PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 9, September 2006, Pages 2719–2728
S 0002-9939(06)08315-8
Article electronically published on March 23, 2006

THE BEDROSIAN IDENTITY FOR THE HILBERT TRANSFORM OF PRODUCT FUNCTIONS

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(Communicated by David R. Larson)

ABSTRACT. We investigate a necessary and sufficient condition which ensures validity of the Bedrosian identity for the Hilbert transform of a product function fg. Convenient sufficient conditions are presented, which cover the classical Bedrosian theorem and provide us with new insightful information.

1. INTRODUCTION

The Hilbert transform, defined for a complex-valued function $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, by

$$(Hf)(x) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy, \tag{1.1}$$

has been widely used in physics, engineering and mathematics (cf. [Co, H]). The value of integral (1.1) is understood as the Cauchy principal value of the singular integral. Specifically,

$$\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy = \lim_{\varepsilon\to 0^+}\int_{B_\varepsilon(x)} \frac{f(y)}{x-y}dy,$$

where $B_\varepsilon(x)$ is defined by

$$B_\varepsilon(x) := \{y \in \mathbb{R} : |y-x| \geq \varepsilon\}.$$

In applications of the Hilbert transform to sinal analysis, we are required to compute the Hilbert transform of the product of two functions. In particular, in the context of using the Hilbert transform for obtaining the instantaneous frequency of a signal [H], people are interested in understanding under what conditions the formula

$$H(a(\cdot)e^{i\theta(\cdot)})(t) = a(t)(H(e^{i\theta(\cdot)})(t) \tag{1.2}$$

holds, where a is the magnitude of a given signal and θ is the phase of the signal. Such a formula is crucial for the theoretical study of signals, as well as practical computation. See [LX] for results regarding piecewise linear frequency.

Received by the editors January 24, 2005 and, in revised form, April 11, 2005.
2000 *Mathematics Subject Classification.* Primary 65R10.
Key words and phrases. Bedrosian identity, Hilbert transform.
The first author was supported in part by the US National Science Foundation under grant 0407476, by the Natural Science Foundation of China under grant 10371122 and by the Chinese Academy of Sciences under the program "One Hundred Distinguished Young Chinese Scientists".

Bedrosian [B] established the following important result: Let f, g be complex-valued functions in $L^2(\mathbb{R})$ of the real variable. If the Fourier transform $\hat{f}(\omega)$ of f vanishes for $|\omega| > a$ and the Fourier transform $\hat{g}(\omega)$ of g vanishes for $|\omega| < a$, where a is a positive constant, then the Hilbert transform of the product of f and g is given by

$$H[f(x)g(x)] = f(x)H[g(x)], \quad x \in \mathbb{R}. \tag{1.3}$$

Formula (1.3) is referred to as the *Bedrosian identity* in the engineering literature, and the above result is called the Bedrosian theorem. It is also well known (cf. [Co]) in time-frequency analysis that if $f, g \in L^2(\mathbb{R})$ are analytic singles, that is,

$$\text{supp}(\hat{f}) \subseteq [0, \infty) \ \text{ and } \ \text{supp}(\hat{g}) \subseteq [0, \infty),$$

then

$$H(fg) = fH(g) = gH(f). \tag{1.4}$$

This can be viewed as another version of the Bedrosian theorem. The hypothesis of these results are rather restricted. Recent studies in the empirical mode decomposition [H] of signals demand a better understanding of the Bedrosian identity, hoping for a weak hypothesis. The classical Bedrosian theorem was reproved in [Br] by a complex analysis approach. A Bedrosian-type theorem in terms of the vanishing moments of the function g was obtained recently in [CHRX].

The primary purpose of this paper is to establish a necessary and sufficient condition for which the Bedrosian identity holds for the product of two functions. We will also derive from this characterization convenient weak sufficient conditions for the Bedrosian identity to hold. This is mainly done by using the Fourier transform. We organize this paper into three sections. In Section 2, we develop a characterization of functions for which the Bedrosian identity is valid. This characterization provides insight for the validity of the Bedrosian identity. In Section 3, we establish several convenient and useful sufficient conditions from the characterization. We also identify the classical Bedrosian theorem as a special corollary of a result presented in the section and demonstrate that the new sufficient conditions offer us more insightful information than the classical Bedrosian theorem.

