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THE BANACH ALGEBRA GENERATED BY A CONTRACTION

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ABSTRACT. Let T be a contraction on a Banach space and A_T the Banach algebra generated by T. Let $\sigma_u(T)$ be the unitary spectrum (i.e., the intersection of $\sigma(T)$ with the unit circle) of T. We prove the following theorem of Katznelson-Tzafriri type: If $\sigma_u(T)$ is at most countable, then the Gelfand transform of $R \in A_T$ vanishes on $\sigma_u(T)$ if and only if $\lim_{n\to\infty} \|T^n R\| = 0$.

Let X be a complex Banach space $B(X)$, the algebra of all bounded linear operators on X, and let I be the identity operator on X. $\sigma(T)$ will denote the spectrum of an operator $T \in B(X)$, and $R_z(T) = (z-T)^{-1}$ will denote the resolvent of T. If A is a uniformly closed subalgebra of $B(X)$ with identity I, then $\sigma_A(T)$ will denote the spectrum of $T \in A$ with respect to A. If $T \in B(X)$, by A_T we will denote the uniformly closed subalgebra of $B(X)$ generated by T and I. A_T is a commutative unital Banach algebra. As is well known, the maximal ideal space of A_T can be identified with $\sigma_{A_T}(T)$. $\hat{R}$ will denote the Gelfand transform of any $R \in A_T$.

Let T be a contraction (i.e., a linear operator of norm ≤ 1) on a Banach space X. Then for every $x \in X$ the limit $\lim_{n\to\infty} \|T^n x\|$ exists and is equal to $\inf_{n\in\mathbb{N}} \|T^n x\|$. Note also that $\sigma(T) \subset \sigma_{A_T}(T) \subset \bar{D}$; $D = \{z \in \mathbb{C} : |z| < 1\}$. Let Γ be the unit circle. $\sigma_u(T) = \sigma(T) \cap \Gamma$ is called the *unitary spectrum* of T. It is easy to see that if $\sigma_u(T) = \varnothing$, then $\lim_{n\to\infty} \|T^n\| = 0$.

It follows from the Y. Katznelson and L. Tzafriri Theorem [4, Theorem 5] that if $\sigma_u(T)$ is at most countable and $f \equiv 0$ on $\sigma_u(T)$, then $\lim_{n\to\infty} \|T^n f(T)\| = 0$, and that $f(z)$ is a function analytic in D, which has absolutely convergent Taylor series. In this note we obtain the following extension of this result.

Theorem 1. *Let T be a contraction on a Banach space such that the unitary spectrum $\sigma_u(T)$ of T is at most countable. Then the Gelfand transform of $R \in A_T$ vanishes on $\sigma_u(T)$ if and only if* $\lim_{n\to\infty} \|T^n R\| = 0$.

For the proof we need some preliminary results.

The proof of the following lemma is similar to that of [7, Lemma 2.1].

Lemma 1. *Let T be a contraction on a Banach space X such that $\sigma(T) \neq \bar{D}$ and $\inf_{n\in\mathbb{N}} \|T^n x\| > 0$ for some $x \in X \setminus \{0\}$. Then there exist a Banach space $Y \neq \{0\}$, a bounded linear operator $J : X \to Y$ with dense range and a surjective isometry S*

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on Y such that:
(i) $\|Jx\| = \lim_{n\to\infty} \|T^n x\|$;
(ii) $SJ = JT$;
(iii) $\sigma(S) \subset \sigma(T)$.

Proof. On X we define the semi-norm p by $p(x) = \lim_{n\to\infty} \|T^n x\|$. Put $E = \ker p$. Then E is a closed invariant subspace of T and $E \neq X$. Let $J : X \to X/E$ be the quotient mapping. Then the semi-norm p induces a norm $\bar{p}$ on $Y_0 = X/E$ by $\bar{p}(Jx) = p(x)$, and we have $\|Jx\| = \lim_{n\to\infty} \|T^n x\|$. Let Y be the completion of Y_0 with respect to the norm $\bar{p}$. Define $S_0 : Y_0 \to Y_0$ by $S_0 J = JT$. Since $\|S_0 Jx\| = \|Jx\|$, S_0 extends to an isometry S on Y. Then we have $SJ = JT$, where $J : X \to Y$ has dense range.

