Jacobsson number. Hence the $\mathbb{T}^2$-knot $K + h$ has the same number as that of the trivial $\mathbb{T}^2$-knot $K_0 + h_0$. □

![Figure 4](#)

By Theorem 2 if there is a 2-knot $K$ such that the Khovanov-Jacobsson number of $K + h$ is non-trivial, then the crossing change is not an unknotting operation for $K$. However, we have no such examples at present.

**Corollary 3.** Let $K$ be a ribbon 2-knot or twist-spun knot. Then for any 1-handle $h$ on $K$, the $\mathbb{T}^2$-knot $K + h$ has trivial Khovanov-Jacobsson number.

**Proof.** This follows from Theorem 2 and the fact that the crossing change is an unknotting operation for every ribbon 2-knot or twist-spun knot (cf. [1, 11]). □

We say that a surface-knot is pseudo-ribbon [7] if it has a diagram without triple points. The notions of ribbon and pseudo-ribbon 2-knots are the same [15] (see also [6]). On the other hand, for $\mathbb{T}^2$-knots, they are not coincident in the sense that the family of pseudo-ribbon $\mathbb{T}^2$-knots properly contains that of ribbon $\mathbb{T}^2$-knots.

**Proposition 4.** Any pseudo-ribbon $\mathbb{T}^2$-knot has the trivial Khovanov-Jacobsson number.
Proof. By the results of Teragaito [14] and Shima [13], every pseudo-ribbon $T^2$-knot $T$ is (i) a ribbon $T^2$-knot, or (ii) a $T^2$-knot obtained from a split union of a Boyle’s turned $T^2$-knot $T’$ [3] and a trivial 2-link $U = U_1 \cup U_2 \cup \cdots \cup U_n$ by surgery along 1-handles $h_1, h_2, \ldots, h_n$ for some $n \geq 0$, where each $h_i$ connects between $T’$ and $U_i$ ($i = 1, 2, \ldots, n$).

For the case (i), there is a ribbon 2-knot $K$ and a 1-handle $h$ such that $T = K + h$. Hence the conclusion follows from Corollary 3.

For the case (ii), we see that $T = (T’ \cup U) + (\bigcup_{i=1}^{n} h_i)$ is ribbon-move equivalent to $T’$. We consider two movies for a classical knot diagram $D$ in a plane, one of which keeps $D$ still and the other twists $D$ by a $2\pi$-rotation of the plane. Then it follows from the definitions [2, 5, 9] that the corresponding Khovanov-Jacobsson homomorphisms $H^*(D) \to H^*(D)$ are the same for these movies. This implies that $T’$ has the same Khovanov-Jacobsson number as that of a non-turned (that is, just spun) $T^2$-knot, which is ribbon. Hence this case reduces to (i). □

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References


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