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# REGULARIZED PRODUCT EXPRESSIONS OF HIGHER RIEMANN ZETA FUNCTIONS

TETSUYA MOMOTANI

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ABSTRACT. As a generalization of recent work by Kurokawa, Matsuda, and Wakayama (2004) we introduce a higher Riemann zeta function for an abstract sequence. Then we explicitly determine its regularized product expression.

## 1. INTRODUCTION

It is known ([De]) that the Riemann zeta function $\zeta(s) = \prod_{p:\text{prime}}(1-p^{-s})^{-1}$ has the regularized product expression

$$\prod_{\rho\in R}\Big(\frac{s-\rho}{2\pi}\Big) = 2^{-1/2}(2\pi)^{-2}\pi^{-s/2}s(s-1)\Gamma\Big(\frac{s}{2}\Big)\zeta(s),$$

where $R$ is the set of the non-trivial zeros of $\zeta(s)$, and $\Gamma(s)$ is the classical gamma function.

On the other hand, a higher Riemann zeta function $\zeta_{l\infty}(s) := \prod_{n=1}^{\infty}\zeta(s+ln)$ has been introduced and studied in the paper [CL] for $l=1$, [KMW] for $l\in\mathbb{Z}_{\geq 1}$. We now consider a generalized higher Riemann zeta function. Let $\Lambda = \{\lambda_k\}_{k\in I}$ be a sequence of complex numbers. Then we define a higher Riemann zeta function for the sequence $\Lambda$ by

$$Z(s,\Lambda) := \prod_{\lambda\in\Lambda}\prod_{p:\text{prime}}(1-p^{-s-\lambda})^{-1} = \prod_{\lambda\in\Lambda}\zeta(s+\lambda).$$

In this paper, we study several properties of the higher Riemann zeta function. For a "regularizable" sequence $\Lambda$, we see that $Z(s,\Lambda)$ has the regularized product expression

$$\coprod_{\lambda\in\Lambda,\ \rho\in R}\Big(\frac{s+\lambda-\rho}{2\pi}\Big) = \text{“ gamma factor ”} \times Z(s,\Lambda).$$

Here $\coprod$ denotes the "dotted" regularized product due to [KW3].

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Further, when $\Lambda$ is given by a lattice $\Omega := \{n_1\omega_1 + \cdots + n_r\omega_r | n_j \in \mathbb{Z}_{\geq 0}\}$, we show that the higher Riemann zeta function

$$Z(s,\Omega) = \prod_{n_1,\cdots,n_r=0}^{\infty} \prod_{p:\text{prime}} (1 - p^{-(s+n_1\omega_1+\cdots+n_r\omega_r)})^{-1}$$

satisfies a certain functional equation of the type $s \leftrightarrow 1 + \omega_1 + \cdots + \omega_r - s$. We remark that the form of $Z(s,\Omega)$ looks like the Selberg zeta function (when $r = 1$) defined for a discrete subgroup of a real rank 1 semi-simple Lie group (cf. [Go] [Ku2]).

## 2. Notation

2.1. **Regularized product.** Let $\Lambda = \{\lambda_k\}_{k\in I}$ be a sequence of complex numbers. We call the sequence $\Lambda$ "regularizable" if $\Lambda$ satisfies the following conditions: (i) $\mathrm{Re}(\lambda_k) \geq 0$ for all $k \in I$, and $\mathrm{Re}(\lambda_k) \to \infty$ as $k \to \infty$. (ii) The series $\sum_{k\in I,\ \lambda_k\neq 0} \lambda_k^{-s}$ converges for sufficiently large $\mathrm{Re}(s)$. (iii) The function $\Theta(x,\Lambda) := \sum_{k\in I} e^{-\lambda_k x}$ has an asymptotic expansion

$$\Theta(x,\Lambda) \sim \sum_{n=0}^{\infty} x^{t_n} T_n(\log x) \quad \text{as } x \to +0,$$

where $t_0 \leq 0$, $t_0 < t_1 < t_2 < \cdots \to +\infty$, and $T_n(z) \in \mathbb{C}[z]$ is a polynomial.