2. A characterization of functions satisfying the Bedrosian identity

In this section, we develop a characterization of functions which satisfy the Bedrosian identity. We first present a simple observation.

Proposition 2.1. *Let $f, g \in L^2(\mathbb{R})$. Then the Hilbert transform of functions fg satisfies the Bedrosian identity*

$$H(fg) = fH(g) \tag{2.1}$$

if and only if

$$\text{p.v.} \int_{\mathbb{R}} \frac{f(y) - f(x)}{x - y} g(y)dy = 0, \quad \textit{for} \ \ x \in \mathbb{R}. \tag{2.2}$$

Proof. By the definition (1.1) of the Hilbert transform, we have for $x \in \mathbb{R}$ that

$$H[fg](x) = f(x)Hg(x) + \frac{1}{\pi}\text{p.v.} \int_{\mathbb{R}} \frac{f(y) - f(x)}{x - y} g(y)dy.$$

It is clear that the Bedrosian identity is satisfied if and only if (2.2) holds. $\square$

Although equation (2.2) looks superficial, it serves as a starting point for further development of a more insightful characterization. To have a more illuminating characterization, we are interested in the version of equation (2.2) in the Fourier domain. To this end, we recall the definition of the Fourier transform in $L^2(\mathbb{R})$. For $f \in L^1(\mathbb{R})$, we define the Fourier transform $\hat{f}$ by

$$\hat{f}(x) = \int_{\mathbb{R}} f(t) e^{-2\pi i x t} dt, \quad x \in \mathbb{R}.$$

When $f \in L^2(\mathbb{R})$, the Fourier transform $\hat{f}$ is considered as the L^2 limit of the sequence $\hat{h}_k$, $k \in \mathbb{N}_0 := \{0, 1, \ldots\}$, where h_k, $k \in \mathbb{N}_0$, is a sequence in $L^1 \cap L^2$ converging to f in the L^2 norm.

We next prove a technical lemma about the Fourier transform of the divided difference. We use $W^{m,p}(\mathbb{R})$ to denote the Sobolev space

$$W^{m,p}(\mathbb{R}) := \{f^{(m)} \in L^p(\mathbb{R}) : \|f\|_{m,p} < \infty\},$$

where

$$\|f\|_{m,p} = \left\{ \sum_{0 \le k \le m} \|f^{(k)}\|_p^p \right\}^{1/p}, \quad 1 \le p < \infty,$$

and the derivative $f^{(m)}$ is understood in the sense of distribution.

Lemma 2.2. *Suppose that $\phi \in W^{1,2}(\mathbb{R})$. Then for any fixed $x \in \mathbb{R}$ the function*

$$\psi(y) := \frac{\phi(x) - \phi(y)}{x - y}, \quad y \in \mathbb{R},$$

is in $L^2(\mathbb{R})$ and

$$\hat{\psi}(\omega) = 2i\pi e^{-2i\pi x \omega} \int_0^1 \frac{\omega}{t^2} e^{2i\pi x \omega / t} \hat{\phi}\left(\frac{\omega}{t}\right) dt. \tag{2.3}$$

Proof. Since $\phi \in W^{1,2}(\mathbb{R})$, the mean-value theorem leads to the formula

$$\psi(y) = \int_0^1 \phi'(ty + (1-t)x) dt.$$

For any fixed $x \in \mathbb{R}$ and $\phi \in W^{1,2}(\mathbb{R})$, we first prove that $\psi \in L^2(\mathbb{R})$. Using the generalized Minkowski inequality, we observe that

$$\begin{aligned} \|\psi\|_2 &= \left(\int_{\mathbb{R}} \left| \int_0^1 \phi'(ty + (1-t)x) dt \right|^2 dy \right)^{1/2} \\ &\le \int_0^1 \left(\int_{\mathbb{R}} |\phi'(ty + (1-t)x)|^2 dy \right)^{1/2} dt \\ &= \|\phi'\|_2 \int_0^1 \frac{1}{\sqrt{t}} dt = 2\|\phi'\|_2 < \infty. \end{aligned}$$

This implies that $\psi \in L^2(\mathbb{R})$, and thus its Fourier transform is well defined. By the definition of the Fourier transform and the condition $\phi \in W^{1,2}(\mathbb{R})$, we obtain that

$$\hat{\psi}(\omega) = \int_0^1 [\phi'(t \cdot + (1-t)x)]\hat{\ }(\omega) dt.$$

Employing formulas of the Fourier transform of derivatives, dilations and translations, we have that

$$\hat{\psi}(\omega) = 2i\pi e^{-2i\pi x\omega} \int_0^1 \frac{\omega}{t^2} e^{2i\pi x\omega/t} \hat{\phi}\left(\frac{\omega}{t}\right) dt,$$

which concludes formula (2.3). □

Next, we use Proposition 2.1 and Lemma 2.2 to establish a useful characterization of the functions that satisfy the Bedrosian identity.

