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A GENERALIZATION OF ANDÔ'S THEOREM AND PARROTT'S EXAMPLE

DAVID OPĚLA

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ABSTRACT. Andô's theorem states that any pair of commuting contractions on a Hilbert space can be dilated to a pair of commuting unitaries. Parrott presented an example showing that an analogous result does not hold for a triple of pairwise commuting contractions. We generalize both of these results as follows. Any n-tuple of contractions that commute according to a graph without a cycle can be dilated to an n-tuple of unitaries that commute according to that graph. Conversely, if the graph contains a cycle, we construct a counterexample.

1. INTRODUCTION

Foiaş and Sz.-Nagy's theory models Hilbert space operators as "parts" of simpler operators. We will begin by recalling its main results; for the proofs see Chapter 10 in [AM-02]. Very detailed treatment is in [SF-70]. Recent books [Pau-02], [Pi-01] contain up-to-date expositions of some aspects of the theory.

If $\mathcal{H} \subset \mathcal{K}$ are two Hilbert spaces, $T \in \mathcal{B}(\mathcal{H})$, $W \in \mathcal{B}(\mathcal{K})$ operators, we say that W is an *extension* of T, if T is the restriction of W to $\mathcal{H}$, i.e., if $Tx = Wx$ for all $x \in \mathcal{H}$. With the same notation, we say that W is a *dilation* of T, or that T is a *compression* of W, if $T^n = PW^n \restriction \mathcal{H}$, for all $n \geq 0$, where P is the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$. By a result of Sarason, we can say equivalently, that W has the following structure:

$$W = \begin{pmatrix} T & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \begin{matrix} \dots & \mathcal{H}, \\ \dots & \tilde{\mathcal{K}} \ominus \mathcal{H}, \\ \dots & \mathcal{K} \ominus \tilde{\mathcal{K}}, \end{matrix}$$

that is, $\mathcal{H}$ is the orthogonal difference of two invariant subspaces for W.

Recall that $A \in \mathcal{B}(\mathcal{H})$ is a *co-isometry*, if A^* is an isometry, that is, if $AA^* = I$.

The first result of the theory, due to Sz.-Nagy, asserts that every contraction $T \in \mathcal{B}(\mathcal{H})$ has a co-isometric extension. In fact, there is a *minimal* co-isometric extension $W_0 \in \mathcal{B}(\mathcal{K}_0)$ of T that is characterized by

$$\mathcal{K}_0 = \overline{\operatorname{span}}\, \{(W_0^*)^n \mathcal{H};\ n \in \mathbb{N}\}.$$

Any two minimal co-isometric extensions W_0 and $\widetilde{W_0}$ are unitarily equivalent via a unitary that restricts to the identity on $\mathcal{H}$ — we denote this by $W_0 \cong \widetilde{W_0}$. For

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any co-isometric extension W of T we have $W \cong W_0 \oplus \widetilde{W}$, where $\widetilde{W}$ is another co-isometry. If T is already a co-isometry, then any of its co-isometric extensions is the direct sum of T and another co-isometry. A minimal co-isometric extension of an isometry is a unitary. If T is a contraction, denote by V any of the co-isometric extensions of the contraction T^*. Let U be a minimal co-isometric extension of the isometry V^*. Then one easily checks (by Sarason's characterization) that U is a dilation of A. The Sz.-Nagy dilation theorem follows — any contraction has a unitary dilation. Given $T \in \mathcal{B}(\mathcal{H})$ there is a *minimal* unitary dilation U_0, unique up to a unitary that restricts to $I_{\mathcal{H}}$, such that any unitary dilation U of T has the form $U \cong U_0 \oplus U_1$.

Let us stress that both the extension and the dilation result above say that a general contraction is a part of a simpler operator. Indeed, unitaries are well-understood via the spectral theorem. As for co-isometries, by the Wold decomposition, any co-isometry is the orthogonal sum of (a finite or infinite number of) copies of the unilateral backward shift and a unitary.