For a regularizable sequence, we can define the regularized product as follows.

**Lemma 2.1** ([Ill], [KW3]). *Assume that the sequence $\Lambda = \{\lambda_k\}_{k\in I}$ is regularizable. Then,* (1) *for* $\mathrm{Re}(z) > 0$, *the function*

$$\zeta(s,z,\Lambda) := \sum_{k\in I} (z+\lambda_k)^{-s}$$

*has an analytic continuation as a meromorphic function on $s \in \mathbb{C}$.* (2) *The "dotted" regularized product*

$$\prod_{k\in I}\!\!\!\!\cdot\,(z+\lambda_k) := \exp\Big\{ -\operatorname*{CT}_{s=0} \frac{\zeta(s,z,\Lambda)}{s} \Big\}$$

*exists, and this is an entire function on $z \in \mathbb{C}$ with zeros at $z = -\lambda_k$ $(k \in I)$. Here $\mathrm{CT}_{s=0} f(s)$ denotes the constant term in the Laurent expansion of $f(s)$ at $s = 0$.*

We remark that the dotted regularized product $\prod\!\!\!\!\cdot_{m\in I,\ n\in J}(z + a_m + b_n)$ exists if $\{a_m\}_{m\in I}$ and $\{b_n\}_{n\in J}$ are regularizable sequences. This follows at once from the relation $\Theta(x, \{a_m + b_n\}_{m\in I,\ n\in J}) = \Theta(x, \{a_m\}_{m\in I}) \times \Theta(x, \{b_n\}_{n\in J})$.

2.2. **Multiple gamma function and multiple sine function.** Let $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_r)$, where $\omega_j \in \mathbb{C}$, $\mathrm{Re}(\omega_j) > 0$, and $\Omega := \{n_1\omega_1 + \cdots + n_r\omega_r | n_j \in \mathbb{Z}_{\geq 0}\}$. We fix $\mathrm{Re}(z) > 0$. Then the multiple zeta function of Barnes ([Ba]) is defined by

$$\zeta(s,z,\boldsymbol{\omega}) := \zeta(s,z,\Omega) = \sum_{n_1,\cdots,n_r=0}^{\infty} (z + n_1\omega_1 + \cdots + n_r\omega_r)^{-s}.$$

This sum converges absolutely for $\mathrm{Re}(s) > r$. It is seen that the sequence $\Omega$ is regularizable, and $\zeta(s,z,\boldsymbol{\omega})$ has a meromorphic continuation to $s \in \mathbb{C}$. Further, for

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$m \in \mathbb{Z}_{\geq 0}$, it is known that

$$\zeta(-m, z, \boldsymbol{\omega}) = \frac{(-1)^m \cdot m! \cdot B_{m+r}(z, \boldsymbol{\omega})}{(m+r)!}.$$

Here $B_n(z, \boldsymbol{\omega})$ is the multiple Bernoulli polynomials given by

$$\frac{x^r e^{-zx}}{(1-e^{-\omega_1 x}) \cdots (1-e^{-\omega_r x})} = \sum_{n=0}^{\infty} \frac{B_n(z, \boldsymbol{\omega})}{n!} x^n,$$

for $|x| < \min_{1 \leq j \leq r} |2\pi/\omega_j|$. We remark that $\zeta(s) = \zeta(s, 1, (1))$ is the Riemann zeta function and $\zeta(-m) = (-1)^m B_{m+1}/(m+1)$, where $B_m = B_m(1, (1))$ is the usual Bernoulli number.

