

**A REMARK ON THE EXISTENCE OF SUITABLE VECTOR  
FIELDS RELATED TO THE DYNAMICS OF SCALAR  
SEMI-LINEAR PARABOLIC EQUATIONS**

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ABSTRACT. In 1992, P. Poláčik showed that one could linearly imbed any vector field into a scalar semi-linear parabolic equation on  $\Omega$  with Neumann boundary condition provided that there exists a smooth vector field  $\Phi = (\phi_1, \dots, \phi_n)$  on  $\overline{\Omega}$  such that

$$\begin{cases} \text{rank}(\Phi(x), \partial_1 \Phi(x), \dots, \partial_n \Phi(x)) = n \text{ for all } x \in \overline{\Omega}, \\ \frac{\partial \Phi}{\partial \nu} = 0 \text{ on } \partial \Omega. \end{cases}$$

In this short paper, we give a classification of all the domains on which one may find such a type of vector field.

Let  $\Omega$  be a bounded smooth domain in  $\mathbb{R}^n$ , let  $\nu$  be the outer normal direction of  $\partial \Omega$  and let  $f \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ . The infinite-dimensional dynamical system defined by

$$(1) \quad \begin{cases} u_t = \Delta u + f(x, u, \nabla u), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega, \end{cases}$$

on suitable Sobolev spaces has attracted much interest (see [A, H] and more recent references at the end of this note). When  $n = 1$ , (1) has rather simple dynamics, and each bounded solution will converge to an equilibrium. The situation is quite different when  $n \geq 2$ ; the solutions of (1) can exhibit very complicated behavior. One relatively easy way to demonstrate the complexity of its dynamical behavior is the realization of ODEs in (1). We refer the reader to the recent survey paper [P4] by P. Poláčik for a quick overview of the progress made up to 2002; more results can be found in [DP, P1, P2, P3, P5, PR, Pr, PrR1, PrR2, PrR3, R] and the references therein.

In particular, the following nice result was proved in [P2]: if there exists a smooth vector field  $\Phi$  on  $\overline{\Omega}$ ,  $\Phi = (\phi_1, \dots, \phi_n)$  such that

$$\begin{cases} \text{rank}(\Phi(x), \partial_1 \Phi(x), \dots, \partial_n \Phi(x)) = n \text{ for all } x \in \overline{\Omega}, \\ \frac{\partial \Phi}{\partial \nu} = 0 \text{ on } \partial \Omega, \end{cases}$$

then for any smooth vector field  $X$  on  $\mathbb{R}^n$ , there exists a smooth function  $f$ , such that the linear space  $\text{span}\{\phi_1, \dots, \phi_n\}$  is invariant under (1) and for any integral curve of  $X$ ,  $c = c(t)$ ,  $u = \sum_{i=1}^n c_i(t) \phi_i(x)$  is a solution to (1). Moreover, it was

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shown that this kind of vector field always exists on a starshaped domain. The main result of this short note is a classification of all the domains on which one may find this type of vector field. More precisely, we have

**Theorem 1.** *Let  $\Omega \subset \mathbb{R}^n$  ( $n \geq 2$ ) be an open bounded smooth domain. Then the necessary and sufficient condition for the existence of a smooth map  $F : \overline{\Omega} \rightarrow \mathbb{R}^n$  with*

$$(2) \quad \begin{cases} \text{rank}(F(x), \partial_1 F(x), \dots, \partial_n F(x)) = n \text{ for any } x \in \overline{\Omega}, \\ \frac{\partial F}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases}$$

is that  $\overline{\Omega}$  is diffeomorphic to  $\overline{B_1}$  or  $\overline{B_2} \setminus B_1$ . Here  $B_1$  and  $B_2$  are two open balls centered at zero with radius 1 and 2, respectively.

*Remark 1.* In fact, if  $\overline{\Omega}$  is diffeomorphic to  $\overline{B_1}$ , then any solution to (2),  $F$ , must have exactly one zero in  $\Omega$ . If  $\overline{\Omega}$  is diffeomorphic to  $\overline{B_2} \setminus B_1$ , then any solution to (2),  $F$ , does not vanish at all. These conclusions will follow from the arguments below.

*Remark 2.* Our theorem improves the linear imbedding result of P. Poláčik [P2], and at the same time, it exhibits the limitation of the linear realization method. On the other hand, realization of ODEs in a nonlinear fashion can be made in a much more general setting. For example, it is shown in [P3, PrR2, PrR3] that any ODE, in any dimension, has an arbitrarily small perturbation that is nonlinearly realizable in a single semi-linear parabolic equation defined on any given open subset of  $\mathbb{R}^n$ ,  $n \geq 2$ .

