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A THEOREM ON REFLEXIVE LARGE RANK OPERATOR SPACES

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Abstract. If every nonzero operator in an n-dimensional operator space $\mathbb{S}$ has rank $\geqslant 2n$, then $\mathbb{S}$ is reflexive.

Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$, and let $\mathcal{L}(\mathbb{V})$ be the space of all linear operators on $\mathbb{V}$. Suppose $\mathbb{S}$ is a finite-dimensional subspace of $\mathcal{L}(\mathbb{V})$. The *reflexive closure* of $\mathbb{S}$ is given by

$$\mathit{ref}\ \mathbb{S} = \{t \in \mathcal{L}(\mathbb{V}) : tx \in \mathbb{S}x \text{ for all } x \in \mathbb{V}\},$$

where $\mathbb{S}x = \{sx : s \in \mathbb{S}\}$, which is called the *cyclic subspace.* We note that every element t of $\mathit{ref}\ \mathbb{S}$ interpolates $\mathbb{S}$, or t is locally in $\mathbb{S}$. It is obvious that $\mathbb{S} \subseteq \mathit{ref}\ \mathbb{S}$. If $\mathbb{S} = \mathit{ref}\ \mathbb{S}$, then $\mathbb{S}$ is said to be *reflexive.* The reflexivity problem was originally introduced as a topological notion by P. Halmos to describe those operator algebras that are determined by their closed invariant subspace lattices of a Hilbert space. Loginov and Sulman extended reflexivity to include operator subspaces which are not necessarily algebras. The purely algebraic version of reflexivity and a unified approach was introduced by Hadwin [4, 5]. When the operator subspace is finite dimensional the topological version and the algebraic version of reflexivity coincide. The reflexivity problem is far from being complete. A historical account can be found in [1, 2, 4, 5, 6, 8].

A well-known result of Larson [6] asserts that for the finite-dimensional $\mathbb{S}$, $\mathit{ref}\ \mathbb{S} = \mathbb{S} + \mathit{ref}\ \mathbb{S}_F$, where $\mathbb{S}_F$ is the collection of all finite rank operators in $\mathbb{S}$. Hence if $\mathbb{S}_F = \{0\}$, then $\mathbb{S}$ is reflexive. In [2, 3] we further find that if the rank of each nonzero operator in $\mathbb{S}$ is large enough, then $\mathbb{S}$ is reflexive. For example, in [2] we proved that if the dimension of $\mathbb{S}$ is n, and every nonzero operator of $\mathbb{S}$ has rank $> 2n^2 - n$, then $\mathbb{S}$ is reflexive. This result was improved to that if every nonzero operator of $\mathbb{S}$ has rank $> n^2$, then $\mathbb{S}$ is reflexive [3]. In this paper we lower the rank to $2n$. This is achieved by making full use of Proposition 1.1 in [3] to prove a generalized version of Theorem 2.1 in [3] and using Theorem 1.2 in [3].

A vector x is a *separating vector* for $\mathbb{S}$ if the map $\mathbb{S} \to \mathbb{S}x$ by sending s to sx is injective. A linear subspace $\mathbb{M}$ of $\mathbb{V}$ is said to be a *separating subspace* for $\mathbb{S}$ if the only member s of $\mathbb{S}$ satisfying $s(\mathbb{M}) = \{0\}$ is $s = 0$. It is easy to see that if $\mathbb{S}$ has a separating vector x, then any subspace containing x is a separating subspace for $\mathbb{S}$. However, if the dimension of a separating subspace is greater than 1, then $\mathbb{S}$ may

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not have a separating vector. Trivially, $\mathbb{V}$ is a separating subspace for $\mathbb{S}$. Of course, if $\mathbb{M}$ is a separating subspace for $\mathbb{S}$, then any subspace containing $\mathbb{M}$ is a separating subspace for $\mathbb{S}$. In [3] we proved the following theorem:

Theorem 1 (Theorem 1.2, [3]). *Suppose the linear subspace* $\mathbb{S} \subseteq \mathcal{L}(\mathbb{V})$ *has a separating vector* x *and a separating subspace* $\mathbb{M}$ *satisfying*

(1) $\dim \mathbb{S} <$ *cardinality of the field* $\mathbb{F}$,
(2) $\mathbb{S}x \cap \mathbb{S}(\mathbb{M}) = \{0\}$.