Let $z \notin \sigma(T)$. From the obvious inequality $p(R_z(T)x) \leq \|R_z(T)\| p(x)$ $(x \in X)$, it follows that $\sigma(S) \subset \sigma(T)$. If S is a non-surjective isometry, then $\sigma(S) = \bar{D}$ [1, p. 27], and therefore $\sigma(T) = \bar{D}$. This contradicts $\sigma(T) \neq \bar{D}$. Hence, S is a surjective isometry. The proof is complete. □

If T is a surjective isometry on a Banach space X, then $\sigma(T) \subset \Gamma$ and

$$R_z(T) = \begin{cases} \sum_{n=0}^{\infty} z^{-n-1} T^n, & |z| > 1, \\ -\sum_{n=1}^{\infty} z^{n-1} T^{-n}, & |z| < 1. \end{cases}$$

It follows that $\|R_z(T)\| \leq ||z| - 1|^{-1}$ $(|z| \neq 1)$. Now let $f \in L^1(\mathbb{Z})$ and

$$\hat{f}(\xi) = \sum_{n\in\mathbb{Z}} f(n) \bar{\xi}^n \quad (\xi \in \Gamma),$$

the Fourier transform of f. We can define $\hat{f}(T) \in B(X)$ by

$$\hat{f}(T) = \sum_{n\in\mathbb{Z}} f(n) T^{-n}.$$

Lemma 2. *Let T be a surjective isometry on a Banach space and let $f \in L^1(\mathbb{Z})$. If $\hat{f}(\xi) = 0$ in a neighborhood of $\sigma(T)$, then $\hat{f}(T) = 0$.*

Proof. Let U be an open set in Γ that contains $\sigma(T)$. Assume that $\hat{f}$ vanishes on U. Then we have

$$\begin{aligned} \hat{f}(T) &= \lim_{r\to 1^-} \sum_{n\in\mathbb{Z}} r^{|n|} f(n) T^{-n} = \lim_{r\to 1^-} \int_{\Gamma} \hat{f}(\xi) \left(\sum_{n\in\mathbb{Z}} r^{|n|} \xi^n T^{-n} \right) d\xi \\ &= \lim_{r\to 1^-} \int_{\Gamma - U} \hat{f}(\xi) \left(TR_{r^{-1}\xi}(T) - TR_{\xi}(T) \right) d\xi \\ &\quad + \lim_{r\to 1^-} \int_{\Gamma - U} \hat{f}(\xi) \left(TR_{\xi}(T) - TR_{r\xi}(T) \right) d\xi = 0. \end{aligned}$$

□

We will also need the following notation.

Recall that $\varphi = \{\varphi(n)\}_{n\in\mathbb{Z}} \in L^\infty(\mathbb{Z})$ is *almost periodic* on $\mathbb{Z}$ if $\{\varphi_m : m \in \mathbb{Z}\}$ is relatively compact in the norm topology of $L^\infty(\mathbb{Z})$, where $\varphi_m(n) = \varphi(n+m)$. We denote by $AP(\mathbb{Z})$ the set of all almost periodic functions on $\mathbb{Z}$. $AP(\mathbb{Z})$ is a closed subalgebra of $L^{\infty}(\mathbb{Z})$. As is well known, there exists a unique $\Phi \in AP(\mathbb{Z})^*$ (which is called *invariant mean* on $AP(\mathbb{Z})$) such that:

(i) $\Phi(\mathbf{1}) = 1$, where $\mathbf{1}(n) \equiv 1$;

(ii) $\Phi(\varphi) \geq 0$ for all $\varphi \geq 0$;

(iii) $\Phi(\varphi_m) = \Phi(\varphi)$ for all $\varphi \in L^{\infty}(\mathbb{Z})$ and $m \in \mathbb{Z}$.

