

THE PROOF OF TCHAKALOFF'S THEOREM

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(Communicated by Andreas Seeger)

ABSTRACT. We provide a simple proof of Tchakaloff's Theorem on the existence of cubature formulas of degree m for Borel measures with moments up to order m . The result improves known results for non-compact support, since we do not need conditions on $(m + 1)$ st moments. In fact, we reduce the classical assertion of Tchakaloff's Theorem to a well-known statement going back to F. Riesz.

We consider the question of existence of cubature formulas of degree m for *Borel measures* μ , i.e. a measure defined on the Borel σ -algebra, where moments up to degree m exist:

Definition 1. Let μ be a positive Borel measure on \mathbb{R}^N and $m \geq 1$ such that

$$\int_{\mathbb{R}^N} \|x\|^k \mu(dx) < \infty$$

for $0 \leq k \leq m$ holds true. A *cubature formula of degree m* for μ is given by an integer $k \geq 1$, points $x_1, \dots, x_k \in \text{supp } \mu$, and weights $\lambda_1, \dots, \lambda_k > 0$ such that

$$\int_{\mathbb{R}^N} P(x) \mu(dx) = \sum_{i=1}^k \lambda_i P(x_i)$$

for all polynomials on \mathbb{R}^N with degree less or equal m , where $\text{supp } \mu$ denotes the closed support of the measure μ , i.e. the complement of the biggest open set $O \subset \mathbb{R}^N$ with $\mu(O) = 0$.

Cubature formulas of degree m have been proved to exist for Borel measures μ , where the $(m + 1)$ st moments exist; see [1] and [6]. The result in the case of compact $\text{supp } \mu$ is classical, and due to Tchakaloff (see [10]), hence we refer to the assertion as Tchakaloff's Theorem.

We collect some basic notions and results from convex analysis (see for instance [9]): fix $N \geq 1$, for some set $S \subset \mathbb{R}^N$ the convex hull of S , i.e. the smallest convex

Received by the editors March 7, 2005 and, in revised form, May 3, 2005.

2000 *Mathematics Subject Classification.* Primary 65D32, 52A21.

Key words and phrases. Quadrature, cubature, truncated moment problem, Tchakaloff's Theorem.

The authors are grateful to Professor Peter Gruber for mentioning the word "Stützebene" in the right moment. The first author acknowledges the support from FWF-Wissenschaftskolleg "Differential Equations" W 800-N05. The second author acknowledges the support from the RTN network HPRN-CT-2002-00281 and from the FWF grant Z-36.

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set in \mathbb{R}^N containing S , is denoted by $\text{conv}(S)$, and the (topological) closure of $\text{conv}(S)$ is denoted by $\overline{\text{conv}}(S)$. The closure of a convex set is convex. Note that the convex hull of a compact set is always closed, but there are closed sets whose convex hull is no longer closed (see [9]).

Closed convex sets can also be described by their *supporting hyperplanes*. Given a convex set C let $y \in \partial C := \overline{C} \setminus \text{int}(C)$ be a boundary point. There is a linear functional l_y and a real number β_y such that the hyperplane defined by $l_y = \beta_y$ contains y , and C is contained in the closed half-space $l_y \leq \beta_y$. Hyperplanes and half-spaces with this property are called supporting hyperplanes and supporting half-spaces, respectively. Moreover, \overline{C} is the intersection of all its supporting half-spaces. Furthermore, if C is not contained in any hyperplane of \mathbb{R}^N (i.e. it has non-empty interior), then a point $x \in C$ is contained in a supporting hyperplane if and only if $x \in C$ is not an interior point of C (see [9], Th. 11.6). This means that we can characterize the boundary of C as those points, which lie at least in one supporting hyperplane of C .

In the case of a convex cone C the supporting hyperplanes can be chosen to be homogeneous, i.e. to be of the form $l_y = 0$. We denote the convex cone generated by some set $A \subset \mathbb{R}^N$ by $\text{cone}(A)$ and its closure by $\overline{\text{cone}}(A)$.

We also introduce the notion of the *relative interior* of a convex set C : a point $x \in C$ lies in the relative interior $\text{ri}(C)$ if for every $y \in C$ there is $\epsilon > 0$ such that $x - \epsilon(y - x) \in C$. In particular we have that the relative interior of a convex set C coincides with the relative interior of its closure \overline{C} . Interior points of C lie in the relative interior; see [9]. This remains true even if C lies in an affine subspace of \mathbb{R}^N , and a point of C lies in the interior with respect to the subspace topology.

Given a measure μ on some measurable space (Ω, \mathcal{F}) and a Borel measurable map $\phi : \Omega \rightarrow \mathbb{R}^N$, we denote by $\phi_*\mu$ the *push-forward Borel measure* on \mathbb{R}^N , which is defined via

$$\phi_*\mu(A) := \mu(\phi^{-1}(A))$$

for all Borel sets $A \subset \mathbb{R}^N$.