Then $\mathbb{S}$ *is reflexive.*

Proposition 2 (Proposition 1.1, [3]). *Suppose* $\mathbb{S}$ *is a linear subspace of* $\mathcal{L}(\mathbb{V})$, *and the dimension of* $\mathbb{S}$ *is less than the cardinality of* $\mathbb{F}$. *Let* x *be a separating vector for* $\mathbb{S}$, *and let* $\mathbb{W}$ *be a linear subspace of* $\mathbb{V}$ *satisfying* $\mathbb{S}x \cap \mathbb{W} = \{0\}$. *Then for each vector* $y \in \mathbb{V}$ *there is a scalar* $\lambda \in \mathbb{F}$ *such that* $y + \lambda x$ *is a separating vector for* $\mathbb{S}$ *and* $\mathbb{S}(y + \lambda x) \cap \mathbb{W} = \{0\}$.

The proof of the proposition is obtained by using the mapping $\phi = E_x^{-1} \cdot P \cdot E_y$ on $\mathbb{S}$ where E_y is an evaluation map, and P is a projection with range $\mathbb{S}x$, and the fact that the number of different eigenvalues of ϕ is less than the cardinality of the field $\mathbb{F}$. The next Theorem 3 is a generalized version of Theorem 2.1 in [3].

Theorem 3. *Suppose* $\mathbb{V}$ *is a vector space over a field* $\mathbb{F}$. *Let* $\mathbb{W}$ *be a finite-dimensional linear subspace of* $\mathbb{V}$, *and let* $\mathbb{S}$ *be an* n*-dimensional linear subspace of* $\mathcal{L}(\mathbb{V})$, *where* n *is less than the cardinality of the field* $\mathbb{F}$. *If every nonzero operator in* $\mathbb{S}$ *has rank greater than or equal to* $n + \dim \mathbb{W}$, *then* $\mathbb{S}$ *has a separating vector* x *such that* $\mathbb{S}x \cap \mathbb{W} = \{0\}$.

Proof. We proceed by induction on the dimension n. For $n = 1, \mathbb{S} = \operatorname{span}\{s\}$, and rank of $s \geqslant 1 + \dim \mathbb{W}$. Choose a vector $y \in$ range of s and $y \notin \mathbb{W}$. Let $sx = y$. Then x separates $\mathbb{S}$ and $\mathbb{S}x \cap \mathbb{W} = \{0\}$.

Assume the statement is true for each $(n-1)$-dimensional subspace of $\mathcal{L}(\mathbb{V})$. We consider n-dimensional $\mathbb{S}$. Let $s_1, \cdots, s_n$ be a basis of $\mathbb{S}$. Denote by $\mathbb{S}'$ the linear span of $\{s_1, \cdots, s_{n-1}\}$. By the induction assumption we get a separating vector x for $\mathbb{S}'$ and $\mathbb{S}'x \cap \mathbb{W} = \{0\}$.

If $s_n x \notin$ linear span $\{\mathbb{S}'x, \mathbb{W}\}$, then for any scalars $\alpha_1, \cdots, \alpha_{n-1}, \alpha_n \in \mathbb{F}$, with $\alpha_1 s_1 x + \cdots + \alpha_{n-1} s_{n-1} x + \alpha_n s_n x \in \mathbb{W}$, we have $\alpha_n s_n x = z - (\alpha_1 s_1 x + \cdots + \alpha_{n-1} s_{n-1} x)$ for some $z \in \mathbb{W}$. Since $s_n x \notin$ linear span $\{\mathbb{S}'x, \mathbb{W}\}$, $\alpha_n s_n x = 0$, and $z - (\alpha_1 s_1 x + \cdots + \alpha_{n-1} s_{n-1} x) = 0$. It follows that $\alpha_n = 0$, and $z = 0$ and $\alpha_1 s_1 x + \cdots + \alpha_{n-1} s_{n-1} x = 0$, since $\mathbb{S}'x \cap \mathbb{W} = \{0\}$. Then $\alpha_1 = \cdots = \alpha_{n-1} = 0$. This proves that $\mathbb{S}x \cap \mathbb{W} = \{0\}$, and $s_1 x, \cdots, s_{n-1} x, s_n x$ are linearly independent, and hence x is a separating vector for $\mathbb{S}$. The proof of the theorem is done in this case. So we need only consider the case that $s_n x \in$ linear span $\{\mathbb{S}'x, \mathbb{W}\}$.

