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EXISTENCE OF POSITIVE SOLUTIONS FOR A SEMILINEAR ELLIPTIC PROBLEM WITH CRITICAL SOBOLEV AND HARDY TERMS

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Abstract. Let $N \geq 4$, let $2^* = 2N/(N-2)$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$. Our purpose in this paper is to consider the existence of solutions of the problem:

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = |u|^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \mu < (\frac{N-2}{2})^2$.

1. Introduction

Let $N \geq 4$, let $2^* = 2N/(N-2)$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$ and $0 \in \Omega$. In the present paper, we consider the existence of solutions of the problem

$$(\mathrm{P}_\mu) \qquad \begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = |u|^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ is a given positive constant. These kind of equations give nonlinear Schrödinger equations with the field having singularity at its origin. The existence of solutions of semilinear elliptic boundary value problems with Hardy terms has been studied by many authors (cf. [4], [5], [6], [8], [10] and [12]). It is known that problem (P_μ) has no nontrivial solution when Ω is star shaped (cf. [1]) for any $\mu \geq 0$. On the other hand, in the case that Ω has nontrivial topology, problem (P_0) has been investigated by several authors. In [9], Kazdon and Warner proved the existence of a nontrivial solution of (P_0) in the case that Ω is an annulus. In [2], Bahri and Coron established the existence of a nontrivial solution of (P_0) in the case that Ω has nontrivial topology. Our purpose in the present paper is to show the existence of a solution of (P_μ) with $\mu > 0$ when domain Ω has nontrivial topology. We now state our main result.

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Theorem 1.1. *Suppose that Ω is not contractible. Then there exists $\mu_0 \in (0,\infty)$ such that for each $\mu \in (0,\mu_0)$, there exists a solution of (P_μ).*

2. Preliminaries

Let $H = H_0^1(\Omega)$. We put $\|v\|^2 = \int_\Omega |\nabla v|^2\,dx$ for $v \in H$ and $|v|_p^p = \int_\Omega |v|^p\,dx$ for $p \geq 2$ and $v \in L^p(\Omega)$. For $u, v \in H$, we put $\langle u, v\rangle = \int_\Omega uv\;dx$. We use the same symbols $\|\cdot\|, ||_p^p$, and $\langle\cdot,\cdot\rangle$ in the case $\Omega = \mathbb{R}^N$. For each $A \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$, $\mathrm{d}(x, A)$ denotes the distance of x from A. We denote by $B_r(x)$ the open ball in $\mathbb{R}^N$ centered at x with radius r. For subsets $A, B \subset \mathbb{R}^N$, $A \cong B$ implies that A and B are homotopy equivalent. For each $d > 0$, Ω_d and Ω_d^i stand for subsets of $\mathbb{R}^N$ defined by $\Omega_d = \left\{x \in \mathbb{R}^N : d(x,\Omega) < d\right\}$ and $\Omega_d^i = \{x \in \Omega : d(x, \partial\Omega) > d\}$, respectively. For each $a \in \mathbb{R}$ and a functional $F : H \to \mathbb{R}$, we denote by F_a the level set $F_a = \{v \in H : F_a(v) \leq a\}$. The Hardy inequality states that

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \leq \int_{\mathbb{R}^N} |\nabla u|^2$$

holds for $u \in D^{1,2}(\mathbb{R}^N) = \left\{v \in L^{2^*}(\mathbb{R}^N) : |\nabla v| \in L^2(\mathbb{R}^N)\right\}$ (cf. [12]). For each bounded domain $U \subset \mathbb{R}^N$ and $\mu \geq 0$, we define a functional $I^{(U,\mu)}$ on $H_0^1(U)$ by

$$I^{(U,\mu)}(u) = \int_U \left(\frac{1}{2}|\nabla u|^2 - \frac{\mu}{2}\frac{|u|^2}{|x|^2} - \frac{1}{2^*}\left|u^+\right|^{2^*}\right)dx \qquad \text{for } u \in H_0^1(U).$$