The *hull* of any ideal $I \subset L^1(\mathbb{Z})$ is $h(I) = \left\{\xi \in \Gamma : \hat{f}(\xi) = 0, f \in I\right\}$. For a closed subset $K \subset \Gamma$, let $I_K = \left\{f \in L^1(\mathbb{Z}) : \hat{f}(K) = \{0\}\right\}$ and $J_K^0 = \{f \in L^1(\mathbb{Z}) : supp \hat{f} \cap K = \varnothing\}$. K is a set of *synthesis* if $I_K = \overline{J_K^0}$. For example, closed countable sets are sets of synthesis. As is well known (Malliavin's theorem), there exists a non-synthesis set (see [5, chap. 8]).

For $\varphi \in L^{\infty}(\mathbb{Z})$ and $f \in L^1(\mathbb{Z})$, $\varphi * f$ will denote the convolution of φ and f. Recall that the w^*-*spectrum* $\sigma_*(\varphi)$ of $\varphi \in L^{\infty}(\mathbb{Z})$ is defined as the hull of the closed ideal $I_{\varphi} = \left\{f \in L^1(\mathbb{Z}) : \varphi * f = 0\right\}$. The well-known theorem of Loomis [6] states that if the w^*-spectrum of $\varphi \in L^{\infty}(\mathbb{Z})$ is at most countable, then $\varphi \in AP(\mathbb{Z})$.

Lemma 3. *Let S be a surjective isometry on a Banach space Y such that $\sigma(S)$ is at most countable. Then for every $\phi \in Y^*$, there exists a Hilbert space H_{ϕ}, a bounded linear operator $J_{\phi} : Y \to H_{\phi}$ with dense range and a unitary operator U_{ϕ} on H_{ϕ} such that:*

(i) $U_{\phi} J_{\phi} = J_{\phi} S$;

(ii) $\sigma\left(U_{\phi}^*\right) \subset \sigma(S)$.

Proof. (i) Let $\phi \in Y^*$. For given $y \in Y$, define the function $\bar{y}_{\phi}$ on $\mathbb{Z}$ by $\bar{y}_{\phi}(n) = \phi(S^n y)$. Since $\|\bar{y}_{\phi}\|_{\infty} \leq \|\phi\| \|y\|$, $\bar{y}_{\phi}$ is a bounded function. We claim that $\sigma_*(\bar{y}_{\phi}) \subset \sigma(S)$. Assume that there exists a $\xi_0 \in \sigma_*(\bar{y}_{\phi})$, but $\xi_0 \notin \sigma(S)$. Then there exists an $f \in L^1(\mathbb{Z})$ such that $\hat{f}(\xi_0) \neq 0$ and $\hat{f}(\xi) = 0$ on some neighborhood of $\sigma(S)$. By Lemma 2, $\hat{f}(S) = 0$ and consequently,

$$0 = \phi\left(S^n \hat{f}(S)\, y\right) = (\bar{y}_{\phi} * f)(n), \quad \text{for all } n \in \mathbb{Z}.$$

Since $\xi_0 \in \sigma_*(\bar{y}_{\phi})$ it follows that $\hat{f}(\xi_0) = 0$. This contradiction proves the claim. Hence, $\sigma_*(\bar{y}_{\phi})$ is at most countable. By the Loomis Theorem [6], $\bar{y}_{\phi} \in AP(\mathbb{Z})$.

Let H_{ϕ}^0 denote the linear set $\{\bar{y}_{\phi} : y \in Y\}$ with the inner product defined by

$$\langle \bar{y}_{\phi}, \bar{z}_{\phi} \rangle = \Phi\left(\left\{\bar{y}_{\phi}(n)\, \overline{\bar{z}_{\phi}(n)}\right\}_{n \in \mathbb{Z}}\right), \quad z \in Y,$$

where Φ is the invariant mean on $AP(\mathbb{Z})$. Let H_{ϕ} be the completion of H_{ϕ}^0 with respect to the norm

$$\|\bar{y}_{\phi}\|_2^2 = \Phi\left(\left\{|\bar{y}_{\phi}(n)|^2\right\}_{n \in \mathbb{Z}}\right).$$

Then H_{ϕ} is a Hilbert space. Note also that $\|\bar{y}_{\phi}\|_2 \leq \|\bar{y}_{\phi}\|_{\infty} \leq \|\phi\| \|y\|$. It follows that the map $J_{\phi} : Y \to H_{\phi}$, defined by $J_{\phi} y = \bar{y}_{\phi}$, is a bounded linear operator with dense range. Now define the map $U_{\phi} : H_{\phi} \to H_{\phi}$, by $U_{\phi} \bar{y}_{\phi} = \left(\overline{Sy}\right)_{\phi}$. It is easy to see that U_{ϕ} is a unitary operator and $U_{\phi} J_{\phi} = J_{\phi} S$. We have proved (i).