Now we define the multiple gamma function $\Gamma(z, \boldsymbol{\omega})$ and the multiple sine function $S(z, \boldsymbol{\omega})$ by using the regularized product as follows ([Ku1]):

$$\begin{aligned} \Gamma(z, \boldsymbol{\omega}) &:= \exp\left\{ \frac{d}{ds} \zeta(s, z, \boldsymbol{\omega}) \Big|_{s=0} \right\} = \prod_{n_1, \cdots, n_r = 0}^{\infty} (z + n_1\omega_1 + \cdots + n_r\omega_r)^{-1}, \\ S(z, \boldsymbol{\omega}) &:= \Gamma(z, \boldsymbol{\omega})^{-1} \cdot \Gamma(\omega_1 + \cdots + \omega_r - z, \boldsymbol{\omega})^{(-1)^r}. \end{aligned}$$

It is seen that these functions satisfy the conditions

$$\begin{aligned} \Gamma(z, (\omega_1, \cdots, \omega_{r-1}, \omega_r)) &= \Gamma(z, (\omega_1, \cdots, \omega_{r-1})) \cdot \Gamma(z + \omega_r, (\omega_1, \cdots, \omega_{r-1}, \omega_r)), \\ S(z, (\omega_1, \cdots, \omega_{r-1}, \omega_r)) &= S(z, (\omega_1, \cdots, \omega_{r-1})) \cdot S(z + \omega_r, (\omega_1, \cdots, \omega_{r-1}, \omega_r)). \end{aligned}$$

We also note that $\Gamma(z, (1)) = (2\pi)^{-1/2}\, \Gamma(z)$ and $S(z, (1)) = 2\sin(\pi z)$.

## 3. Higher Riemann zeta function

3.1. **Definition and analytic continuation.** First of all, we introduce a higher Riemann zeta function as follows.

**Definition 3.1.** Define a higher Riemann zeta function for $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ by

$$Z(s, \Lambda) = Z(s, \{\lambda_k\}_{k=0}^{\infty}) := \prod_{k=0}^{\infty} \zeta(s + \lambda_k) = \prod_{k=0}^{\infty} \prod_{p:\text{prime}} (1 - p^{-(s+\lambda_k)})^{-1}.$$

To show the absolute convergence of $Z(s, \Lambda)$, we prepare the following lemma.

**Lemma 3.2.** *Fix a positive number $x \geq 2$. If the sequence $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ satisfies* (i) $0 \leq \mathrm{Re}(\lambda_0) \leq \mathrm{Re}(\lambda_1) \leq \cdots \to \infty$, *and* (ii) *the sum $\sum_{k \in I,\, \lambda_k \neq 0} \lambda_k^{-s}$ converges absolutely for large* $\mathrm{Re}(s)$, *then we have*

$$\Big| \sum_{k=j}^{\infty} x^{-\lambda_k} \Big| \ll_j x^{-\mathrm{Re}(\lambda_j)}, \qquad \textit{for all } j \geq 0\ .$$

*Proof.* We observe that

$$\begin{aligned} \Big| \sum_{k=j}^{\infty} x^{-\lambda_k} \Big| &= \Big| x^{-\lambda_j} \cdot \sum_{k=j}^{\infty} x^{-(\lambda_k - \lambda_j)} \Big| \\ &\leq x^{-\mathrm{Re}(\lambda_j)} \cdot \sum_{k=j}^{\infty} 2^{-(\mathrm{Re}(\lambda_k) - \mathrm{Re}(\lambda_j))}. \end{aligned}$$

Under condition (ii), the sum of the second factor is bounded. □

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**Theorem 3.3.** *Assume that the sequence* $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ *satisfies conditions* (i) *and* (ii) *in the previous lemma. Then* $Z(s,\Lambda)$ *is defined for* $\mathrm{Re}(s) > 1 - \mathrm{Re}(\lambda_0)$. *Moreover* $Z(s,\Lambda)$ *has an analytic continuation as a meromorphic function on* $s \in \mathbb{C}$.