Theorem 2.3. *Let* $f \in W^{1,2}(\mathbb{R})$ *and* $g \in L^2(\mathbb{R})$*. Then the Hilbert transform of function* fg *satisfies the Bedrosian identity* (2.1) *if and only if*

$$\int_{-1}^0 \int_{\mathbb{R}} \frac{\omega}{t^2} e^{2i\pi x\omega(t+1)/t} \hat{f}\left(\frac{\omega}{t}\right) \hat{g}(\omega) d\omega dt = 0. \tag{2.4}$$

Proof. Proposition 2.1 ensures that the Bedrosian identity holds if and only if equation (2.2) holds. Since $f \in W^{1,2}(\mathbb{R})$, using Lemma 2.2, we find that the divided difference $\frac{f(x)-f(\cdot)}{x-\cdot}$ is an L^2 function. Thus, the integration involved in equation (2.2) is now in the usual sense. By the Plancherel-Parseval theorem, equation (2.2) holds if and only if

$$\int_{\mathbb{R}} \left[\frac{f(x)-f(\cdot)}{x-\cdot}\right]^{\wedge} (\omega) \overline{\hat{\overline{g}}(\omega)} d\omega = 0. \tag{2.5}$$

A straightforward computation leads to the formula $\overline{\hat{\overline{g}}(\omega)} = \hat{g}(-\omega)$. The hypotheses of this theorem allows us to use Lemma 2.2 to conclude that (2.5) is equivalent to equation

$$\int_{\mathbb{R}} 2i\pi e^{-2i\pi x\omega} \int_0^1 \frac{\omega}{t^2} e^{2i\pi x\omega/t} \hat{f}\left(\frac{\omega}{t}\right) dt \hat{g}(-\omega) d\omega = 0. \tag{2.6}$$

In order to obtain equation (2.4), we introduce a function of two variables by

$$h(T) := \frac{\omega}{t^2} e^{-2i\pi x\omega(t-1)/t} \hat{f}\left(\frac{\omega}{t}\right) \hat{g}(-\omega), \quad T := (\omega, t) \in \mathbb{R} \times (0,1),$$

and will prove that $h \in L^1(\mathbb{R} \times (0,1))$.

To this end, we note that the transformation $(\omega, t) = \left(\omega, \frac{1}{y}\right)$ maps the domain $\mathbb{T} := \mathbb{R} \times (0,1)$ onto $\mathbb{Y} := \mathbb{R} \times (1, +\infty)$, and the corresponding measure is mapped from $dT := d\omega dt$ to $dY := d\omega dy$. Consequently, we have that

$$\int_{\mathbb{T}} |h(T)|\, dT = \int_{\mathbb{Y}} \left|\omega \hat{f}\,(y\omega)\, \hat{g}(-\omega)\right| dY,$$

where $Y := (\omega, y)$. For $\sigma \in (0,1)$, we let

$$I_1 := \int_{\mathbb{Y}} \left|\omega y^{\frac{1+\sigma}{2}} \hat{f}(y\omega)\right|^2 dY \quad \text{and} \quad I_2 := \int_{\mathbb{Y}} \left|y^{-\frac{1+\sigma}{2}} \hat{g}(-\omega)\right|^2 dY.$$

By another transformation $(\lambda, y) = (y\omega, y)$ we find that

$$\begin{aligned} I_1 &= \int_{\mathbb{R}} \int_1^{\infty} \left|\lambda \hat{f}(\lambda)\right|^2 y^{\sigma-2} dy d\lambda = \frac{1}{(2\pi)^2(1-\sigma)} \int_{\mathbb{R}} \left|2\pi i\lambda \hat{f}(\lambda)\right|^2 d\lambda \\ &= \frac{1}{(2\pi)^2(1-\sigma)} \|f'\|_2^2. \end{aligned}$$