Next we prove (ii). We have

$$S^* J_{\phi}^* = J_{\phi}^* U_{\phi}^*. \tag{1}$$

Assume that there exists $\xi \in \sigma\left(U_\phi^*\right)$, but $\xi \notin \sigma(S) = \sigma(S^*)$. Put $\delta = \|(S^* - \xi)^{-1}\|^{-1}$. Choose $\varepsilon > 0$ such that $\varepsilon < \delta$. Let $\Gamma_\varepsilon = \{z \in \mathbb{C} : |z - \xi| < \varepsilon\} \cap \Gamma$ and let $E(\cdot)$ be the spectral measure for U_ϕ^*. Since $\sigma\left(U_\phi^*\right) \cap \Gamma_\varepsilon \neq \varnothing$, we have $E(\Delta_\varepsilon) \neq 0$. Let $h \in E(\Delta_\varepsilon) H_\phi$ be such that $\|h\| = 1$. From the identity

$$\left(U_\phi^* - \xi\right)^n h = \int_{\Gamma_\varepsilon} (t - \xi)^n dE(t)\, h, \quad n \in \mathbb{N},$$

we have

$$\left\|\left(U_\phi^* - \xi\right)^n h\right\| \leq \varepsilon^n.$$

On the other hand from (1) we can write

$$(S^* - \xi)^n J_\phi^* h = J_\phi^* \left(U_\phi^* - \xi\right)^n h.$$

It follows that

$$\left\|(S^* - \xi)^n J_\phi^* h\right\| \leq \varepsilon^n \left\|J_\phi^*\right\|.$$

Consequently, we have

$$\left\|J_\phi^* h\right\| \leq \left\|(S^* - \xi)^{-n}\right\| \left\|(S^* - \xi)^n J_\phi^* h\right\| \leq \left(\frac{\varepsilon}{\delta}\right)^n \left\|J_\phi^*\right\| \to 0, \text{ as } n \to \infty.$$

Hence, $J_\phi^* h = 0$. Since J_ϕ^* has zero kernel, we obtain $h = 0$. This is a contradiction. The proof is complete. □

Proof of Theorem 1. Let $R \in A_T$. Assume that $\lim_{n\to\infty} \|T^n R\| = 0$. Then for any $\xi \in \sigma_u(T)$, $\left|\hat{T}^n(\xi)\hat{R}(\xi)\right| \to 0$, as $n \to \infty$. Since $\left|\hat{T}(\xi)\right| \equiv 1$, it follows that $\hat{R}(\xi) = 0$. Now assume that $\sigma_u(T)$ is at most countable and $\hat{R}(\xi) \equiv 0$ on $\sigma_u(T)$. It is enough to prove that $\lim_{n\to\infty} \|T^n R x\| = 0$ for all $x \in X$. Indeed, suppose that this is proved. For $C \in B(X)$, let L_C be the left multiplication operator on $B(X)$ defined by $L_C F = CF$. Then L_T is a contraction. Moreover, the maximal ideal spaces of A_{L_T} and A_T are the same, and $\sigma(L_T) = \sigma(T)$. Note also that $L_R \in A_{L_T}$ and the Gelfand transform of L_R vanishes on $\sigma_u(L_T)$. Therefore,

$$\lim_{n\to\infty} \|L_T^n L_R F\| = 0,$$

for all $F \in B(X)$. Taking $F = I$, we get the desired conclusion.