$I^{(\mathbb{R}^N,\mu)}(v)$ is defined by the same way as above with $U = \mathbb{R}^N$ and $v \in D^{1,2}(\mathbb{R}^N)$. Here $u^+(x) = \max\{u(x), 0\}$ for $x \in U$. Then the solutions of (P_μ) correspond to critical points of the functional $I^{(\Omega,\mu)}$. Throughout the rest of this paper, we assume that $0 \leq \mu < \overline{\mu} = (\frac{N-2}{2})^2$. Denote $\beta = \sqrt{\overline{\mu} - \mu}$. For each $(z,\varepsilon) \in \mathbb{R}^N \times \mathbb{R}^+$, we put

$$u_{(z,\varepsilon)}^{(\mu)}(x) = \frac{C\varepsilon^{\frac{N-2}{4}}}{|x-z|^{\frac{N-2}{2}-\beta}\left(\varepsilon + |x-z|^{\frac{4\beta}{N-2}}\right)^{\frac{N-2}{2}}} \qquad \text{for } x \in \mathbb{R}^N,$$

where C is a suitable positive constant. Terracini [12] showed that for each $\varepsilon > 0$, $u_{(0,\varepsilon)}^{(\mu)}$ is a solution of problem (P_μ) with $\Omega = \mathbb{R}^N$. That is, for each $\varepsilon > 0$, $u_{(0,\varepsilon)}^{(\mu)}$ is a critical point of $I^{(\mathbb{R}^N,\mu)}$ with the same critical value. We put $c_\mu = I^{(\mathbb{R}^N,\mu)}(u_{(0,\varepsilon)}^{(\mu)})$ for each $\mu \geq 0$ and $\varepsilon > 0$. It is also known that for each $z \in \mathbb{R}^N$ and $\varepsilon > 0$, $u_{(z,\varepsilon)}^{(0)}$ is a critical point of $I^{(\mathbb{R}^N,0)}$ with the same critical value c_0. One can see that $c_\mu < c_0$ for each $\mu > 0$. We put

$$\mathcal{S}_\mu = \left\{v \in D^{1,2}(\mathbb{R}^N)\setminus\{0\} : \int_{\mathbb{R}^N}\left(|\nabla v|^2 - \frac{\mu|v|^2}{|x|^2}\right)dx = \int_{\mathbb{R}^N}|v|^{2^*}\,dx\right\}$$

for each $\mu \geq 0$. We put $\mathcal{S}_\mu(\Omega) = \mathcal{S}_\mu \cap H$. One can see that for each $\mu \geq 0$ and $v \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}$, there exists $t_{v,\mu} > 0$ such that $t_{v,\mu}v \in \mathcal{S}_\mu$ (cf. [7]). From the definition of $\mathcal{S}_\mu$, we find that

$$\left\langle \nabla I^{(\Omega,\mu)}(v), v\right\rangle = 0 \qquad \text{for all } v \in \mathcal{S}_\mu. \tag{2.1}$$

Then each nontrivial critical point of $I^{(\Omega,\mu)}$ is contained in $\mathcal{S}_\mu$. For each μ, we have

$$c_\mu = \inf\left\{\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 - \frac{\mu\,|v|^2}{|x|^2})dx - \frac{1}{2^*}\int_{\mathbb{R}^N}|v|^{2^*}\,dx : v \in \mathcal{S}_\mu\right\}.$$

From the definition of $\mathcal{S}_\mu$, we also find

$$c_\mu = \inf\left\{m_0\int_{\mathbb{R}^N}(|\nabla v|^2 - \frac{\mu\,|v|^2}{|x|^2}) : v \in \mathcal{S}_\mu\right\}, \tag{2.2}$$

where $m_0 = \frac{2^*-2}{2\cdot 2^*}$. It also follows from the Hardy inequality that $c_\mu \longrightarrow c_0$ as $\mu \longrightarrow 0$. Then we can choose $\mu' \in (0,\overline{\mu})$ such that $2c_\mu > c_0$ for each $\mu \in (0,\mu')$.