Moreover, a direct computation confirms that $I_2 = \frac{1}{\sigma}\|g\|_2^2$. By applying the Hölder inequality, we conclude that for any $\sigma \in (0,1)$

$$\int_{\mathbb{Y}} \left|\omega \hat{f}\,(y\omega)\,\hat{g}(-\omega)\right| dY \leq \frac{1}{2\pi\sqrt{(1-\sigma)\sigma}}\|f'\|_2\|g\|_2.$$

This proves that $h \in L^1(\mathbb{T})$. Using the Fubini theorem in equation (2.6) yields that

$$\int_0^1 \int_{\mathbb{R}} \frac{\omega}{t^2} e^{-2i\pi x\omega(t-1)/t} \hat{f}\left(\frac{\omega}{t}\right)\hat{g}(-\omega)d\omega dt = 0.$$

Making the change of variables $(-\omega, -t) \to (\omega, t)$ in the above equation leads to the desired result. □

3. Convenient sufficient conditions

In this section we derive from Theorem 2.3 useful and convenient sufficient conditions for functions which admit the Bedrosian identity. These sufficient conditions are proposed in terms of *supports* of the Fourier transforms $\hat{f}$ and $\hat{g}$.

To state our first sufficient condition and its corollary, for a nonempty set $\Omega \subseteq \mathbb{R}$ and a real number t, we let

$$t\,\Omega := \{tx : x \in \Omega\}.$$

Let u be a complex-valued continuous function defined on $\mathbb{R}$. Note that a point $x \in \operatorname{supp}\left(u\left(\frac{\cdot}{t}\right)\right)$ if and only if $\frac{x}{t} \in \operatorname{supp}(u)$, which is equivalent to $x \in t\operatorname{supp}(u)$. Hence, for any nonzero real number t, we have that

$$\operatorname{supp}\left(u\left(\frac{\cdot}{t}\right)\right) = t\operatorname{supp}(u). \tag{3.1}$$

For a nonempty set $\Omega \subset \mathbb{R}$ and the unit interval $I := [0,1]$, we define the product set $I \cdot \Omega$ by

$$I \cdot \Omega := \bigcup_{t\in[0,1]} t\Omega.$$

We characterize the set $I \cdot \Omega$ in the next lemma.

Lemma 3.1. *Suppose that Ω is a nonempty closed set in $\mathbb{R}$. Let*

$$\ell := \inf\{x \in \Omega\}, \quad L := \sup\{x \in \Omega\}, \quad \ell^* := \min\{0, \ell\} \quad \text{and} \quad L^* := \max\{0, L\}.$$

The following statements hold:

(1) *If both L and ℓ are finite, then $I \cdot \Omega = [\ell^*, L^*]$.*
(2) *If ℓ is finite and $L = \infty$, then $I \cdot \Omega = [\ell^*, \infty)$.*
(3) *If $\ell = -\infty$ and L is finite and $\ell = -\infty$, then $I \cdot \Omega = (-\infty, L^*]$.*
(4) *If $\ell = -\infty$ and $L = \infty$, then $I \cdot \Omega = (-\infty, \infty)$.*

Moreover, if $\Omega_1 \subseteq \Omega_2$, then $I \cdot \Omega_1 \subseteq I \cdot \Omega_2$.

Proof. (1) Since Ω is closed and both L and ℓ are finite, we have that $\ell, L \in \Omega$. Let $\Omega' := [\ell^*, L^*]$. We observe that if $\ell \geq 0$, $\Omega' = [0, L]$, if $L \leq 0$, $\Omega' = [\ell, 0]$, and if $\ell < 0$ and $L > 0$, $\Omega' = [\ell, L]$.

Let $x \in I \cdot \Omega$. By the definition of $I \cdot \Omega$, there exists some $t \in I$ such that $x \in t\Omega$. That is, $x \in [t\ell, tL]$. If $\ell \geq 0$, $x \in [0, L]$, if $L \leq 0$, $x \in [\ell, 0]$, and if $\ell < 0$ and $L > 0$, $x \in [\ell, L]$. Hence, $x \in \Omega'$.