To prove our main result, we first show that Neumann boundary condition in (2) can be relaxed.

**Lemma 1.** *Let  $\Omega \subset \mathbb{R}^n$  ( $n \geq 2$ ) be an open bounded smooth domain. If there exists a smooth map  $G : \overline{\Omega} \rightarrow \mathbb{R}^n$  such that*

$$\text{rank}(G(x), \partial_1 G(x), \dots, \partial_n G(x)) = n \text{ for any } x \in \overline{\Omega}$$

and

$$\dim \text{span} \left\{ G(x), \text{im}(G|_{\partial\Omega})_{*,x} \right\} = n \text{ for any } x \in \partial\Omega$$

(here  $(G|_{\partial\Omega})_{*,x}$  denotes the tangent map of  $G|_{\partial\Omega}$  at  $x$ ), then we may find a smooth map  $F : \overline{\Omega} \rightarrow \mathbb{R}^n$  satisfying (2).

*Proof.* Let  $\varepsilon > 0$  be small enough such that the map

$$\phi : \partial\Omega \times [0, 3\varepsilon] \rightarrow \{y \in \overline{\Omega} : \text{dist}(y, \partial\Omega) \leq 3\varepsilon\}$$

defined by

$$\phi(x, t) = x - t\nu(x)$$

is a diffeomorphism. Let  $P = G \circ \phi$ . For any  $t \in [0, 3\varepsilon]$ , let  $P_t(x) = P(x, t)$  for  $x \in \partial\Omega$ . Then we may assume  $\varepsilon$  is small enough such that for any  $t \in [0, 3\varepsilon]$ ,

$$\dim \text{span} \left\{ P_t(x), \text{im}(P_t)_{*,x} \right\} = n \text{ for all } x \in \partial\Omega.$$

Let  $\eta : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function such that

$$\eta(t) = \begin{cases} \varepsilon, & \text{when } t \leq \varepsilon/2, \\ t, & \text{when } t \geq 3\varepsilon/2, \end{cases}$$

and  $\eta'(t) \geq 0$  for all  $t$ . Define

$$Q_t(x) = Q(x, t) = P(x, \eta(t)) \text{ for } x \in \partial\Omega, 0 \leq t \leq 3\varepsilon.$$

Then it is clear that for any  $t \in [0, 3\varepsilon]$ ,

$$\dim \text{span} \{ Q_t(x), \text{im}(Q_t)_{*,x} \} = n \text{ for any } x \in \partial\Omega.$$

Let

$$F(y) = \begin{cases} G(y) & \text{if } y \in \overline{\Omega}, \text{dist}(y, \partial\Omega) \geq 2\varepsilon, \\ Q(\phi^{-1}(y)) & \text{if } y \in \overline{\Omega}, \text{dist}(y, \partial\Omega) \leq 3\varepsilon. \end{cases}$$

Then it is easy to check that  $F$  satisfies all the requirements. □

Since the existence of a smooth vector field  $G$  in Lemma 1 is a property of the  $\overline{\Omega}$  which is preserved under diffeomorphisms, we conclude

**Corollary 1.** *Assume  $\overline{\Omega}_1$  is diffeomorphic to  $\overline{\Omega}_2$ , and for  $\Omega_1$  we may find a solution to (2); then we may find a solution to (2) for  $\Omega_2$ , too.*

To derive the necessary condition for the existence of a vector field satisfying (2), we will need

**Lemma 2.** *Let  $\Omega \subset \mathbb{R}^n$  ( $n \geq 2$ ) be an open bounded smooth domain, if there exists a smooth map  $H : \overline{\Omega} \rightarrow S^{n-1}$  such that*

$$\text{rank}(\partial_1 H(x), \dots, \partial_n H(x)) = n - 1 \text{ for any } x \in \overline{\Omega}$$

and

$$\dim \text{im}(H|_{\partial\Omega})_{*,x} = n - 1 \text{ for any } x \in \partial\Omega.$$

Then  $\overline{\Omega}$  is diffeomorphic to  $\overline{B}_2 \setminus B_1$ .

*Proof.* First we claim that each path connected component of  $\partial\Omega$  is diffeomorphic to  $S^{n-1}$ . This is clear when  $n = 2$ . If  $n \geq 3$ , since  $H|_{\partial\Omega}$  has full rank everywhere and  $\partial\Omega$  is compact,

$$H|_{\partial\Omega} : \partial\Omega \rightarrow S^{n-1}$$

is a covering map (see [M]). Since  $S^{n-1}$  is simply connected, we see that each path connected component of  $\partial\Omega$  must be diffeomorphic to  $S^{n-1}$ . Indeed, the restriction of  $H$  to such a component serves as a diffeomorphism.