Lemma 2.1. *Let $\mu \in (0,\mu')$. If $q \in (0,1)$ and $v \in D^{1,2}(\mathbb{R}^N)$ such that*

$$\int_{\mathbb{R}^N}(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2})dx \le q\frac{c_\mu}{m_0},$$

then

$$\int_{\mathbb{R}^N}(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2} - |v|^{2^*}) \ge (1 - q^{-(2-2^*)/2})\int_{\mathbb{R}^N}(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2}).$$

Proof. Let $\mu \in (0,\mu')$ and $q \in (0,1)$. Let $v \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}$ such that

$$\int_{\mathbb{R}^N}(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2}) \le q\frac{c_\mu}{m_0}.$$

We put $t = t_{v,\mu}$. Then by the definition,

$$t^2\int_{\mathbb{R}^N}(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2}) = t^{2^*}\,|v|_{2^*}^{2^*} \ge \frac{c_\mu}{m_0}$$

and then $t \ge q^{-1/2}$. Therefore

$$\begin{aligned}\int_{\mathbb{R}^N}(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2} - |v|^{2^*}) &= (1 - t^{2-2^*})\int_{\mathbb{R}^N}(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2})\\ &\ge (1 - q^{-(2-2^*)/2})\int_{\mathbb{R}^N}(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2}).\end{aligned}$$ □

Lemma 2.2. *Let $\mu \in (0,\mu')$ and assume that there exists no solution of (P_μ). Let $\{v_n\} \subset \mathcal{S}_\mu(\Omega)$ satisfy*

$$\lim_{n\to\infty} I^{(\Omega,\mu)}(v_n) = c < c_0 \text{ and } \lim_{n\longrightarrow\infty}\nabla I^{(\Omega,\mu)}(v_n) = 0. \tag{2.3}$$

Then there exist $\{\varepsilon_n\}\subset\mathbb{R}^+$ and $\{x_n\}\subset\mathbb{R}^N$ such that $\lim_{n\longrightarrow\infty}|x_n| = 0$, $\lim_{n\longrightarrow\infty}\varepsilon_n = 0$ *and* $\lim_{n\to\infty}\left\|v_n - u^{(\mu)}_{(x_n,\varepsilon_n)}\right\| = 0$.

Proof. The proof of the assertion is based on the standard argument for the case that $\mu = 0$ (cf. Struwe [11]). Fix $\mu \in (0,\mu')$ and assume that there exists no solution of (P_μ). Let $\{v_n\} \subset \mathcal{S}_\mu(\Omega)$ be a sequence satisfying (2.3). From the definition of $\mathcal{S}_\mu$,

$$I^{(\Omega,\mu)}(v_n) = m_0\int_\Omega(|\nabla v_n|^2 - \mu\frac{|v_n|^2}{|x|^2})dx \qquad \text{for } n \ge 1.$$

Then since $\lim_{n\to\infty} I^{(\Omega,\mu)}(v_n) = c$, we have by the Hardy inequality that $\{v_n\}$ is bounded in H. Then we may assume that $v_n \longrightarrow v$ weakly in H, $v_n \longrightarrow v$ strongly in $L^{2^*-1}(\Omega)$ and $v_n \longrightarrow v$ a.e. on Ω. Then we find that v is a solution of problem (P_μ). If $v \neq 0$, this contradicts the assumption. Therefore we have that $v_n \longrightarrow 0$ strongly in $L^{2^*-1}(\Omega)$ and $v_n \longrightarrow 0$ a.e. on Ω. Let k_0, k_1 be positive integers such that $B_2(0)$ can be covered by k_0 balls with radius 1, Ω can be covered by k_1 balls with radius 1, and $c_0 < 2k_0 < k_1$. Let $\{(x_n, r_n)\} \subset \mathbb{R}^N \times \mathbb{R}^+$ be a sequence such that