If $\lim_{n\to\infty} \|T^n x\| = 0$ for all $x \in X$, then there is nothing to prove. Hence, we may assume that $\lim_{n\to\infty} \|T^n x\| > 0$ for some $x \neq 0$. On the other hand, since $\sigma_u(T)$ is at most countable, $\sigma(T) \neq \bar{D}$. In view of Lemma 1 there exists a Banach space $Y \neq \{0\}$, a bounded linear operator $J : X \to Y$ with dense range and a surjective isometry S on Y such that:

(i) $\|Jx\| = \lim_{n\to\infty} \|T^n x\|$;
(ii) $SJ = JT$;
(iii) $\sigma(S) \subset \sigma(T)$.

It follows from (iii) that $\sigma(S) \subset \sigma_u(T)$ and therefore, $\sigma(S)$ is at most countable. Now let $\phi \in Y^*$ be given. By Lemma 3, there exists a Hilbert space H_ϕ, a bounded linear operator $J_\phi : Y \to H_\phi$ with dense range and a unitary operator U_ϕ on H_ϕ such that

$$U_\phi J_\phi = J_\phi S \tag{2}$$

and $\sigma\left(U_{\phi}^{*}\right) \subset \sigma(S) \subset \sigma_u(T)$. Moreover, from (ii) and (2) we obtain

$$U_{\phi} J_{\phi} J = J_{\phi} J T. \tag{3}$$

Further, since $R \in A_T$, there exists a sequence of polynomials $\{P_n(z)\}_{n\in\mathbb{N}}$ such that $\|P_n(T) - R\| \to 0$. Also since the Gelfand transform of R vanishes on $\sigma_u(T)$, the sequence $\{P_n(z)\}_{n\in\mathbb{N}}$ uniformly converges to zero on $\sigma_u(T)$. Hence, the sequence $\{P_n(z)\}_{n\in\mathbb{N}}$ uniformly converges to zero on $\sigma(U_{\phi}^{*})$. It follows that $\|P_n\left(U_{\phi}^{*}\right)\| \to 0$. On the other hand, from the identity (3) we can write

$$J^{*} J_{\phi}^{*} P_n\left(U_{\phi}^{*}\right) = P_n(T^{*}) J^{*} J_{\phi}^{*}.$$

This clearly implies that $R^{*} J^{*} J_{\phi}^{*} = 0$ and so $J_{\phi} J R = 0$. Hence, $\phi(JRx) = 0$ for all $\phi \in Y^{*}$ and $x \in X$. Thus, we obtain that

$$0 = \|JRx\| = \lim_{n\to\infty} \|T^n R x\|,$$

for all $x \in X$. This completes the proof. □

We do not know whether Theorem 1 true if $\sigma_u(T)$ is a synthesis set.

Remark 1. Theorem 1 remains valid if $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$. Indeed, in this case $|||x||| = \sup_{n\in\mathbb{N}} \|T^n x\|$ is an equivalent norm on X with respect to which T becomes a contraction.

In contrast with the unitary operator on a Hilbert space, there exists a surjective isometry on a Banach space that generates a non-semisimple algebra (see [3]). But surjective isometry on a Banach space with countable spectrum generated a semisimple algebra [3]. The following example shows that even on a Hilbert space there exists a contraction with countable unitary spectrum that generates a non-semisimple algebra: Let V be the Volterra operator on $L^2[0,1]$ defined by $(Vf)(t) = \int_0^t f(s)ds$ and let $T = (I+V)^{-1}$. Then $\|T^n\| = 1$ for all $n \in \mathbb{N}$ and $\sigma(T) = \{1\}$. But $T \neq I$.

Recall that a contraction T on a Banach space X is said to be a C_1-*contraction* if $\inf_{n\in\mathbb{N}} \|T^n x\| > 0$ for all $x \in X \setminus \{0\}$ [1, p. 250].

Corollary 1. *Let T be a C_1-contraction on a Banach space such that the unitary spectrum $\sigma_u(T)$ of T is at most countable. If the Gelfand transform of $R \in A_T$ vanishes on $\sigma_u(T)$, then $R = 0$. In particular, A_T is semisimple.*

For contractions on a Hilbert space, the Katznelson-Tzafriri theorem can be improved as follows [2]: If T is a contraction on a Hilbert space and $f \in A(D)$ vanishes on $\sigma_u(T)$, then

$$\lim_{n\to\infty} \|T^n f(T)\| = 0;$$

$A(D)$ is the disc algebra. Now let $R \in A_T$ be such that $\hat{R}(\xi) \equiv 0$ on $\sigma_u(T)$. Is it then true that $\lim_{n\to\infty} \|T^n R\| = 0$? We do not know the answer to this question.