*Proof.* We see from the lemma that

$$|\sum_{k=0}^{\infty}\sum_{p} p^{-(s+\lambda_k)}| \ll \sum_{p} p^{-\mathrm{Re}(s)-\mathrm{Re}(\lambda_0)}.$$

The right side converges in $\mathrm{Re}(s) > 1 - \mathrm{Re}(\lambda_0)$. Hence $Z(s,\Lambda)$ is holomorphic on this domain. Further, we have

$$Z(s,\{\lambda_k\}_{k=0}^{\infty}) = \prod_{k=0}^{j-1}\zeta(s+\lambda_k)\cdot\prod_{k=j}^{\infty}\zeta(s+\lambda_k) = \prod_{k=0}^{j-1}\zeta(s+\lambda_k)\times Z(s,\{\lambda_k\}_{k=j}^{\infty})$$

for any $j$. From the previous lemma, we see that the second factor is now defined for $\mathrm{Re}(s) > 1 - \mathrm{Re}(\lambda_j)$. This provides a meromorphic continuation to $\mathbb{C}$. □

We remark that the higher Riemann zeta function has the Dirichlet series expression $Z(s,\Lambda) = \sum_{n=1}^{\infty} g_\Lambda(n) n^{-s}$, where

$$g_\Lambda(n) = \sum_{\substack{n_0\cdot n_1\cdot n_2\cdots = n \\ n_0\geq 1,\ n_1\geq 1,\cdots}} n_0^{-\lambda_0}\cdot n_1^{-\lambda_1}\cdot n_2^{-\lambda_2}\cdots .$$

From this expression, we see that $g_\Lambda(n)$ is multiplicative and $Z(s,\Lambda)$ has the Euler product

$$Z(s,\Lambda) = \prod_{p:\text{prime}}\sum_{m=0}^{\infty}\frac{g_\Lambda(p^m)}{p^{ms}}, \qquad \text{for } \mathrm{Re}(s) > 1 - \mathrm{Re}(\lambda_0).$$

Moreover, using the Tauberian theorem (cf. [Mu]), we obtain the behavior of $g_\Lambda(n)$ as follows.

**Corollary 3.4.** *If* $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ *satisfies* (i) *and* (ii), *and* $0 \leq \lambda_0 = \cdots = \lambda_{K-1} < \lambda_K \leq \cdots$, *then we have*

$$\sum_{n\leq x} g_\Lambda(n) = \big(c_\Lambda + o(1)\big)x^{1-\lambda_0}\cdot\log^{K-1}x \qquad \text{as } x\to\infty,$$

*where* $c_\Lambda$ *is a constant which is given by*

$$c_\Lambda := \frac{1}{(K-1)!}\lim_{s\to 1-\lambda_0}(s-1+\lambda_0)^K Z(s,\Lambda) = \frac{1}{(K-1)!}Z(1-\lambda_0,\{\lambda_k\}_{k=K}^{\infty}). \qquad \square$$

3.2. **Regularized product expression.** We first recall the regularized product expression of the Riemann zeta function.

**Lemma 3.5** ([De]). *For* $\mathrm{Re}(z) > 1$ *and* $\mathrm{Re}(s) > 1$, *we have*

$$\sum_{\rho\in R}(z-\rho)^{-s} = z^{-s} + (z-1)^{-s} - \sum_{n=0}^{\infty}(z+2n)^{-s} - \frac{1}{\Gamma(s)}\sum_{k=0}^{\infty}\sum_{p:\,prime}\frac{\log p}{p^{nz}}(\log p^n)^{s-1}, \tag{3.1}$$

*where* $R$ *denotes the set of the non-trivial zeros of the Riemann zeta function.* □

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This relation follows by applying Weil's explicit formula. For a proof, see [De]. From the lemma we see that the function $\sum_{\rho\in R}(z-\rho)^{-s}$ has a meromorphic continuation to $s\in\mathbb{C}$ and the Riemann zeta function has the regularized product expression

$$\prod_{\rho\in R}\Big(\frac{s-\rho}{2\pi}\Big)=\frac{s}{2\pi}\cdot\frac{s-1}{2\pi}\cdot\prod_{n=0}^{\infty}\Big(\frac{s+2n}{2\pi}\Big)^{-1}\cdot\zeta(s).$$

Further, using properties of the Hurwitz zeta function and the gamma function, we have

$$\prod_{\rho\in R}\Big(\frac{s-\rho}{2\pi}\Big)=2^{-1/2}\,(2\pi)^{-2}\cdot s(s-1)\cdot\pi^{-s/2}\cdot\Gamma(\frac{s}{2})\cdot\zeta(s).$$

We remark that this function is invariant under $s\leftrightarrow 1-s$.