$$r_n = \min\left\{ r > 0 : m_0 \int_{B_r(x)} (|\nabla v_n|^2 - \mu\frac{|v_n|^2}{|x|^2}) = \frac{c_0}{k_1} \quad \text{for some } x \in \mathbb{R}^N \right\}$$

and

$$m_0 \int_{B_{r_n}(x_n)} (|\nabla v_n|^2 - \mu\frac{|v_n|^2}{|x|^2}) = \frac{c_0}{k_1}$$

for each $n \geq 1$. Since $v_n \longrightarrow 0$ a.e. on Ω, one can see that $r_n \longrightarrow 0$ as $n \longrightarrow \infty$. Here we put

$$w_n(x) = r_n^{(N-2)/2} v_n(x_n + r_n x) \qquad \text{for } x \in \mathbb{R}^N.$$

Then $w_n \in H_0^1(\Omega_n)$, where $\Omega_n = \left\{x \in \mathbb{R}^N : x_n + r_n x \in \Omega\right\}$. It also follows from (2.3) that

$$-\Delta w_n - \mu\frac{r_n w_n}{|x_n + r_n x|^2} - w_n^{2^*-1} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

We also have that

$$\int_{B_{r_n}(x_n)} (|\nabla v_n|^2 - \mu\frac{|v_n|^2}{|x|^2}) = \int_{B_1(0)} (|\nabla w_n|^2 - \mu\frac{r_n^2\,|w_n|^2}{|x_n + r_n x|^2}) = \frac{c_0}{k_1 m_0}$$

for each $n \geq 1$. Then since $r_n \leq 1$ for each $n \geq 1$, $w_n \longrightarrow w_0 \in D^{1,2}(\mathbb{R}^N)$ weakly in $D^{1,2}(\mathbb{R}^N)$. We will see that $w_n \longrightarrow w_0$ strongly in $D^{1,2}(\Omega')$ for any bounded domain $\Omega' \subset \mathbb{R}^N$. It is sufficient to show the case that $\Omega' = B_1(0)$. For each $n \geq 1$, we can construct functions $\{\varphi_n\}$ such that $\varphi_n = w_n - w_0$ on $B_\rho(0)$ for each $n \geq 1$, where $\rho \in (1,2)$ and $\|\varphi_n\|_{D^{1,2}(\mathbb{R}^N \setminus B_\rho(0))} \longrightarrow 0$ as $n \longrightarrow \infty$ (cf. the proof of Lemma 3.3 of [11]). Then by using the Brezis-Lieb lemma we have that

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \int_{\mathbb{R}^N} (|\nabla \varphi_n|^2 & - \mu\frac{r_n^2\,|\varphi_n|^2}{|x_n + r_n x|^2}) \\
&= \lim_{n\longrightarrow\infty} \int_{B_\rho(0)} (|\nabla \varphi_n|^2 - \mu\frac{r_n^2\,|\varphi_n|^2}{|x_n + r_n x|^2}) \\
&= \lim_{n\longrightarrow\infty} \int_{B_\rho(0)} (|\nabla w_n|^2 - \mu\frac{r_n^2\,|w_n|^2}{|x_n + r_n x|^2}) \\
&\leq \lim_{n\longrightarrow\infty} \int_{B_{2r_n}(x_n)} (|\nabla v_n|^2 - \mu\frac{|v_n|^2}{|x|^2}) \\
&\leq \frac{k_0 c_0}{k_1 m_0}.
\end{aligned}$$