Let $s_n x = (\alpha_1 s_1 x + \cdots + \alpha_{n-1} s_{n-1} x) + w$, for some scalars $\alpha_1, \cdots, \alpha_{n-1} \in \mathbb{F}$, and $w \in \mathbb{W}$. Replacing s_n by $s_n' = s_n - \alpha_1 s_1 - \cdots - \alpha_{n-1} s_{n-1}$, we may assume $s_n x \in \mathbb{W}$.

Since rank of $s_n \geqslant n + \dim \mathbb{W}$, there is $y \in \mathbb{V}$ such that $s_n y \notin$ linear span of $\{\mathbb{S}'x, \mathbb{W}\}$. Note that $\mathbb{S}'x \cap \mathbb{W} = \{0\}$, so this also implies that $\mathbb{S}'x \cap$ linear span $\{s_n y, \mathbb{W}\} = \{0\}$. By Proposition 2, there is scalar λ so that $y + \lambda x$ separates $\mathbb{S}'$, and $\mathbb{S}'(y + \lambda x) \cap$ linear span $\{s_n y, \mathbb{W}\} = \{0\}$. We next claim $y + \lambda x$ is a separating vector for $\mathbb{S}$.

Let $t \in \mathbb{S}'$ and $\alpha \in \mathbb{F}$ with $(t + \alpha s_n)(y + \lambda x) = 0$. Then

$$t(y + \lambda x) = -\alpha s_n(y + \lambda x) = -\alpha s_n y - \alpha\lambda s_n x,$$

where $\alpha\lambda s_n x \in \mathbb{W}$. By the disjointness of $\mathbb{S}'(y + \lambda x)$ and linear span $\{s_n y, \mathbb{W}\}$, we have $t(y + \lambda x) = 0$, and $\alpha s_n y = -\alpha\lambda s_n x$. Hence $t = 0$, and $\alpha = 0$, since $s_n y \notin \mathbb{W}$. This completes the proof of the claim.

Finally, we show that $\mathbb{S}(y + \lambda x) \cap \mathbb{W} = \{0\}$. Indeed, suppose there are scalars $\alpha_1, \cdots, \alpha_n$ satisfying

$$\alpha_1 s_1(y + \lambda x) + \cdots + \alpha_{n-1} s_{n-1}(y + \lambda x) + \alpha_n s_n(y + \lambda x) \in \mathbb{W}.$$

Then $\alpha_1 s_1(y+\lambda x) + \cdots + \alpha_{n-1} s_{n-1}(y+\lambda x) + \alpha_n s_n(y+\lambda x) = z$ for some $z \in \mathbb{W}$. It follows that

$$-\alpha_n s_n y = (\alpha_1 s_1(y + \lambda x) + \cdots + \alpha_{n-1} s_{n-1}(y + \lambda x)) + (\alpha_n \lambda s_n x - z).$$

Since $\alpha_n \lambda s_n x - z \in \mathbb{W}$ and $s_n y \notin$ linear span of $\{\mathbb{S}'x, \mathbb{W}\}$, we conclude that $\alpha_n s_n y = 0$. It follows that $\alpha_n = 0$. Again, the disjointness of $\mathbb{S}'(y + \lambda x) \cap \mathbb{W} = \{0\}$ implies that $z = 0$, and $\alpha_1 s_1(y + \lambda x) + \cdots + \alpha_{n-1} s_{n-1}(y + \lambda x) = 0$. Therefore,

$$\alpha_1 = \cdots = \alpha_{n-1} = 0.$$

□

Before stating our next result we set the notation below. If $\mathbb{M}$ is a subspace of $\mathbb{V}$, we write $\mathbb{S}(\mathbb{M})$ for the linear span of $\{s(\mathbb{M})\colon s \in \mathbb{S}\}$. We also assume that the cardinality of the field $\mathbb{F}$ is greater than the dimension of $\mathbb{S}$. Combining Theorem 3 and Theorem 1, we immediately have the following conclusion.