Next we describe the regularized product expression of the higher Riemann zeta function as follows.

**Theorem 3.6.** *Let $\Lambda$ be a regularizable sequence. Then the higher Riemann zeta function $Z(s,\Lambda)$ has the following regularized product expression:*

$$\tag{3.2}\prod_{\lambda\in\Lambda,\,\rho\in R}\Big(\frac{s+\lambda-\rho}{2\pi}\Big)=\prod_{\lambda\in\Lambda}\Big(\frac{s+\lambda}{2\pi}\Big)\cdot\prod_{\lambda\in\Lambda}\Big(\frac{s-1+\lambda}{2\pi}\Big)\cdot\prod_{\lambda\in\Lambda,\,n\geq 0}\Big(\frac{s+\lambda+2n}{2\pi}\Big)^{-1}\cdot Z(s,\Lambda).$$

*Proof.* From (3.1), we see that

$$\sum_{\lambda\in\Lambda}\sum_{\rho\in R}(z+\lambda-\rho)^{-s}=\sum_{\lambda\in\Lambda}(z+\lambda)^{-s}+\sum_{\lambda\in\Lambda}(z+\lambda-1)^{-s}-\sum_{\lambda\in\Lambda}\sum_{n=0}^{\infty}(z+\lambda+2n)^{-s}-\frac{1}{\Gamma(s)}\sum_{\lambda\in\Lambda}\sum_{n=1}^{\infty}\sum_{p}\frac{\log p}{p^{n(z+\lambda)}}(\log p^n)^{s-1},$$

for $\mathrm{Re}(z)>1$ and large $\mathrm{Re}(s)$. We observe that the right-hand side is now meromorphic for any $s\in\mathbb{C}$. Therefore the dotted product $\prod_{\lambda,\,\rho}\{(z+\lambda-\rho)/2\pi\}$ exists. Further we obtain the equation (3.2) by using the relation

$$\log Z(z,\Lambda)=\sum_{\lambda\in\Lambda}\sum_{p}\sum_{n=1}^{\infty}\frac{1}{np^{n(z+\lambda)}}=\underset{s=0}{\mathrm{CT}}\,\frac{1}{s}\Big\{\frac{(2\pi)^s}{\Gamma(s)}\sum_{\lambda\in\Lambda}\sum_{n=1}^{\infty}\sum_{p}\frac{\log p}{p^{n(z+\lambda)}}(\log p^n)^{s-1}\Big\}.$$

This completes the proof. □

3.3. **Semi-lattice and functional equation.** Let $\Omega:=\{n_1\omega_1+\cdots+n_r\omega_r|n_j\in\mathbb{Z}_{\geq 0}\}$ with $\mathrm{Re}(\omega_j)>0$. Then it is seen that $\Omega$ is a regularizable sequence. Now, we consider the higher Riemann zeta function for the semi-lattice $\Omega$.

**Definition 3.7.** Define the higher Riemann zeta function of the weight $\boldsymbol{\omega}=(\omega_1\cdots\omega_r)$ by

$$Z(s,\boldsymbol{\omega}):=Z(s,\Omega)=\prod_{n_1,\cdots,n_r=0}^{\infty}\prod_{p:\mathrm{prime}}(1-p^{-(s+n_1\omega_1+\cdots+n_r\omega_r)})^{-1}.$$

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We see that this product absolutely for $\mathrm{Re}(s) > 1$ and $Z(s, \boldsymbol{\omega})$ has an analytic continuation as a meromorphic function on $s \in \mathbb{C}$. Obviously, this is invariant under the arrangement of $\omega_j$'s.