Both of the above results can be generalized to a pair of commuting contractions. Andô's theorem [A-63] states that given a pair of commuting contractions, we can extend each of them to a co-isometry such that the two co-isometries commute. As in the single operator case, if the contractions are isometries, the co-isometries can be constructed to be unitaries. The dilation version asserts that for any pair of commuting contractions A, B, we can find commuting unitaries U, V such that

$$PU^mV^n \restriction \mathcal{H} = A^mB^n, \text{ for all } m, n \geq 0.$$

Note that the equality above implies that U (resp. V) is a dilation of A (resp. B) by taking $n = 0$ ($m = 0$, respectively). However, not every pair of dilations satisfies this property. This more restrictive relation is more desirable, since it implies that the map $U \mapsto A, V \mapsto B$ extends to an algebra homomorphism between the operator algebras generated by U, V and A, B, respectively.

There is a related theorem whose modification we will use. The commutant lifting theorem of Foiaş and Sz.-Nagy asserts that given a pair of commuting contractions and a co-isometric extension (or unitary dilation) of one of them, one can extend (or dilate) the other one to a contraction that commutes with the given co-isometric extension (unitary dilation, respectively).

Surprisingly, Andô's theorem cannot be generalized to three (or more) commuting contractions. The first counterexample was constructed by S. Parrott; see [Par-70]. There are some sufficient conditions on when an n-tuple of commuting contractions can be dilated to an n-tuple of commuting unitaries, e.g., if the operators doubly commute; see Theorem 12.10 in [Pau-02].

There are generalizations of Andô's result to an n-tuple of contractions. Gaşpar and Rácz assume only that the n-tuple is *cyclic commutative*; see [GR-69]. Their result was further generalized by G. Popescu [Po-86]. Instead of starting with n-tuples of contractions, one can work with the so-called *row contractions*, that is, with n-tuples satisfying $\sum_j T_jT_j^* \leq I$. This case has been extensively studied; see the recent survey [B-02] and the references therein.

We derive a different generalization of Andô's theorem, namely, we assume that only some of the $\binom{n}{2}$ pairs commute; see the definition below.

Before we will be able to formulate our results, let us recall a few basic definitions and facts from graph theory. A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a set (the elements of $V(G)$ are called *vertices* of G) and $E(G)$ is a set of unordered pairs of distinct vertices — these are called *edges*. A graph G' is a *subgraph* of a graph G, if $V(G') \subset V(G)$ and $E(G') \subset E(G)$. A *cycle of length* n is the graph G with $V(G) = \{v_1, \dots, v_n\}$ and $E(G) = \{(v_j, v_{j+1});\ 1 \le j \le n-1\} \cup \{(v_1, v_n)\}$. A graph G is *connected,* if for any two vertices v, w there exists a sequence of vertices $\{v_k\}_{k=0}^m \subset V(G)$ with $v_0 = v$, $v_m = w$ and such that $(v_j, v_{j+1}) \in E(G)$ for $j = 0, \dots, m-1$. A graph is *acyclic*, if it does not contain a cycle as a subgraph, and a connected acyclic graph is called a *tree*. Every tree has a vertex that lies on exactly one edge.

Definition 1.1. Let $A_1, A_2, \dots, A_n \in \mathcal{B}(\mathcal{H})$ be an n-tuple of operators, and let G be a graph on the vertices $\{1, 2, \dots, n\}$. We say that the operators $A_1, \dots, A_n$ *commute according to* G, if $A_i A_j = A_j A_i$ whenever (i, j) is an edge of G.

We prove that given G, every n-tuple of contractions commuting according to G has unitary dilations that commute according to G, if and only if G is acyclic. Andô's theorem is a special case, when G is the acyclic graph consisting of two vertices and an edge connecting them. The case of three commuting contractions, Parrott's example, corresponds to the cycle of length three. If G is a graph with no edges, the result is also known; see Exercise 5.4, p. 71 in [Pau-02].

2. The main result

Our main result is Theorem 2.3. We will make use of the following lemma that can be regarded as a version of the commutant lifting theorem.

