PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 9, September 2006, Pages 2779–2783
S 0002-9939(06)08288-8
Article electronically published on April 10, 2006

RIBBON-MOVES FOR 2-KNOTS WITH 1-HANDLES ATTACHED AND KHOVANOV-JACOBSSON NUMBERS

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(Communicated by Ronald A. Fintushel)

ABSTRACT. We prove that a crossing change along a double point circle on a 2-knot is realized by ribbon-moves for a knotted torus obtained from the 2-knot by attaching a 1-handle. It follows that any 2-knots for which the crossing change is an unknotting operation, such as ribbon 2-knots and twist-spun knots, have trivial Khovanov-Jacobsson number.

A *surface-knot* or *-link* is a closed surface embedded in 4-space $\mathbb{R}^4$ locally flatly. Throughout this note, we always assume that all surface-knots are oriented. A *ribbon-move* (cf. [10]) is a local operation for (a diagram of) a surface-knot as shown in Figure 1. We say that surface-knots F and F' are *ribbon-move equivalent*, denoted by $F \sim F'$, if F' is obtained from F by a finite sequence of ribbon-moves.

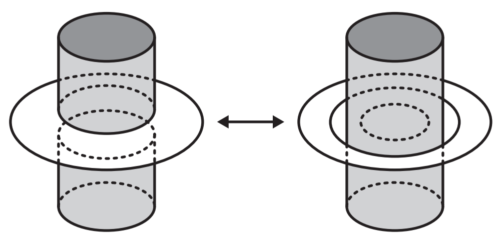

FIGURE 1.

The ribbon-move is a special case of the *crossing change*: Assume that a surface-knot F has a double point circle L in a diagram such that (i) L has no self-intersection, and (ii) at every triple point on L, the sheet transverse to L is either top or bottom (not middle). The condition (i) means that L does not go through the same triple point twice. When L satisfies these conditions, we can perform a crossing change along L by exchanging the roles of over- and under-sheets as indicated in Figure 2 (cf. [16]). See [4] for details on diagrams of surface-knots.

Received by the editors October 19, 2004 and, in revised form, April 14, 2005.
2000 *Mathematics Subject Classification.* Primary 57Q45; Secondary 57Q35.
Key words and phrases. Khovanov homology, 2-knot, ribbon-move, twist-spun knot, crossing change.
The first author was supported in part by NSF Grant DMS #0301095.
The second author was supported in part by NSF Grant DMS #0301089.
The third author was supported in part by JSPS Postdoctoral Fellowships for Research Abroad.

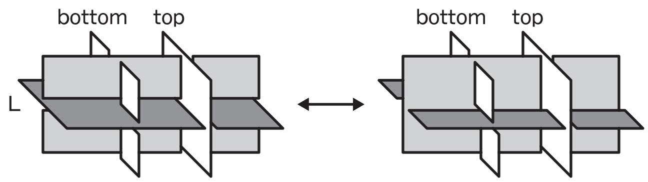

FIGURE 2.

For a 2-knot K (a knotted sphere in $\mathbb{R}^4$), a crossing change is not necessarily realized by ribbon-moves; indeed, a ribbon-move does not change the Farber-Levine pairing of K, but a crossing change might (cf. [10]). On the other hand, when we consider the $\mathbb{T}^2$-knot (knotted torus in $\mathbb{R}^4$) $K + h$ obtained from K by attaching a 1-handle h on K, we obtain the following.

Theorem 1. *Let K and K' be 2-knots such that K' is obtained from K by a crossing change. Then for any 1-handles h and h' on K and K', respectively, the $\mathbb{T}^2$-knot $K + h$ is ribbon-move equivalent to $K' + h'$.*

Proof. Along the double point circle L for which we perform the crossing change, there is a neighborhood N identified with $(B^3, t) \times S^1$, where (B^3, t) is a tangle with two strings as shown on the left of Figure 3. In the figure, the orientations of tangles are induced from that of K, and all bands are attached in an orientation-compatible manner. For an interval I in S^1, we take a 1-handle $h_1 = b_1 \times I$ on K, where b_1 is a band as indicated in the figure.

We observe that $K + h_1$ is ambient isotopic to $(K' \cup T) + h_2$ (cf. [12]), where $T = m \times S^1$ is a $\mathbb{T}^2$-knot linking with K', and the 1-handle $h_2 = b_2 \times I$ connects between K' and T. See the center of Figure 3.

Consider a 1-handle $h_3 = b_3 \times I$ on $K' \cup T$. Since both h_2 and h_3 connect between K' and T, the $\mathbb{T}^2$-knot $(K' \cup T) + h_2$ is ribbon-move equivalent to $(K' \cup T) + h_3$.

Finally we see that $(K' \cup T) + h_3$ is ambient isotopic to $K' + h_4$, where $h_4 = b_4 \times I$ is the 1-handle on K' as shown on the right of the figure. Thus we obtain

$$K + h \sim K + h_1 = (K' \cup T) + h_2 \sim (K' \cup T) + h_3 = K' + h_4 \sim K' + h'.$$

This completes the proof. $\square$

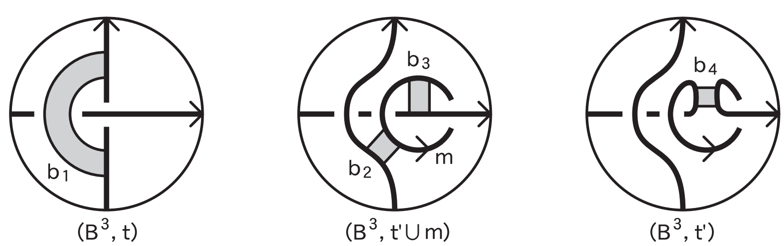

FIGURE 3.