**Proposition 3.8.** *The higher Riemann zeta function $Z(s, \boldsymbol{\omega})$ defined for $Re(s) > 1$ has a meromorphic continuation to the whole complex plane and satisfies*

(3.3)

$$Z(s, (\omega_1, \cdots, \omega_{r-1}, \omega_r)) = Z(s, (\omega_1, \cdots, \omega_{r-1})) \cdot Z(s + \omega_r, (\omega_1, \cdots, \omega_{r-1}, \omega_r)),$$

$$Z(s, \boldsymbol{\omega}) \times \prod_{\substack{1 \le k_1 < \cdots < k_j \le r \\ 1 \le j \le r}} Z(s + \omega_{k_1} + \cdots + \omega_{k_j}, \boldsymbol{\omega})^{(-1)^j} = \zeta(s).$$

*Proof.* We observe that

$$\begin{aligned} Z(s, (\omega_1, \cdots, \omega_{r-1}, \omega_r)) &= \prod_{n=0}^{\infty} Z(s + n\omega_r, (\omega_1, \cdots, \omega_{r-1})) \\ &= Z(s, (\omega_1, \cdots, \omega_{r-1})) \cdot \sum_{n=0}^{\infty} Z(s + n\omega_r + \omega_r, (\omega_1, \cdots, \omega_{r-1})) \\ &= Z(s, (\omega_1, \cdots, \omega_{r-1})) \cdot Z(s + \omega_r, (\omega_1, \cdots, \omega_{r-1}, \omega_r)). \end{aligned}$$

Thus the relation (3.3) holds. By using (3.3) repeatedly, we have

$$\begin{aligned} \zeta(s) &= Z(s, (\omega_1)) \cdot Z(s + \omega_1, (\omega_1))^{-1} \\ &= Z(s, (\omega_1, \omega_2)) Z(s + \omega_2, (\omega_1, \omega_2))^{-1} \\ &\quad \cdot Z(s + \omega_1, (\omega_1, \omega_2))^{-1} Z(s + \omega_1 + \omega_2, (\omega_1, \omega_2)) \\ &= \cdots = Z(s, \boldsymbol{\omega}) \times \prod_{\substack{1 \le k_1 < \cdots < k_j \le r \\ 1 \le j \le r}} Z(s + \omega_{k_1} + \cdots + \omega_{k_j}, \boldsymbol{\omega})^{(-1)^j}. \end{aligned}$$

This completes the proof. □

Next we show the regularized product expression of $Z(s, \boldsymbol{\omega})$ as follows.

**Theorem 3.9.** *We have*

$$\begin{aligned} &\prod_{n_1, \cdots, n_r \ge 0,\ \rho \in R} \left( \frac{s + n_1\omega_1 + \cdots + n_r\omega_r - \rho}{2\pi} \right) \\ &= \exp\left\{ \left( -\frac{B_r(s, \boldsymbol{\omega})}{r!} - \frac{B_r(s-1, \boldsymbol{\omega})}{r!} + \frac{B_{r+1}(s, (2, \omega_1, \cdots, \omega_r))}{(r+1)!} \right) \log(2\pi) \right\} \\ &\times \Gamma(s, \boldsymbol{\omega})^{-1} \cdot \Gamma(s-1, \boldsymbol{\omega})^{-1} \cdot \Gamma(s, (2, \omega_1, \cdots, \omega_r)) \cdot Z(s, \boldsymbol{\omega}). \end{aligned}$$

*Here $B_n(s, \boldsymbol{\omega})$ is the multiple Bernoulli polynomial and $\Gamma(s, \boldsymbol{\omega})$ is the multiple gamma function.*