Since $\frac{k_0 c_0}{k_1 m_0} < \frac{c_0}{2m_0} < \frac{c_\mu}{m_0}$, we have by Lemma 2.1 that there exists $C > 0$ such that

$$\begin{aligned}
0 &= \lim_{n\longrightarrow\infty} \int_{\mathbb{R}^N} (\bigtriangledown w_n \bigtriangledown \varphi_n - \mu \frac{r_n^2 w_n \varphi_n}{|x_n + r_n x|^2} - w_n^{2^*-1}\varphi_n) \\
&= \lim_{n\longrightarrow\infty} \int_{B_\rho(0)} (|\bigtriangledown (w_n - w_0)|^2 - \mu \frac{r_n^2 |w_n - w_0|^2}{|x_n + r_n x|^2} - |w_n - w_0|^{2^*}) \\
&= \lim_{n\longrightarrow\infty} \int_{B_\rho(0)} (|\bigtriangledown \varphi_n|^2 - \mu \frac{r_n^2 |\varphi_n|^2}{|x_n + r_n x|^2} - |\varphi_n|^{2^*}) \\
&= \lim_{n\longrightarrow\infty} \int_{\mathbb{R}^N} (|\bigtriangledown \varphi_n|^2 - \mu \frac{|\varphi_n|^2}{\left|\frac{x_n}{r_n} + x\right|^2} - |\varphi_n|^{2^*}) \\
&\geq \lim_{n\longrightarrow\infty} C \int_{\mathbb{R}^N} (|\bigtriangledown \varphi_n|^2 - \mu \frac{|\varphi_n|^2}{\left|\frac{x_n}{r_n} + x\right|^2}).
\end{aligned}$$

Then we have by the Hardy inequality that $\lim_{n\longrightarrow\infty} |\bigtriangledown \varphi_n|_2^2 = 0$ and then $w_n \longrightarrow w_0$ strongly in $D^{1,2}(B_1(0))$. Therefore we have that $w_n \longrightarrow w$ strongly in $H_0^1(\Omega')$ for each bounded domain $\Omega' \subset \mathbb{R}^N$. We may assume by subtracting subsequences that $x_n/r_n \longrightarrow x_0 \in \mathbb{R}^N$ or $|x_n/r_n| \longrightarrow \infty$ holds as $n \longrightarrow \infty$. If $|x_n/r_n| \longrightarrow \infty$ holds, then recalling that

$$-\Delta w_n - \mu \frac{w_n}{\left|\frac{x_n}{r_n} + x\right|^2} - w_n^{2^*-1} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{2.4}$$

we have $w_0 \in D^{1,2}(\mathbb{R}^N)$ is a solution of the problem

$$-\Delta w = w^{2^*-1} \qquad \text{on } \mathbb{R}^N.$$

Then we find

$$c_0 > \lim_{n\rightarrow\infty} I^{(\Omega,\mu)}(v_n) \geq \lim_{n\longrightarrow\infty} I^{(\mathbb{R}^N,0)}(w_0) = c_0.$$

This is a contradiction. Therefore we have that $x_n/r_n \longrightarrow x_0 \in \mathbb{R}^N$. This implies that $\lim_{n\longrightarrow\infty} |x_n| = 0$, and w_0 is a positive solution of problem

$$-\Delta w - \mu \frac{w}{|x_0 + x|^2} = w^{2^*-1} \qquad \text{on } \mathbb{R}^N.$$

By the translation, we may assume without any loss of generality that $x_0 = 0$. Then $\lim_{n\longrightarrow\infty} I^{(\mathbb{R}^N,\mu)}(w_n) \geq I^{(\mathbb{R}^N,\mu)}(w_0) = c_\mu$. Suppose that $\lim_{n\longrightarrow\infty} |\bigtriangledown (w_n - w_0)|_2^2 > 0$. Then noting that

$$\begin{gathered}
\lim_{n\longrightarrow\infty} |\bigtriangledown (w_n - w_0)|_2^2 + |\bigtriangledown w_0|_2^2 = \lim_{n\longrightarrow\infty} |\bigtriangledown w_n|_2^2, \\
\lim_{n\longrightarrow\infty} |w_n - w_0|_{2^*}^{2^*} + |w_0|_{2^*}^{2^*} = \lim_{n\longrightarrow\infty} |w_n|_{2^*}^{2^*},
\end{gathered}$$

and

$$\lim_{n\longrightarrow\infty} \int_{\mathbb{R}^N} \frac{|w_n - w_0|^2}{|x|^2} + \int_{\mathbb{R}^N} \frac{|w_0|^2}{|x|^2} = \lim_{n\longrightarrow\infty} \int_\Omega \frac{|w_n|^2}{|x|^2},$$

we find

$$\lim_{n\longrightarrow\infty}\int_{\mathbb{R}^N}(|\bigtriangledown(w_n-w_0)|^2-\mu\frac{|w_n-w_0|^2}{|x|^2}-|w_n-w_0|^{2^*})=0.$$