Conversely, we let $x \in \Omega'$. If $\ell \geq 0$, there exists $t \in I$ such that $x = tL$ and thus, $x \in t\Omega$. If $\ell < 0$ and $L \leq 0$, there exists $t \in [0,1]$ such that $x = t\ell$. If $\ell < 0$ and

$L > 0$, we have that $x \in [\ell, L]$. When $x \leq 0$, there exists $t \in I$ such that $x = t\ell$ and when $x > 0$, there exists $t \in I$ such that $x = tL$. That is, $x \in I \cdot \Omega$.

(2) Clearly, we have that $I \cdot \Omega \subseteq [\ell^*, \infty)$. It suffices to prove that $I \cdot \Omega \supseteq [\ell^*, \infty)$. If $\ell < 0$, then $[\ell^*, \infty) = [\ell, \infty)$. We let $x \in [\ell, \infty)$. When $\ell \leq x \leq 0$, there exists $t \in [0, 1]$ such that $x = t\ell$, and when $x > 0$, since $L = \infty$, there exists $y \in \Omega$ such that $x \leq y$. Set $t = x/y \in I$ and thus, $x \in t\Omega$. If $\ell \geq 0$, then we can use the same method to prove that $I \cdot \Omega \supseteq [\ell^*, \infty)$.

Proofs for (3) and (4) are similar, and thus we omit the details. □

We remark that $I \cdot \Omega$ is an *interval* having the origin as either an interior point or an end point, and there holds

$$I \cdot (I \cdot \Omega) = I \cdot \Omega.$$

Let $I^- := [-1, 0]$. Then,

$$I^- \cdot \Omega := \bigcup_{t \in I^-} t\Omega.$$

If $I \cdot \Omega = [a, b]$, then $I^- \cdot \Omega = (-1)(I \cdot \Omega) = [-b, -a]$. In other words, both $I \cdot \Omega$ and $I^- \cdot \Omega$ are intervals symmetric with respect to the origin.

Next we present a sufficient condition for functions f and g which admit the Bedrosian identity in terms of the supports of their Fourier transform.

Proposition 3.2. *Let $f \in W^{1,2}(\mathbb{R})$ and $g \in L^2(\mathbb{R})$. If*

$$(I^- \cdot \operatorname{supp}(\hat{f})) \cap \operatorname{supp}(\hat{g}) = \emptyset, \tag{3.2}$$

then the Hilbert transform of function fg satisfies the Bedrosian identity (2.1).

Proof. Because of condition (3.2), we observe that

$$(t \operatorname{supp}(\hat{f})) \cap \operatorname{supp}(\hat{g}) = \emptyset, \quad \text{for all} \quad t \in I^-,$$

which, by (3.1), is equivalent to

$$\operatorname{supp}\left(\hat{f}\left(\frac{\cdot}{t}\right)\right) \cap \operatorname{supp}(\hat{g}) = \emptyset, \quad \text{for all} \quad t \in I^-.$$

We then conclude that equation (2.4) holds. Thus, by Theorem 2.3, the Hilbert transform of function fg satisfies the Bedrosian identity (2.1). □

The classical Bedrosian theorem follows immediately from Proposition 3.2. We present it in the next corollary.

Corollary 3.3. *Let $a > 0$ and suppose that $f, g \in L^2(\mathbb{R})$ with*

$$\operatorname{supp}(\hat{f}) \subseteq [-a, a] \quad \textit{and} \quad \operatorname{supp}(\hat{g}) \subseteq (-\infty, -a) \cup (a, \infty).$$

Then, the Hilbert transform of function fg satisfies the Bedrosian identity (2.1).

Proof. Since $\operatorname{supp}(\hat{f}) \subseteq (-a, a)$, it follows from the Paley-Wiener theorem that $f \in C^\infty(\mathbb{R})$, and thus $f \in W^{1,2}(\mathbb{R})$. Moreover, by Lemma 3.1, we see that

$$I^- \cdot [-a, a] = [-a, a].$$

Thus,

$$I^- \cdot \operatorname{supp}(\hat{f}) \subseteq [-a, a].$$

Hence, the condition of Proposition 3.2 holds, and consequently, the Hilbert transform of function fg satisfies the Bedrosian identity (2.1). □

We remark that Proposition 3.2 is a nontrivial extension of the classical Bedrosian theorem. It should be pointed out that for the classical Bedrosian theorem, $\mathrm{supp}(\hat{f})$ must be supported in a symmetric interval such as $(-a, a)$, for some positive constant a. For example, the following corollary of Proposition 3.2 is not covered by the classical Bedrosian theorem.