*Proof.* We note that

$$\begin{aligned} &\prod_{n_1, \cdots, n_r = 0}^{\infty} \left( \frac{z + n_1\omega_1 + \cdots + n_r\omega_r}{2\pi} \right) \\ &= (2\pi)^{-\zeta(0, z, \boldsymbol{\omega})} \prod_{n_1, \cdots, n_r = 0}^{\infty} (z + n_1\omega_1 + \cdots + n_r\omega_r). \end{aligned}$$

Then, the result follows from Theorem 3.6. □

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From this theorem, we see that the function

$$\hat{Z}(s,\boldsymbol{\omega}) := \exp\Big\{\Big(-\frac{B_r(s,\boldsymbol{\omega})}{r!} - \frac{B_r(s-1,\boldsymbol{\omega})}{r!} + \frac{B_{r+1}(s,(2,\omega_1,\cdots,\omega_r))}{(r+1)!}\Big)\log(2\pi)\Big\}$$
$$\times\Gamma(s,\boldsymbol{\omega})^{-1}\cdot\Gamma(s-1,\boldsymbol{\omega})^{-1}\cdot\Gamma(s,(2,\omega_1,\cdots,\omega_r))\cdot Z(s,\boldsymbol{\omega})$$

is an entire function of order $r+1$ with zeros at $s=\rho-n_1\omega_1-\cdots-n_r\omega_r$. Further $\hat{Z}(s,\boldsymbol{\omega})$ satisfies the relation

$$\hat{Z}(s,(\omega_1,\cdots,\omega_{r-1},\omega_r)) = \hat{Z}(s,(\omega_1,\cdots,\omega_{r-1}))\cdot\hat{Z}(s+\omega_r,(\omega_1,\cdots,\omega_{r-1},\omega_r)).$$

Finally, we give the functional equation of the higher Riemann zeta function $Z(s,\boldsymbol{\omega})$ as follows.

**Theorem 3.10.** *Define the function* $\Lambda(s,\boldsymbol{\omega})$ *by*

$$\Lambda(s,\boldsymbol{\omega}) := \hat{Z}(s,\boldsymbol{\omega})\cdot\hat{Z}(1+\omega_1+\cdots+\omega_r-s,\boldsymbol{\omega})^{(-1)^{r+1}}.$$

*Then we have*

(3.4)

$$\Lambda(s,(\omega_1,\cdots,\omega_{r-1},\omega_r)) = \Lambda(s,(\omega_1,\cdots,\omega_{r-1}))\cdot\Lambda(s+\omega_r,(\omega_1,\cdots,\omega_{r-1},\omega_r)).$$

*Moreover,* $\Lambda(s,\boldsymbol{\omega})$ *satisfies the functional equation*

$$\Lambda(s,\boldsymbol{\omega})\times\prod_{\substack{1\le k_1<\cdots<k_j\le r\\ 1\le j\le r}}\Lambda(s+\omega_{k_1}+\cdots+\omega_{k_j},\boldsymbol{\omega})^{(-1)^j} = 1. \tag{3.5}$$

*Proof.* We observe (3.3) and

$$\hat{Z}(1+\omega_1+\cdots+\omega_{r-1}+\omega_r-s,(\omega_1,\cdots,+\omega_{r-1},\omega_r))$$
$$=\frac{\hat{Z}(1+\omega_1+\cdots+\omega_{r-1}+\omega_r-(s+\omega_r),(\omega_1,\cdots,+\omega_{r-1},\omega_r))}{\hat{Z}(1+\omega_1+\cdots+\omega_{r-1}-s,(\omega_1,\cdots,\omega_{r-1}))}.$$

Then the equation (3.4) follows immediately.