Lemma 2.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be commuting contractions and let $\tilde{X} \in \mathcal{B}(\tilde{\mathcal{K}})$ be a co-isometric extension of A. Then there exists a Hilbert space $\mathcal{K}$ containing $\tilde{\mathcal{K}}$ and commuting co-isometries $X, Y \in \mathcal{B}(\mathcal{K})$ such that X extends $\tilde{X}$ and Y extends B. Moreover, if A, B are isometries and $\tilde{X}$ is a unitary, we can construct X and Y to be unitary.*

Proof. Let $X_0 \in \mathcal{B}(\mathcal{K}_0)$ be a minimal co-isometric extension of A. Then $\tilde{X} \cong X_0 \oplus X_1$, where $X_1 \in \mathcal{B}(\mathcal{K}_1)$ is a co-isometry. By taking a different minimal co-isometric extension, we may assume that $\tilde{X} = X_0 \oplus X_1$. By Andô's theorem there exist commuting co-isometries $X_A, Y_A \in \mathcal{B}(\mathcal{K}_A)$ extending A, B, respectively. Hence, $X_A = X_0 \oplus X_2$, where $X_2 \in \mathcal{B}(\mathcal{K}_2)$ is a co-isometry and Y_A has a corresponding decomposition

$$Y_A = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}.$$

We set $\mathcal{K} := \mathcal{K}_0 \oplus \mathcal{K}_2 \oplus \mathcal{K}_1$,

$$X := X_0 \oplus X_2 \oplus X_1, \qquad \text{and} \qquad Y := \begin{pmatrix} Y_{11} & Y_{12} & 0 \\ Y_{21} & Y_{22} & 0 \\ 0 & 0 & I_{\mathcal{K}_1} \end{pmatrix}.$$

Clearly, X, Y extend $\tilde{X}, B$, respectively. Also, it is easy to check that X and Y are commuting co-isometries.

To prove the second part, one follows the above proof, uses the fact that X_A, Y_A can be chosen to be unitary and then verifies that X, Y will also be unitary. $\square$

We will construct the unitary dilations using co-isometric extensions in a way sketched in the Introduction for a single contraction. Thus we need the following lemma.

Lemma 2.2. *Let G be a graph without a cycle on vertices $\{1, 2, \ldots, n\}$ and let $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ an n-tuple of contractions that commute according to G. Then there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and an n-tuple of co-isometries $X_1, \ldots, X_n \in \mathcal{B}(\mathcal{K})$ that commute according to G and such that X_j extends A_j, for $j = 1, 2, \ldots, n$. Moreover, if the A_j's are all isometries, the X_j's can be chosen to be unitaries.*

Proof. We may assume that G is connected since the result for trees easily implies the general case. Indeed, otherwise we consider each component separately to get co-isometric extensions $\tilde{X}_j \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K}_i)$ for j lying in the i-th component of G. Then we denote $\mathcal{K} := \mathcal{H} \oplus \bigoplus_{i=1}^{k} \mathcal{K}_i$, where k is the number of components of G. We define $X_j := \tilde{X}_j \oplus \bigoplus_{l \neq i} I_{\mathcal{K}_l}$, where again, j lies in the i-th component of G. One easily checks that the X_j's are co-isometries that commute according to G.

We will proceed by induction on n. If $n = 1$, both statements are true by the results about a single contraction. For the induction step assume without loss of generality that n is a vertex with only one neighbor $n - 1$. Let $\tilde{G}$ be the graph obtained from G by deleting the vertex n and the edge $(n - 1, n)$. Then $\tilde{G}$ is a tree, and so we can apply the induction hypothesis to $\tilde{G}$ and $A_1, \ldots, A_{n-1}$ to get co-isometric extensions $\tilde{X}_1, \ldots, \tilde{X}_{n-1} \in \mathcal{B}(\tilde{\mathcal{K}})$ of $A_1, \ldots, A_{n-1}$ that commute according to $\tilde{G}$. Now we apply Lemma 2.1 to A_{n-1}, A_n and $\tilde{X}_{n-1}$ to get commuting co-isometries $X_{n-1}, X_n \in \mathcal{B}(\mathcal{K})$ that extend $\tilde{X}_{n-1}, A_n$, respectively. Since $\tilde{X}_{n-1}$ is a co-isometry, we have $X_{n-1} = \tilde{X}_{n-1} \oplus X$ and we let $X_j := \tilde{X}_j \oplus I_{\mathcal{K} \ominus \tilde{\mathcal{K}}} \in \mathcal{B}(\mathcal{K})$, for $j = 1, 2, \ldots, n-2$. It is now easy to verify that $X_1, \ldots, X_n$ commute according to G.