To proceed, we observe that from the assumption on  $H$ , it follows from implicit function theorem that for any  $\xi \in S^{n-1}$ ,  $H^{-1}(\xi)$  is a smooth one-dimensional submanifold of  $\overline{\Omega}$ . Moreover  $H : \overline{\Omega} \rightarrow S^{n-1}$  is a smooth fiber bundle (see [M]). Fix a point  $x_0 \in \partial\Omega$ , and let  $\xi_0 = H(x_0)$  and  $\Gamma = H^{-1}(\xi_0)$ . Then we have an exact sequence (see Theorem 6.7 of chapter VII in [B])

$$\pi_{n-1}(\Gamma, x_0) \rightarrow \pi_{n-1}(\overline{\Omega}, x_0) \rightarrow \pi_{n-1}(S^{n-1}, \xi_0) \rightarrow \pi_{n-2}(\Gamma, x_0).$$

If  $n \geq 3$ , then both  $\pi_{n-1}(\Gamma, x_0)$  and  $\pi_{n-2}(\Gamma, x_0)$  vanish. This shows that

$$\pi_{n-1}(\overline{\Omega}, x_0) \cong \mathbb{Z}$$

and hence  $\overline{\Omega}$  is diffeomorphic to  $\overline{B}_2 \setminus B_1$ . If  $n = 2$ , then since  $\pi_1(\Gamma, x_0)$  vanishes and  $\pi_0(\Gamma, x_0)$  is finite, we see  $\pi_1(\overline{\Omega}, x_0)$  is again isomorphic to  $\mathbb{Z}$ . This shows that  $\overline{\Omega}$  must be diffeomorphic to  $\overline{B}_2 \setminus B_1$ . □

Now we are ready to prove the main theorem.

*Proof of Theorem 1.* First if  $\Omega = B_1$  or  $B_2 \setminus \overline{B_1}$ , then  $G(x) = x$  satisfies the assumption in Lemma 1; half of the theorem follows from Corollary 1. On the other hand, assume for some  $\Omega$  that we may find a smooth map  $F$  satisfying (2). For  $x \in \partial\Omega$ , choose a base for the tangent space of  $\partial\Omega$  at  $x$ , namely  $e_1, \dots, e_{n-1}$ . Then

$$\begin{aligned} & \text{rank}(F(x), \partial_1 F(x), \dots, \partial_n F(x)) \\ &= \text{rank}(F(x), F_* e_1, \dots, F_* e_{n-1}, F_* \nu) \\ &= \text{rank}(F(x), F_* e_1, \dots, F_* e_{n-1}) = n; \end{aligned}$$

hence  $F(x) \neq 0$  for any  $x \in \partial\Omega$ . Moreover, it follows from the fact

$$\text{rank}(F(x), \partial_1 F(x), \dots, \partial_n F(x)) = n \text{ for any } x \in \overline{\Omega}$$

that the zeroes of  $F$  in  $\Omega$  must be isolated. Hence  $F$  has at most finitely many zeroes, say  $x_1, \dots, x_m$ ; here  $m \geq 0$ . For  $\varepsilon > 0$  small enough, let

$$U = \Omega \setminus \bigcup_{i=1}^m \overline{B_\varepsilon(x_i)}$$

if  $m \geq 1$  and  $U = \Omega$  if  $m = 0$ . Then we have

$$\text{rank}(F(x), \partial_1 F(x), \dots, \partial_n F(x)) = n \text{ for any } x \in \overline{U}$$

and

$$\dim \text{span} \left\{ F(x), \text{im}(F|_{\partial U})_{*,x} \right\} = n \text{ for any } x \in \partial U.$$

Let

$$H(x) = \frac{F(x)}{|F(x)|} \text{ for } x \in \overline{U}.$$

Then clearly

$$\text{rank}(\partial_1 H(x), \dots, \partial_n H(x)) = n - 1 \text{ for any } x \in \overline{U}$$

and

$$\dim \text{im}(H|_{\partial U})_{*,x} = n - 1 \text{ for any } x \in \partial U.$$

It follows from Lemma 2 that  $\overline{U}$  must be diffeomorphic to  $\overline{B_2} \setminus B_1$ , hence  $m \leq 1$  and  $\overline{\Omega}$  must be diffeomorphic to either  $\overline{B_1}$  or  $\overline{B_2} \setminus B_1$ .  $\square$

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