Then we have

$$\lim_{n\longrightarrow\infty}\inf I^{(\mathbb{R}^N,\mu)}(w_n-w_0)\geq c_\mu.$$

Therefore we find that

$$\lim_{n\longrightarrow\infty}\inf I^{(\mathbb{R}^N,\mu)}(w_n)\geq I^{(\mathbb{R}^N,\mu)}(w_0)+\lim_{n\longrightarrow\infty}\inf I^{(\mathbb{R}^N,\mu)}(w_n-w_0)\geq 2c_\mu.$$

This is a contradiction. Thus we have $w_n \longrightarrow w_0$ strongly in $D^{1,2}(\mathbb{R}^N)$ and $I^{(\mathbb{R}^N,\mu)}(w_0)=c_\mu$. Therefore $w_0=u^{(\mu)}_{(0,\varepsilon)}$ for some $\varepsilon>0$. Then from the definition of w_n, we find that the assertion holds. $\square$

In case $\mu=0$, it is known that analogous results to Lemma 2.2 hold. We state a lemma for the case $\mu=0$. The proof is quite same as that of Lemma 2.2 above.

Lemma 2.3. *Let* $\{v_n\}\subset H$ *be a sequence such that*

$$\lim_{n\longrightarrow\infty}|\bigtriangledown v_n|^2=\lim_{n\longrightarrow\infty}|v_n|^{2^*}_{2^*}\quad \textit{and}\quad \lim_{n\longrightarrow\infty}I^{(\Omega,0)}(v_n)\leq c_0.$$

Then there exists a sequence $\{(z_n,\varepsilon_n)\}\subset\mathbb{R}^N\times\mathbb{R}^+$ *such that* $\lim_{n\longrightarrow\infty}\varepsilon_n=0$ *and* $\lim_{n\longrightarrow\infty}\left\|v_n-u^{(0)}_{(z_n,\varepsilon_n)}\right\|=0$.

Here we choose a positive number $d>0$ such that $\Omega_d\cong\Omega$. We put

$$\beta(v)=\frac{\int_\Omega x\,|\nabla v(x)|^2\,dx}{\int_\Omega|\nabla v(x)|^2\,dx}\qquad\text{for each } v\in H\setminus\{0\}\,.$$

Lemma 2.4. *There exists* $\mu_0\in(0,\mu')$ *such that for each* $\mu\in(0,\mu_0)$, *and each* $v\in\mathcal{S}_\mu(\Omega)$ *with* $I^{(\Omega,\mu)}(v)<c_0$, $\beta(v)\in\Omega_d$.

Proof. Suppose to the contrary that there exists a sequence $\{(v_n,\mu_n)\}\subset H\times\mathbb{R}^+$ such that $\lim_{n\longrightarrow\infty}\mu_n=0$, $v_n\in S_{\mu_n}$, $I^{(\Omega,\mu_n)}(v_n)<c_0$ and $\beta(v_n)\notin\Omega_d$ for all $n\geq1$. We may assume without any loss of generality that $\beta(v_n)\longrightarrow z_0\in\mathbb{R}^N\diagdown\Omega_d$. Since $\mu_n\longrightarrow0$, we have by Hardy's inequality that

$$\mu_n\int_\Omega\frac{v_n^2}{|x|^2}\longrightarrow0\qquad\text{as } n\longrightarrow\infty.$$