Next, using (3.4) repeatedly, we have

$$\Lambda(s,\boldsymbol{\omega})\times\prod_{\substack{1\le k_1<\cdots<k_j\le r\\ 1\le j\le r}}\Lambda(s+\omega_{k_1}+\cdots+\omega_{k_j},\boldsymbol{\omega})^{(-1)^j} = \hat{\zeta}(s)\cdot\hat{\zeta}(1-s)^{-1} = 1,$$

where

$$\hat{\zeta}(s) := \prod_{\rho\in R}\big(\frac{s-\rho}{2\pi}\big) = 2^{-1/2}\,(2\pi)^{-2}\cdot s(s-1)\cdot\pi^{-s/2}\cdot\Gamma(\frac{s}{2})\cdot\zeta(s).$$

Hence the theorem follows. □

## References


[Ba] E.W.Barnes, *On the theory of the multiple gamma function.* Trans. Cambridge Philos. Soc. **19** (1904), 374-425.

[CL] H.Cohen, H.W.Lenstra, *Heuristics on class groups of number fields.* Lecture Notes in Math. **1068**, Springer, Berlin (1984), 33-62. MR0756082 (85j:11144)

[De] C.Deninger, *Local L-factors of motives and regularized determinants.* Invent. Math. **107** (1992), 135-150. MR1135468 (93a:11056)

[Go] Y.Gon, *Gamma factors of Selberg zeta functions and functional equation of Ruelle zeta functions.* Math. Ann. **308** (1997), 251-278. MR1464902 (98h:11109)

[HKW] M.Hirano, N.Kurokawa, M.Wakayama, *Half zeta functions.* J. Ramanujan Math. Soc. **18** (2003), 195-209. MR1995867 (2004m:11136)


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[HW] Y.Hashimoto, M.Wakayama, *Hierarchy of the Selberg zeta functions.* Preprint (2004).

[Ill] G.Illies, *Regularized products and determinants.* Comm. Math. Phys. **220** no.1 (2001), 69-94. MR1882400 (2003f:11131)

[JL] J.Jorgenson, S.Lang, *Basic analysis of regularized series and products.* Lecture Notes in Math. **1564**, Springer, Berlin (1993). MR1284924 (95e:11094)

[KK] N.Kurokawa, S.Koyama, *Multiple sine functions.* Forum Math. **15** (2003), 839-876. MR2010282 (2004g:11077)

[Ku1] N.Kurokawa, *Lectures on multiple sine functions.* Notes by Shinya Koyama, Univ. of Tokyo (1991), April-July.

[Ku2] N.Kurokawa, *Gamma factors and Plancherel measures.* Proc. Japan Acad. **68** Ser. A (1992), 256-260. MR1202627 (94b:11044)

[KMW] N.Kurokawa, S.Matsuda, M.Wakayama, *Gamma factors and functional equations of higher Riemann zeta functions.* Preprint (2004).

[KW1] N.Kurokawa, M.Wakayama, *Zeta extensions.* Proc. Japan Acad. **78** Ser. A (2002), 126-130. MR1930216 (2003h:11112)

[KW2] N.Kurokawa, M.Wakayama, *Higher Selberg zeta functions.* Comm. Math. Phys. **247** (2004), 447-466. MR2063268 (2005h:11200)

[KW3] N.Kurokawa, M.Wakayama, *Generalized zeta regularizations, quantum class number formulas, and Appell's O functions.* to appear in The Ramanujan J.

[Mo] T.Momotani, *Higher Selberg zeta functions for congruence subgroups.* Preprint (2004).

[Mu] M.R.Murty, V.K.Murty, *Non-vanishing of L-functions and applications.* Progress in Mathematics **157** (1997). MR1482805 (98h:11106)

[Pa] S.J.Patterson, *An introduction to the theory of the Riemann zeta function.* Cambridge Studies in Advanced Mathematics **14** (1988). MR0933558 (89d:11072)

[We] A.Weil, *Sur les "formules explicites" de la théorie des nombers premiers.* Comm. Sém. Math. Univ. Lund (1952), 252-265. MR0053152 (14:727e)



Graduate School of Mathematics, Kyushu University 6-10-1, Hakozaki Fukuoka 812-8581, Japan

*E-mail address*: momo@math.kyushu-u.ac.jp


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