For the statement involving isometries one can verify that the induction argument works and produces unitaries. □

Theorem 2.3. *Let G be an acyclic graph on n vertices $\{1, 2, \ldots, n\}$. Then for any n-tuple of contractions $A_1, A_2, \ldots, A_n$ on a Hilbert space $\mathcal{H}$ that commute according to G, there exists an n-tuple of unitaries $U_1, U_2, \ldots, U_n$ on a Hilbert space $\mathcal{K}$ that commute according to G and such that*

$$PU_{j_1}U_{j_2}\ldots U_{j_k} \restriction \mathcal{H} = A_{j_1}A_{j_2}\ldots A_{j_k}, \tag{D}$$

for all $k \in \mathbb{N}$, $j_i \in \{1, 2, \ldots, n\}$, $1 \leq i \leq k$. Here $P : \mathcal{K} \to \mathcal{H}$ is the orthogonal projection.

Conversely, if G contains a cycle, there exists an n-tuple of contractions that commute according to G with no n-tuple of unitaries dilating them that also commute according to G.

Proof. To prove the first statement apply Lemma 2.2 to the contractions $A_1^*, \ldots, A_n^*$ — they commute according to G. We obtain an n-tuple of co-isometries $X_1, \ldots, X_n \in \mathcal{B}(\mathcal{K}_1)$ commuting according to G and such that X_j extends A_j^* for all j. In other words, we have

$$X_j = \begin{pmatrix} A_j^* & * \\ 0 & * \end{pmatrix}, \qquad \text{so that} \qquad X_j^* = \begin{pmatrix} A_j & 0 \\ * & * \end{pmatrix}.$$

Now we apply the second statement of Lemma 2.2 to the n-tuple of isometries $X_1^*, \ldots, X_n^*$. We obtain unitaries $U_1, \ldots, U_n \in \mathcal{B}(\mathcal{K})$ that commute according to G

and extend the X_j^*'s, that is,

$$U_j = \begin{pmatrix} A_j & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \qquad \text{for } j = 1, 2, \dots, n.$$

An easy computation with matrices reveals that the U_j's satisfy the condition (D). Thus we are done with the first part.

For the converse, we may assume that G is a cycle. Indeed, otherwise we extend the example below by taking the remaining A_i's equal to the identity operator. So, without loss of generality, the edges of G are $(i, i+1)$, for $i = 1, \dots, n-1$ and $(n, 1)$ with $n \geq 3$. Let $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_0$, where $\mathcal{H}_0$ is (an at least two-dimensional) Hilbert space. Let

$$A_j := \begin{pmatrix} 0 & 0 \\ B_j & 0 \end{pmatrix},$$

where $B_j \in \mathcal{B}(\mathcal{H}_0)$ is $I_{\mathcal{H}_0}$ for $j \neq 1, n$, $B_1 = R$, $B_n = T$ and R, T are non-commuting unitaries. Then $A_i A_j = 0$ for all i, j and so the n-tuple is commuting (and thus commuting according to G). It is well known (and easy to check) that the minimal unitary dilation of B_2 is the "inflated" bilateral shift $\mathcal{S}$ on $\mathcal{K}_0 := \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_0$. Let us describe this in more detail. We identify the space $\mathcal{H}$ with the sum of the zeroth and the first copy of $\mathcal{H}_0$ in $\mathcal{K}_0$, that is, $(x, y) \in \mathcal{H}$ is equal to $\bigoplus_{k \in \mathbb{Z}} x_k$, with $x_0 = x$, $x_1 = y$ and $x_k = 0$, for $k \neq 0, 1$. Then we have

$$\mathcal{S} \bigoplus_{k \in \mathbb{Z}} x_k = \bigoplus_{k \in \mathbb{Z}} x_{k-1}.$$

Assume that there exist unitaries $U_1, \dots, U_n \in \mathcal{B}(\mathcal{K})$ dilating the n-tuple and that they commute according to G. Then $U_2 \cong \mathcal{S} \oplus \tilde{U}_2$ and U_3 has a corresponding decomposition

$$U_3 = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{matrix} \dots & \mathcal{K}_0, \\ \dots & \mathcal{K} \ominus \mathcal{K}_0. \end{matrix}$$