Then we find that

$$\lim_{n\longrightarrow\infty}|\bigtriangledown v_n|^2=\lim_{n\longrightarrow\infty}|v_n|^{2^*}_{2^*}\quad\text{and}\quad\lim_{n\longrightarrow\infty}I^{(\Omega,0)}(v_n)\leq c_0.$$

Then by Lemma 2.3, we find that there exists a sequence $\{(z_n,\varepsilon_n)\}\subset\mathbb{R}^N\times\mathbb{R}^+$ such that $\lim_{n\longrightarrow\infty}\varepsilon_n=0$ and $\lim_{n\longrightarrow\infty}\left\|v_n-u^{(0)}_{(z_n,\varepsilon_n)}\right\|=0$. From the assumption, we have that $\lim_{n\longrightarrow\infty}z_n=z_0$. But since $z_0\notin\Omega_d$, we have that $\lim_{n\longrightarrow\infty}\left\|v_n-u_{(z_n,\varepsilon_n)}\right\|>0$. This is a contradiction. $\square$

Here we choose $d_1>0$ such that $\Omega\cong\Omega^i_{d_1}$. We put

$$\lambda=\inf\left\{\mu/\,|x|^2:x\in\Omega_d\right\}>0.$$

We fix a function $\varphi \in C^\infty(\mathbb{R}^+ : [0,1])$ such that $\varphi(x) = 1$ for $x \in [0, d_1/2]$ and $\varphi(x) = 0$ for $x \in [d_1, \infty)$. For each $(z,\varepsilon) \in \mathbb{R}^N \times \mathbb{R}^+$, we define a function $v_{(z,\varepsilon)}$ by

$$v_{(z,\varepsilon)}(x) = \tau_{(z,\varepsilon)}\varphi(x-z)u^{(0)}_{(z,\varepsilon)}(x) \qquad \text{for } x \in \mathbb{R}^N,$$

where $\tau_{(z,\varepsilon)}$ is a positive constant such that

$$\left\|v_{(x,\varepsilon)}\right\|^2 - \lambda \left|v_{(x,\varepsilon)}\right|_2^2 = \left|v_{(x,\varepsilon)}\right|_{2^*}^{2^*}.$$

Lemma 2.5. *Let $\mu \in (0,\mu_0)$. Then there exists $\varepsilon > 0$ such that*

$$\sup\left\{ I^{(\Omega,\mu)}(t_{v_{(z,\varepsilon)},\mu} v_{(z,\varepsilon)}) : z \in \Omega^i_{d_1}\right\} < c_0.$$

Proof. We put

$$Q(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 - \lambda |v|^2) - \frac{1}{2^*}\int_\Omega |v|^{2^*}$$

for $v \in H$. Then by the results of Brezis and Nirenberg [3], we have that

$$Q(v_{(z,\varepsilon)}) = \begin{cases} c_0 - \lambda C\varepsilon + o(\varepsilon) & \text{for } N \geq 5, \\ c_0 - \lambda C\varepsilon \log\varepsilon + O(\varepsilon) & \text{for } N = 4. \end{cases} \tag{2.5}$$

Then we can choose $\varepsilon > 0$ sufficiently small so that $Q(v_{(z,\varepsilon)}) < c_0$ for all $z \in \Omega^i_{d_1}$. Let $z \in \mathbb{R}^N$. We put $t = t_{v_{(z,\varepsilon)},\mu}$ Then noting that $\mu/|x|^2 > \lambda$ for $x \in \Omega$, we have that

$$\begin{aligned} \frac{t^2}{2}\int_\Omega (\left|\nabla v_{(z,\varepsilon)}\right|^2 - \lambda \left|v_{(z,\varepsilon)}\right|^2) &> \frac{t^2}{2}\int_\Omega (\left|\nabla v_{(z,\varepsilon)}\right|^2 - \mu\frac{\left|v_{(z,\varepsilon)}\right|^2}{|x|^2}) \\ &= \frac{t^{2^*}}{2}\int_\Omega \left|v_{(z,\varepsilon)}\right|^{2^*} \\ &= \frac{t^{2^*}}{2}\int_\Omega (\left|\nabla v_{(z,\varepsilon)}\right|^2 - \lambda \left|v_{(z,\varepsilon)}\right|^2). \end{aligned}$$