However, we prove the following.

Theorem 2. *Let H be a Hilbert space and let T be a contraction on H. If the Gelfand transform of $R \in A_T$ vanishes on $\sigma_u(T)$, then*

$$\lim_{n\to\infty} \|T^n R x\| = 0,$$

for all $x \in H$.

Proof. Let $R \in A_T$ be such that $\hat{R}(\xi) \equiv 0$ on $\sigma_u(T)$. We define a new inner product on H by the formula

$$[x, y] = \lim_{n\to\infty} \langle T^n x, T^n y\rangle$$

(it is easy to see that the limit on the right-hand side exists). This induced a semi-norm on H defined by

$$p(x) = \left(\lim_{n\to\infty} \|T^n x\|^2\right)^{1/2}.$$

Let $E = \ker p$. It is clear that E is a closed invariant subspace of T. If $E = H$, then there is nothing to prove. Hence, we may assume that $E \neq H$. Let $J : H \to H/E$ be the quotient mapping. Then the semi-norm p induces a norm $\bar{p}$ on $K_0 = H/E$ by $\bar{p}(Jx) = p(x)$, and we have

$$\|Jx\| = \left(\lim_{n\to\infty} \|T^n x\|^2\right)^{1/2}.$$

Let K be the completion of K_0 with respect to the norm $\bar{p}$. Define $U_0 : K_0 \to K_0$ by $U_0 J = JT$. Since $\|U_0 Jx\| \leq \|T\| \|Jx\|$, U_0 extends to a bounded operator U on K. Then we have $UJ = JT$, where $J : H \to K$ has dense range. Also since

$$[Up(x), Up(y)] = [p(x), p(y)], \quad x, y \in H,$$

U is an isometry on K. As in the proof of Lemma 1 we can see that $\sigma(U) \subset \sigma(T)$.

Now assume that U is a non-surjective isometry. Then $\sigma(U) = \bar{D}$ and consequently, $\sigma(T) = \bar{D}$. Hence, $\sigma_u(T) = \Gamma$. Since $R \in A_T$, there exists a sequence $\{P_n(z)\}_{n\in\mathbb{N}}$ of polynomials such that $\|P_n(T) - R\| \to 0$. It follows that $P_n(z) \to 0$ uniformly on Γ. By the von Neumann inequality,

$$\|P_n(T)\| \leq \sup_{\xi\in\Gamma} |P_n(\xi)| \to 0,$$

and so $R = 0$. Hence, we may assume that U is a unitary operator. As above, there exists a sequence $\{P_n(z)\}_{n\in\mathbb{N}}$ of polynomials such that $\|P_n(T) - R\| \to 0$. It follows that $P_n(z) \to 0$ uniformly on $\sigma_u(T)$. Since $\sigma(U) \subset \sigma_u(T)$, we have $\|P_n(U)\| \to 0$. Now from the identity $P_n(U) J = JP_n(T)$ we obtain that $JR = 0$. Hence we have that $\lim_{n\to\infty} \|T^n Rx\| = 0$ for all $x \in H$. The proof is complete. □

A similar result holds for the power-bounded operators.

Theorem 3. *Let T be a power-bounded operator on a Hilbert space H. If the Gelfand transform of $R \in A_T$ vanishes on $\sigma_u(T)$, then for all $x \in H$,*

$$l.i.m. \|T^n Rx\| = 0,$$

where l.i.m. is a Banach limit on $\mathbb{N}$.

Corollary 2. *Let T be a contraction on a Hilbert space. If $R \in A_T$ is a compact operator and $\hat{R}(\xi) \equiv 0$ on $\sigma_u(T)$, then*

$$\lim_{n\to\infty} \|T^n R\| = 0.$$

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