Since U_2 and U_3 commute, it follows that $\mathcal{S}$ commutes with Y_{11} and, consequently, Y_{11} is a Laurent operator on $\mathcal{K}_0 = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_0$. Thus Y_{11} has constant diagonals. Hence, it has $I_{\mathcal{H}_0}$ on the diagonal just below the main diagonal. Since U_3 is a unitary, Y_{11} is a contraction, and so all the other entries of Y_{11} are zero. Thus $Y_{11} = \mathcal{S}$, and so it is a unitary. Since U_3 is a unitary, Y_{12} and Y_{21} must be both zero and so U_3 reduces $\mathcal{K}_0$, i.e., $U_3 = \mathcal{S} \oplus \tilde{U}_3$.

Applying the same argument repeatedly implies $U_4 = \mathcal{S} \oplus \tilde{U}_4, \dots, U_{n-1} = \mathcal{S} \oplus \tilde{U}_{n-1}$. For U_1 and U_n we get $U_1 = \mathcal{R} \oplus \tilde{U}_1$ and $U_n = \mathcal{T} \oplus \tilde{U}_n$, where $\mathcal{R}, \mathcal{T} \in \mathcal{B}(\mathcal{K}_0)$ also have constant diagonal. More precisely,

$$\mathcal{R} \bigoplus_{k \in \mathbb{Z}} x_k = \bigoplus_{k \in \mathbb{Z}} R x_{k-1}, \qquad \mathcal{T} \bigoplus_{k \in \mathbb{Z}} x_k = \bigoplus_{k \in \mathbb{Z}} T x_{k-1}.$$

Now we are done. Indeed, R, T do not commute, hence $\mathcal{R}, \mathcal{T}$ do not commute, and so U_1, U_n also do not commute, a contradiction. □

Remark 2.4. In the construction of the example for the converse part, we modified the original idea of Parrott [Par-70]. G. Pisier pointed out to us a related construction; see Example 25.19 in [Pi-03].

Remark 2.5. The above theorem holds for infinite graphs.

Proof. Since cycles are always finite, the second part can be proven in exactly the same manner.

To prove the first part, we use the same strategy as in the finite case. We first observe that we may assume that G is connected by the same argument. Thus we only need to prove an infinite version of Lemma 2.2. We will use Zorn's lemma. Fix a family of contractions $\{A_v\}_{v\in V(G)} \subset \mathcal{B}(\mathcal{H})$ that commute according to G. The partially ordered set $\mathcal{S}$ is the set of all pairs

$$p = (\tilde{G}, \{X_v\}_{v\in V(\tilde{G})}),$$

such that $\tilde{G}$ is a connected subgraph of G, $X_v \in \mathcal{B}(\mathcal{K}_p)$ is a co-isometric extension of A_v for every vertex v of $\tilde{G}$, and the operators $\{X_v\}_{v\in V(\tilde{G})}$ commute according to $\tilde{G}$. The partial order on $\mathcal{S}$ is given by $(\tilde{G}, \{X_v\}_{v\in V(\tilde{G})}) \prec (G', \{Y_v\}_{v\in V(G')})$, if and only if $\tilde{G}$ is a proper subgraph of G' and Y_v extends X_v for every $v \in V(\tilde{G})$. Let $\{p_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary chain in $\mathcal{S}$, where $p_\lambda = (G_\lambda, \{X_v^\lambda\}_{v\in V(G_\lambda)})$ and $X_v^\lambda \in \mathcal{B}(\mathcal{K}_\lambda)$. Consider the pair $p := (\tilde{G}, \{X_v\}_{v\in V(\tilde{G})})$, where $\tilde{G} = \bigcup_\lambda G_\lambda$ and X_v is the common extension of $\{X_v^\lambda\}_{\lambda\in\Lambda}$, for each $v \in V(\tilde{G})$. Since the X_v^λ's are co-isometries, so are the X_v's. Thus the pair $(\tilde{G}, \{X_v\}_{v\in V(\tilde{G})})$ is an upper bound of our chain. Hence every chain in $\mathcal{S}$ has an upper bound, and so Zorn's lemma guarantees the existence of a maximal element $p_{\max} = (G_{\max}, \{X_v^{\max}\}_{v\in V(G_{\max})})$ in $\mathcal{S}$. We claim that $G_{\max} = G$ (which will complete the proof). Indeed, if $G_{\max} \subsetneq G$, then we can find a vertex $w \in V(G)$ that lies on an edge e whose other end-point lies in $V(G_{\max})$. One can now find a majorant of $p_{\max}$ in the same manner as the induction step in Lemma 2.2 was proved (the corresponding graph is obtained from $G_{\max}$ by adding w to the set of vertices and e to the set of edges).