Corollary 3.4. *Let $a \leq 0$, $b \geq 0$ and suppose that $f, g \in L^2(\mathbb{R})$ with*

$$\mathrm{supp}(\hat{f}) \subseteq [a, b] \quad \textit{and} \quad \mathrm{supp}(\hat{g}) \subseteq \mathbb{R} \backslash (-\infty, -b) \cup (-a, \infty).$$

Then, the Hilbert transform of function fg satisfies the Bedrosian identity (2.1).

We will use $\mu(\Omega)$ to denote the Lebesgue measure of the set Ω. We remark that condition (3.2) in Proposition 3.2 may be weakened as

$$\mu((I^- \cdot \mathrm{supp}(\hat{f})) \cap \mathrm{supp}(\hat{g})) = 0. \tag{3.3}$$

Our next task is to relax the smooth condition of Proposition 3.2 on the function f. To this end, we denote by $\mathcal{D}(\mathbb{R})$ the space of functions in C^∞ having bounded supports. It is well known (cf. [SW]) that $\mathcal{D}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. In the next lemma we establish a density result of functions in $\mathcal{D}(\mathbb{R})$ with an additional requirement on supports. This new requirement is central for our specific application later in this paper.

Lemma 3.5. *If $f \in L^2(\mathbb{R})$ satisfies the condition that*

$$(I^- \cdot \mathrm{supp}(f)) \cap K = \emptyset \tag{3.4}$$

for some closed set K, then for any $\varepsilon > 0$, there exists $\varphi \in \mathcal{D}(\mathbb{R})$ such that

$$\|f - \varphi\|_2 < \varepsilon \tag{3.5}$$

and

$$(I^- \cdot \mathrm{supp}(\varphi)) \cap K = \emptyset. \tag{3.6}$$

Proof. We prove this result by constructing a function φ having the desired property. For this purpose, we set $B := \mathrm{supp}(f)$. Since $f \in L^2(\mathbb{R})$, for $\varepsilon > 0$, there exists an $N_0 \in \mathbb{N}$ satisfying that

$$\int_{B \cap (\mathbb{R} \backslash [-N_0, N_0])} |f(x)|^2 dx < \frac{\varepsilon^2}{4}. \tag{3.7}$$

We introduce the greatest lower bound and the smallest upper bound of the set $B \cap [-N_0, N_0]$ by setting

$$\ell := \min\{x \in B \cap [-N_0, N_0]\} \quad \text{and} \quad L := \max\{x \in B \cap [-N_0, N_0]\},$$

and we consider the closed interval $[\ell, L]$. Clearly, we have that

$$I^- \cdot [\ell, L] \subseteq I^- \cdot \mathrm{supp}(f).$$

Since K is a closed set and

$$(I^- \cdot \mathrm{supp}(f)) \cap K = \emptyset,$$

we have that

$$I^- \cdot [\ell, L] \cap K = \emptyset.$$

We now construct a function $\varphi \in C^\infty(\mathbb{R})$ with a support contained in $[\ell, L]$. For this purpose, we introduce functions

$$f_0 := \chi_{[\ell, L]} f \quad \text{and} \quad f_\delta := \chi_{(\ell+\delta, b-\delta)} f, \quad \text{for} \quad \delta > 0.$$

We choose $\rho \in \mathcal{D}(\mathbb{R})$ satisfying

$$\rho \geq 0, \quad \mathrm{supp}(\rho) \subseteq (-\delta, \delta) \quad \text{and} \quad \|\rho\|_1 = 1.$$

We define φ as the convolution of f_δ and ρ, that is, $\varphi := f_\delta * \rho$. Then we have that $\varphi \in \mathcal{D}(\mathbb{R})$ and

$$\mathrm{supp}(\varphi) \subset \mathrm{supp}(f_\delta) + \mathrm{supp}(\rho) \subset [\ell, L].$$

Thus,

$$I^- \cdot \mathrm{supp}(\varphi) \subset I^- \cdot [\ell, L] \subseteq (I^- \cdot \mathrm{supp}(f)).$$

This ensures that condition (3.6) holds. Now we choose the parameters δ such that

$$\|f_0 - f_\delta\|_2 \leq \frac{\varepsilon}{4} \quad \text{and} \quad \|f_\delta - \varphi\|_2 \leq \frac{\varepsilon}{4}.$$