Then $t < 1$. Therefore we have that

$$\begin{aligned} I^{(\Omega,\mu)}(tv_{(z,\varepsilon)}) &= \frac{t^2(2^*-2)}{2\cdot 2^*}(\left|\nabla v_{(z,\varepsilon)}\right|_2^2 - \mu\int_\Omega \frac{\left|v_{(z,\varepsilon)}\right|^2}{|x|^2}) \\ &\leq \frac{(2^*-2)}{2\cdot 2^*}(\left|\nabla v_{(z,\varepsilon)}\right|_2^2 - \lambda \left|v_{(z,\varepsilon)}\right|_2^2) \\ &= Q(v_{(z,\varepsilon)}) < c_0, \end{aligned}$$

and this completes the proof. □

Proof of Theorem 1.1. Fix $\mu \in (0,\mu_0)$. Let $\rho : \mathcal{S}_\mu(\Omega) \times [0,\infty) \longrightarrow \mathcal{S}_\mu(\Omega)$ be a pseudo-gradient flow associated with $I^{(\Omega,\mu)}$ (cf. [11]). That is, ρ satisfies:

(1) For $s,t \in \mathbb{R}^+$ with $t > s$ and $v \in \mathcal{S}_\mu$ with $\nabla I^{(\Omega,\mu)}(v) \neq 0$,

$$I^{(\Omega,\mu)}(\rho(t,v)) < I^{(\Omega,\mu)}(\rho(s,v));$$

(2) $\lim_{t\longrightarrow\infty} I^{(\Omega,\mu)}(\rho(t,v)) > -\infty$ implies that $\lim_{t\longrightarrow\infty} \nabla I^{(\Omega,\mu)}((t,v)) = 0$.

Here we put $V = \left\{t_{v_{(z,\varepsilon)},\mu} v_{(z,\varepsilon)} : z \in \Omega^i_{d_1}\right\}$, where ε is the positive constant obtained in Lemma 2.5. From the definition of $v_{(z,\tau)}$, we have that $v_{(z,\varepsilon)} \in H^1_0(\Omega)$ for $z \in \Omega^i_{d_1}$. Then $V \subset \mathcal{S}_\mu(\Omega)$. Then since $\left\{I^{(\Omega,\mu)}(\rho(t,v)) : t \geq 0\right\}$ is bounded from below for each $v \in V$, we have that $\lim_{n\longrightarrow\infty} \nabla I^{(\Omega,\mu)}(\rho(t,v)) = 0$ for each $v \in V$.

Then we have by Lemma 2.2 that for each $v \in V$, there exists $\{(z_t, \varepsilon_t)\}_{t \geq 0} \subset \Omega \times \mathbb{R}^+$ such that $\lim_{t \longrightarrow \infty} |z_t| = 0$ and $\lim_{t \longrightarrow \infty} \left\| \rho(t,v) - u^{(\mu)}_{(z_t,\varepsilon_t)} \right\| = 0$. This implies that

$$\lim_{t \longrightarrow \infty} \beta(\rho(t,v)) = 0 \in \Omega \qquad \text{for all } v \in V.$$

We also have by Lemma 2.4 that

$$\{\beta(\rho(t,v)) : v \in V\} \subset \Omega_d.$$

Since $\{\beta(\rho(0,v)) : v \in V\} = \Omega^i_{d_1}$, we have that $\Omega^i_{d_1}$ is contractible in Ω_d. This contradicts the assumption that $\Omega^i_{d_1} \cong \Omega \cong \Omega_d$ and that Ω is not contractible. Then we obtain that there exists a positive solution u of (P_μ) in S_μ. □

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