Theorem 4. *Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$ and $\mathbb{S}$ an n-dimensional subspace of $\mathcal{L}(\mathbb{V})$. Suppose $\mathbb{M}$ is a separating subspace for $\mathbb{S}$ and $\mathbb{S}(\mathbb{M})$ is finite dimensional. If every nonzero operator in $\mathbb{S}$ has rank greater than or equal to $n + \dim \mathbb{S}(\mathbb{M})$, then $\mathbb{S}$ is reflexive.*

Proof. By Theorem 3 there is a separating vector x for $\mathbb{S}$ such that $\mathbb{S}x \cap \mathbb{S}(\mathbb{M}) = \{0\}$. The conclusion follows immediately from Theorem 1. □

In [2, 3] we actually obtained several separating subspaces. For example, in Theorem 2.3 [3] we assume every nonzero operator in $\mathbb{S}$ has rank greater than n^2. Then $\mathbb{S}$ has a separating vector x. Let $\mathbb{W}$ be a vector space complement of $\mathbb{S}x$ in $\mathbb{V}$. Then $\mathbb{U} = \{u \in \mathbb{V} : \mathbb{S}u \subseteq \mathbb{W}\}$ is a separating subspace for $\mathbb{S}$. Theorem 4 opens a door to lower the rank requirement. A much better improvement of lowering the rank is ready now. Before we state the improved result we give a corollary of Theorem 3.

Corollary 5. *Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$ and suppose $\mathbb{S}$ is an n-dimensional linear subspace of $\mathcal{L}(\mathbb{V})$, where n is less than the cardinality of the field $\mathbb{F}$. Then if every nonzero operator in $\mathbb{S}$ has rank $\geqslant n$, then $\mathbb{S}$ has a separating vector.*

Proof. In Theorem 3 we choose $\mathbb{W} = \{0\}$. The proof follows. □

Remark 6. This corollary recovers Theorem 2.1 of [3].

Theorem 7. *Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$. Suppose $\mathbb{S}$ is an n-dimensional linear subspace of $\mathcal{L}(\mathbb{V})$, where n is less than the cardinality of the field $\mathbb{F}$. Then, if every nonzero operator in $\mathbb{S}$ has rank $\geqslant 2n$, then $\mathbb{S}$ is reflexive.*

Proof. By Corollary 5 there is a separating vector x for $\mathbb{S}$. Then the cyclic subspace $\mathbb{S}x$ has dimension n. Note that the separating vector x generates a one-dimensional separating subspace for $\mathbb{S}$, and this matches the assumption of $\mathbb{S}(\mathbb{M})$ of Theorem 4. Hence the proof is completed. $\square$

Remark 8. In the proof we see that if the rank $\geqslant 2n$, then we obtain a disjoint pair of separating vectors which provides an example for the main result Theorem 2.1 in [2].

Remark 9. The contrapositive of Corollary 5 states that for n linear operators $s_1, \cdots, s_n$ on $\mathbb{V}$, if $s_1u, \cdots, s_nu$ are linearly dependent for every vector $u \in \mathbb{V}$, then there are scalars $\alpha_1, \cdots, \alpha_n$, not all zero, such that $s = \alpha_1 s_1 + \cdots + \alpha_n s_n$ has rank $\leqslant n-1$. As pointed out in Remark 2.2 of [3] this is Aupetit's well-known improvement of Kaplansky's lemma which states that if a linear operator on a linear space is locally algebraic, and the degrees of the related polynomials form a bounded set, then the operator is algebraic.

Remark 10. After this paper was accepted, J. Li and Z. Pan [7] showed that $2n$ can be reduced to $2n-1$ in Theorem 6 when the scalar field is $\mathbb{C}$.

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