This implies that

$$\|f_0 - \varphi\|_2 \leq \|f_0 - f_\delta\|_2 + \|f_\delta - \varphi\|_2 \leq \frac{\varepsilon}{2}. \tag{3.8}$$

Next let us estimate the L^2 norm of $f - \varphi$. To this end, we let $A := [\ell, L]$ and find that

$$\|f - \varphi\|_2^2 = \int_A |f(x) - \varphi(x)|^2 dx + \int_{\mathbb{R}\setminus A} |f(x)|^2 dx.$$

We consider an estimate of the first term on the right-hand side of the last equation. By inequality (3.8), we have that

$$\int_A |f(x) - \varphi(x)|^2 dx = \|f_0 - \varphi\|_2^2 \leq \frac{\varepsilon^2}{4}.$$

It remains to estimate the second term. Noting that

$$\mathbb{R} \setminus A \subseteq \mathbb{R} \setminus (B \cap [-N_0, N_0]) = (\mathbb{R} \setminus [-N_0, N_0]) \cup (\mathbb{R} \setminus B),$$

we observe that

$$\begin{aligned} \int_{\mathbb{R}\setminus A} |f(x)|^2 dx &\leq \int_{\mathbb{R}\setminus(B\cap[-N_0,N_0])} |f(x)|^2 dx \\ &\leq \int_{\mathbb{R}\setminus[-N_0,N_0]} |f(x)|^2 dx + \int_{\mathbb{R}\setminus B} |f(x)|^2 dx \\ &= \int_{B\cap(\mathbb{R}\setminus[-N_0,N_0])} |f(x)|^2 dx. \end{aligned}$$

Combining the inequality above and estimate (3.7) we have that

$$\int_{\mathbb{R}\setminus A} |f(x)|^2 dx \leq \frac{\varepsilon^2}{4}.$$

Consequently, the estimate (3.5) holds, which proves the result. □

We now use this density result (Lemma 3.5) and Proposition 3.2 to conclude the next theorem which provides useful and convenient sufficient conditions for product functions which satisfy the Bedrosian identity.

Theorem 3.6. *Let $f, g \in L^2(\mathbb{R})$. Suppose that $\hat{f}$ and $\hat{g}$ satisfy condition* (3.2). *Then, the Hilbert transform of function fg satisfies the Bedrosian identity* (2.1).

Proof. Since $f \in L^2(\mathbb{R})$, we have that $\hat{f} \in L^2(\mathbb{R})$. Because $\operatorname{supp}(\hat{g})$ is closed, by Lemma 3.5, there exists a sequence of functions $\varphi_n \in \mathcal{D}(\mathbb{R})$, $n \in \mathbb{N}$, such that

$$\left\|\varphi_n - \hat{f}\right\|_2 \to 0, \quad \text{as} \quad n \to \infty,$$

and

$$(I^- \cdot \operatorname{supp}(\varphi_n)) \cap \operatorname{supp}(\hat{g}) = \emptyset.$$

We choose f_n as the inverse Fourier transform of φ_n, that is, $f_n := \check{\varphi}_n$. Consequently, we have that

$$\left\|\hat{f}_n - \hat{f}\right\|_2 \to 0, \quad \text{as} \quad n \to \infty, \tag{3.9}$$

and

$$\left(I^- \cdot \operatorname{supp}(\hat{f}_n)\right) \cap \operatorname{supp}(\hat{g}) = \emptyset.$$

By the Paley-Wiener theorem, we find that $f_n \in C^\infty(\mathbb{R})$, and thus $f_n \in W^{1,2}(\mathbb{R})$. It follows from Proposition 3.2 that for each $n \in \mathbb{N}$ there holds the Bedrosian identity

$$H(f_n g) = f_n H(g). \tag{3.10}$$

Note that both fg and $f_n g$ are in $L^1(\mathbb{R})$. We define the set

$$\Omega_n := \left\{x : \left|H[(f_n g) - (fg)](x)\right| \geq \varepsilon\right\}.$$

Because of the weak-type (1,1) boundedness of the Hilbert transform (cf. [D]), for any $\varepsilon > 0$, there exists a positive constant c such that

$$\mu(\Omega_n) \leq \frac{c}{\varepsilon}\left\|(f_n - f)g\right\|_1 \leq \frac{c}{\varepsilon}\left\|f_n - f\right\|_2 \|g\|_2 \to 0, \text{ as } n \to \infty.$$