The infinite version of the 'isometric' part of Lemma 2.2 can be proved by repeating the above argument with the obvious changes. $\square$

Remark 2.6. Using the main result, we can understand the representation theory of some universal operator algebras. Let G be a graph without a cycle and let us denote by $\mathcal{A}_G$ the universal operator algebra for unitaries commuting according to G. That is, $\mathcal{A}_G$ is the norm-closed operator algebra generated by the unitaries $\{\bigoplus_\lambda U_v^\lambda\}_{v\in V(G)}$, where $\{\{U_v^\lambda\}_{v\in V(G)}\}_{\lambda\in\Lambda}$ are all the families of unitaries that commute according to G. By definition, every contractive representation of $\mathcal{A}_G$ is of the form $\bigoplus_\lambda U_v^\lambda \mapsto A_v$, where $\{A_v\}_{v\in V(G)}$ is a family of contractions commuting according to G. Conversely, if $\{A_v\}_{v\in V(G)}$ is an arbitrary family of contractions that commute according to G, then we can dilate it to unitaries $\{U_v\}_{v\in V(G)}$ commuting according to G, so the map $U_v \mapsto A_v$, for all $v \in V(G)$, extends to a contractive homomorphism from the operator algebra $\mathcal{A}$ generated by $\{U_v\}_{v\in V(G)}$ to the operator algebra generated by $\{A_v\}_{v\in V(G)}$. Composing with the contractive homomorphism from $\mathcal{A}_G$ to $\mathcal{A}$ given by $\bigoplus_\lambda U_v^\lambda \mapsto U_v$, we get a contractive representation of $\mathcal{A}_G$. In fact, the argument above shows that the representation is completely contractive. Hence every contractive representation of $\mathcal{A}_G$ is completely contractive, and thus, by Arveson's dilation theorem (Corollary 7.7 in [Pau-02]), can be dilated to a $*$-representation of $\mathcal{C}_G^*$ — the universal C*-algebra for unitaries commuting according to G.

It is possible to obtain an extension of Lemma 2.2 which we will now describe. To that end, we need to explain the following natural graph construction. Suppose we are given n graphs $G_1, \dots, G_n$ (with mutually disjoint sets of vertices), a graph G on vertices $\{1, \dots, n\}$, and for each edge $e = (k, l) \in E(G)$ a pair of vertices v_1^e, v_2^e with $v_1^e \in G_k$ and $v_2^e \in G_l$. We can then construct a graph $\hat{G}$ given by $V(\hat{G}) = \bigcup_{i=1}^n V(G_i)$ and $E(\hat{G}) = \bigcup_{i=1}^n E(G_i) \ \cup \{(v_1^e, v_2^e);\ e \in E(G)\}$. In other words, $\hat{G}$ is obtained by joining the G_i's together according to G via edges joining the vertices v_1^e's to the corresponding v_2^e's.

Proposition 2.7. *Suppose we are given graphs $\{G_i\}_{i=1}^n$ and G, vertices v_1^e, v_2^e for each $e \in E(G)$ as in the previous paragraph, and contractions $\{\{A_v^i\}_{v \in V(G_i)}\}_{i=1}^n \subset \mathcal{B}(\mathcal{H})$ that commute according to $\hat{G}$. Suppose further that the graph G is acyclic and that for each fixed i, there exists co-isometries $\{X_v^i\}_{v \in V(G_i)} \subset \mathcal{B}(\mathcal{K}_i)$ that commute according to G_i and such that X_v^i extends A_v^i, for each $v \in V(G_i)$. Then there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, and co-isometries $\{\{\hat{X}_v^i\}_{v \in V(G_i)}\}_{i=1}^n \subset \mathcal{B}(\mathcal{K})$ that commute according to $\hat{G}$ and such that the $\hat{X}_v^i$'s extend the A_v^i's.*