We say that the crossing change is an *unknotting operation* for a surface-knot F if the trivial surface-knot is obtained from F by a finite sequence of crossing changes. It is still unknown whether the crossing change is an unknotting operation for *any* surface-knot.

Khovanov [8] introduced a categorification of the Jones polynomial, that is, a chain complex for a given classical knot diagram such that its graded Euler characteristic is the Jones polynomial. Khovanov [9] and Jacobsson [5] proved that it defines an invariant for cobordisms (relative to boundary diagrams). Specifically, a cobordism between two knot diagrams gives rise to a chain map (we call it a Khovanov-Jacobsson homomorphism) between corresponding chain complexes that is invariant under equivalence of cobordisms of diagrams. See also [2]. In particular, a diagram of a $\mathbb{T}^2$-knot is a cobordism between empty diagrams, giving rise to a homomorphism $\mathbb{Z} \to \mathbb{Z}$ defined up to sign, a multiplication by a constant. We call this constant the *Khovanov-Jacobsson number.*

Theorem 2. *Let K be a 2-knot for which a crossing change is an unknotting operation. Then for any 1-handle h on K, the $\mathbb{T}^2$-knot $K+h$ has trivial Khovanov-Jacobsson number.*

Proof. Let K_0 be the trivial 2-knot and h_0 the trivial 1-handle on K_0. By assumption and Theorem 1, the $\mathbb{T}^2$-knot $K+h$ is ribbon-move equivalent to K_0+h_0, which is the trivial $\mathbb{T}^2$-knot.

Consider two movies as shown in Figure 4. It is seen from the definitions [2, 5] that the corresponding Khovanov-Jacobsson homomorphisms $H^*(|\bigcirc) \to H^*(\bigcirc|)$ are the same for these movies. This implies that a ribbon-move does not change the Khovanov-Jacobsson number. Hence the $\mathbb{T}^2$-knot $K+h$ has the same number as that of the trivial $\mathbb{T}^2$-knot K_0+h_0. □

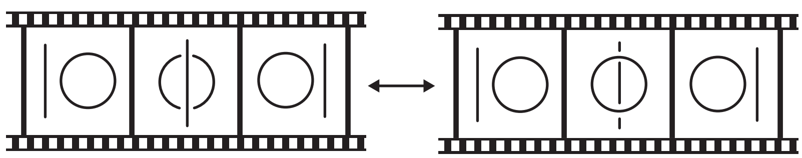

FIGURE 4.

By Theorem 2, if there is a 2-knot K such that the Khovanov-Jacobsson number of $K+h$ is non-trivial, then the crossing change is not an unknotting operation for K. However, we have no such examples at present.

Corollary 3. *Let K be a ribbon 2-knot or twist-spun knot. Then for any 1-handle h on K, the $\mathbb{T}^2$-knot $K+h$ has trivial Khovanov-Jacobsson number.*

Proof. This follows from Theorem 2 and the fact that the crossing change is an unknotting operation for every ribbon 2-knot or twist-spun knot (cf. [1, 11]). □

We say that a surface-knot is *pseudo-ribbon* [7] if it has a diagram without triple points. The notions of ribbon and pseudo-ribbon 2-knots are the same [15] (see also [6]). On the other hand, for $\mathbb{T}^2$-knots, they are not coincident in the sense that the family of pseudo-ribbon $\mathbb{T}^2$-knots properly contains that of ribbon $\mathbb{T}^2$-knots.

Proposition 4. *Any pseudo-ribbon $\mathbb{T}^2$-knot has the trivial Khovanov-Jacobsson number.*

Proof. By the results of Teragaito [14] and Shima [13], every pseudo-ribbon $\mathbb{T}^2$-knot T is (i) a ribbon $\mathbb{T}^2$-knot, or (ii) a $\mathbb{T}^2$-knot obtained from a split union of a Boyle's turned $\mathbb{T}^2$-knot T' [3] and a trivial 2-link $U = U_1 \cup U_2 \cup \cdots \cup U_n$ by surgery along 1-handles $h_1, h_2, \ldots, h_n$ for some $n \geq 0$, where each h_i connects between T' and h_i $(i = 1, 2, \ldots, n)$.

For the case (i), there is a ribbon 2-knot K and a 1-handle h such that $T = K + h$. Hence the conclusion follows from Corollary 3.

For the case (ii), we see that $T = (T' \cup U) + (\bigcup_{i=1}^{n} h_i)$ is ribbon-move equivalent to T'. We consider two movies for a classical knot diagram D in a plane, one of which keeps D still and the other twists D by a 2π-rotation of the plane. Then it follows from the definitions [2, 5, 9] that the corresponding Khovanov-Jacobsson homomorphisms $H^*(D) \to H^*(D)$ are the same for these movies. This implies that T' has the same Khovanov-Jacobsson number as that of a non-turned (that is, just spun) $\mathbb{T}^2$-knot, which is ribbon. Hence this case reduces to (i). □

Acknowledgement

The third author expresses his gratitude for the hospitality of the University of South Florida and the University of South Alabama. The first author thanks the organizers of Knots in Washington, May 2004; this conference stimulated our interest in the subject.

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