We thus conclude that $H(f_n g)$ converges to $H(fg)$ in measure. By invoking the Riesz theorem (cf. [HS]), there exists a subsequence f_{n_k} such that $H(f_{n_k} g)$ converges to $H(fg)$ almost everywhere in $\mathbb{R}$. Moreover, because of (3.9), we have that

$$\|f_n - f\|_2 \to 0, \quad \text{as} \quad n \to \infty,$$

which implies naturally that

$$\|f_{n_k} - f\|_2 \to 0, \quad \text{as} \quad k \to \infty.$$

This implies that f_{n_k} converges to f in measure. Again, by employing the Riesz theorem, there exists a subsequence $f_{n_{k_j}}$ such that $f_{n_{k_j}}$ converges to f almost everywhere in $\mathbb{R}$. By using equation (3.10), we conclude that

$$fH(g) = \lim_{j \to \infty} f_{n_{k_j}} H(g) = \lim_{j \to \infty} H(f_{n_{k_j}} g) = H(fg), \quad \text{a.e.},$$

which completes the proof of this theorem. □

As we have mentioned in Section 1, if $f, g \in L^2(\mathbb{R})$ are analytic singles, then formula (1.4) holds. This is a special case of the next corollary of Theorem 3.6.

Corollary 3.7. *Let $a \geq 0$, $b \leq 0$. Suppose that $f, g \in L^2(\mathbb{R})$, where one of the following conditions holds:*

$$\operatorname{supp}(\hat{f}) \subseteq (-\infty, a] \quad \textit{and} \quad \operatorname{supp}(\hat{g}) \subseteq (-\infty, -a]$$

or

$$\operatorname{supp}(\hat{f}) \subseteq [b, \infty) \quad \textit{and} \quad \operatorname{supp}(\hat{g}) \subseteq [-b, \infty).$$

Then, the Hilbert transform of function fg satisfies the Bedrosian identity (2.1).

The next corollary follows directly from Theorem 3.6 and a remark made after Corollary 3.4.

Corollary 3.8. *Let $f, g \in L^2(\mathbb{R})$. Suppose that $\hat{f}$ and $\hat{g}$ satisfy the following condition:*

$$\mu((I^- \cdot \operatorname{supp}(\hat{f})) \cap \operatorname{supp}(\hat{g})) = 0.$$

Then, the Hilbert transform of function fg satisfies the Bedrosian identity (2.1).

Acknowledgment

The authors are indebted to Professor Tao Qian for his comments which improved the presentation of the paper. In particular, his suggestion leads to weakening the hypothesis of Theorem 3.6.

References

[B] E. Bedrosian, A product theorem for Hilbert transform, *Proc. IEEE* **51** (1963), 868-869.

[Br] J. L. Brown, Jr., Analytic signals and product theorems for Hilbert transforms, *IEEE Transactions on Circuits and Systems,* November (1974), 790-792. MR0405009 (53:8805)

[CHRX] Q. Chen, N. Huang, S. Riemenschneider and Y. Xu, A B-spline approach for empirical mode decompositions, *Adv. Comput. Math.*, to appear.

[Co] L. Cohen, *Time-Frequency Analysis: Theory and Application*, Prentice Hall, Englewood Cliffs, NJ, 1995.

[D] J. Duoandikoetxea, *Fourier Analysis*, Graduate Studies in Mathematics, Vol. 29, Amer. Math. Soc., Providence, 2001. MR1800316 (2001k:42001)

[GW] C. Gasquet and P. Witomski, *Fourier Analysis and Applications*, Springer, New York, 1995. MR1657104 (99h:42003)

[HS] E. Hewitt and K. Stromberg, *Real and Abstract Analysis: A Modern Treatment of the Theory of Functions of A Real Variable*, Springer-Verlag, New York, 1975. MR0367121 (51:3363)

[H] N. Huang, et al., The empirical mode decomposition and the Hilbert spectrum for non-linear and non-stationary time series analysis, *Proc. R. Soc. London*, A (1998), 454, 903-995. MR1631591 (99d:76082)

[LX] Y. Liu and Y. Xu, Piecewise linear spectral sequences, Proc. Amer. Math. Soc. **133** (2005), 2297–2308. MR2138872

[SW] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princton University Press, Princeton, New Jersey, 1971. MR0304972 (46:4102)

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