This result is proved by induction analogously to the proof of Lemma 2.2. The role of Lemma 2.1 is played by the following fact. Given G_i's, G, A_v^i's and v_i^e's as in the above proposition with G being the (unique) connected graph on two vertices and co-isometric extensions $\{\tilde{X}_v^1\}_{v \in V(G_1)} \subset \mathcal{B}(\tilde{\mathcal{K}})$ of $\{\tilde{A}_v^1\}_{v \in V(G_1)}$ that commute according to G_1, there exist $\mathcal{K} \supset \tilde{\mathcal{K}}$ and X_v^i's as in the proposition and such that X_v^1 extends $\tilde{X}_v^1$ for each $v \in V(G_1)$. This fact is deduced from Lemma 2.1 in essentially the same way as the induction step of the proof of Lemma 2.2.

The proposition above is more general than Lemma 2.2, as the following example shows. Let G be the graph on vertices $\{1, 2\}$ with one edge e (that joins them), let G_1 be the cycle of length 3 on vertices $\{w_1, w_2, w_3\}$, and let G_2 be the graph with one vertex w_4. Suppose that $v_1^e = w_1$ and $v_2^e = w_4$, and that we are given contractions $\{A_{w_i}\}_{i=1}^4$ that commute according to $\hat{G}$ and such that $\{A_{w_i}\}_{i=1}^3$ possess commuting co-isometric extensions. Then, by the proposition, $\{A_{w_i}\}_{i=1}^4$ have co-isometric extensions which commute according to $\hat{G}$. This does not follow from Lemma 2.2 since $\hat{G}$ contains a cycle.

Remark 2.8. One could also prove a unitary-dilation version of Proposition 2.7 — in both the statement and the proof, one would replace each occurrence of 'co-isometric extension(s)' by 'unitary dilation(s)'. The downside is that one would not be able to conclude that the property (D) holds, even if it was assumed to hold on each G_i separately. This is, indeed, the reason we proved the main theorem in two steps using co-isometric extensions.

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References

[A-63] T. Andô; *On a Pair of Commuting Contractions*; Acta Sci. Math. (Szeged) **(24)** (1963), 88–90. MR0155193 (27:5132)

[AM-02] J. Agler, J. E. McCarthy; *Pick Interpolation and Hilbert Function Spaces*; Graduate Studies in Mathematics 44, AMS, 2002. MR1882259 (2003b:47001)

[B-02] T. Bhattacharyya; *Dilation of Contractive Tuples: A Survey*; Proc. Centre Math. Appl. Austral. Nat. Univ. **40** (2002), 89–126. MR1953481 (2003k:47014)

[GR-69] D. Gaşpar, A. Rácz; *An Extension of a Theorem of T. Andô*; Michigan Math. J. **(16)** (1969), 377–380. MR0251586 (40:4813)

[Par-70] S. Parrott; *Unitary Dilations For Commuting Contractions*; Pacific J. of Math. **(34)** (1970), no.2, 481–490. MR0268710 (42:3607)

[Pau-02] V. Paulsen; *Completely Bounded Maps and Operator Algebras*; Cambridge University Press, 2002. MR1976867 (2004c:46118)

[Pi-01] G. Pisier; *Similarity Problems and Completely Bounded Maps*; 2nd, expanded edition, LNM **1618** (2001), Springer-Verlag. MR1818047 (2001m:47002)

[Pi-03] G. Pisier; *Introduction to Operator Space Theory*; Cambridge University Press, 2003. MR2006539 (2004k:46097)

[Po-86] G. Popescu; *Isometric Dilations of $\mathcal{P}$-Commuting Contractions*; Rev. Roumaine Math. Pures Appl. **(31)** (1986), 383–393. MR0856214 (87k:47012)

[SF-70] B. Sz.-Nagy, C. Foiaş; *Harmonic Analysis of Operators on Hilbert Space*; North-Holland, 1970. MR0275190 (43:947)

Department of Mathematics, Campus Box 1146, Washington University in Saint Louis, Saint Louis, Missouri 63130

E-mail address: opela@math.wustl.edu