Theorem 1. *Let μ be a positive Borel measure on \mathbb{R}^N , such that the first moments exist, i.e.*

$$\int_{\mathbb{R}^N} \|x\| \mu(dx) < \infty,$$

and let $A \subset \mathbb{R}^N$ be a measurable set with $\mu(\mathbb{R}^N \setminus A) = 0$. Then the first moment $E = \int_{\mathbb{R}^N} x\mu(dx)$, where x denotes the vector (x_1, \dots, x_N) , is contained in $\text{cone}(A)$.

Proof. We first assume that there is no $B \subset A$ with $\mu(A \setminus B) = 0$ such that B is contained in a hyperplane, since otherwise we could work in a lower-dimensional space instead (with A replaced by B). Fix some $y \in \overline{K} \setminus \text{int}(K)$ in the boundary of $K = \text{cone}(A)$. Then all linear functionals $l_y : \mathbb{R}^N \rightarrow \mathbb{R}$ corresponding to the supporting half-spaces $l_y \leq 0$ at y are certainly integrable and we have

$$l_y(E) = \int_{\mathbb{R}^N} l_y(x)\mu(dx) \leq 0;$$

consequently $E \in \overline{\text{cone}}(A)$.

By existence of the first moments, for each $\delta > 0$ we have $\mu(\mathbb{R}^N \setminus B(0, \delta)) < \infty$, where $B(0, \delta)$ denotes the centered ball with radius δ . Given l_y as above, we may conclude that $\mu(\{x \in A | l_y(x) < 0\}) > 0$, since otherwise the complement in A of the

intersection of A with the hyperplane $l_y = 0$ would have measure 0, a contradiction to the assumption above. Then we can find $\epsilon > 0$ such that

$$0 < \mu(\{x \in A | l_y(x) \leq -\epsilon\}) < \infty$$

and get

$$l_y(E) = \int_{\mathbb{R}^N} l_y(x)\mu(dx) \leq -\epsilon\mu(\{x \in A | l_y(x) \leq -\epsilon\}) < 0.$$

Hence, $E \in \overline{\text{cone}}(A)$ is an interior point of $\overline{\text{cone}}(A)$. In particular, $E \in \text{cone}(A)$, since the interior lies in the convex cone hull of A . If the first condition is not satisfied, we obtain that E is an interior point of $\text{cone}(A)$ in an affine subspace of \mathbb{R}^N (where the first condition is satisfied), but then E lies in the relative interior of $\text{cone}(A)$ in \mathbb{R}^N , which is the desired result. \square

Corollary 1. *Let μ be a positive Borel measure on \mathbb{R}^N concentrated in $A \subset \mathbb{R}^N$, i.e. $\mu(\mathbb{R}^N \setminus A) = 0$, such that the first moments exist, i.e.*

$$\int_{\mathbb{R}^N} \|x\| \mu(dx) < \infty.$$

Then there exist an integer $1 \leq k \leq N$, points $x_1, \dots, x_k \in A$ and weights $\lambda_1, \dots, \lambda_k > 0$ such that

$$\int_{\mathbb{R}^N} f(x)\mu(dx) = \sum_{i=1}^k \lambda_i f(x_i)$$

for any monomial f on \mathbb{R}^N of degree 1.

Proof. The corollary follows immediately from Theorem 1 and Caratheodory's Theorem (see [9], Th. 17.1 and Cor. 17.1.2). \square

Corollary 2. *Let μ be a positive measure on the measurable space (Ω, \mathcal{F}) concentrated in $A \in \mathcal{F}$, i.e. $\mu(\Omega \setminus A) = 0$, and $\phi : \Omega \rightarrow \mathbb{R}^N$ a Borel measurable map. Assume that the first moments of $\phi_*\mu$ exist, i.e.*

$$\int_{\mathbb{R}^N} \|x\| \phi_*\mu(dx) < \infty.$$

Then there exist an integer $1 \leq k \leq N$, points $\omega_1, \dots, \omega_k \in A$ and weights $\lambda_1, \dots, \lambda_k > 0$ such that

$$\int_{\Omega} \phi_j(\omega)\mu(d\omega) = \sum_{i=1}^k \lambda_i \phi_j(\omega_i)$$

for $1 \leq j \leq N$, where ϕ_j denotes the j -th component of ϕ .

Remark 1. In other words, $A \in \mathcal{F}$ such that $\mu(\Omega \setminus A) = 0$ correspond to $B \subset \phi(\Omega)$ such that $\phi_*\mu(\mathbb{R}^N \setminus B) = 0$.

Remark 2. Note that $\mu(\Omega) = \infty$ is also possible, since we only speak about integrability of N measurable functions ϕ_1, \dots, ϕ_N . If we have $\mu(\Omega) < \infty$, we could add $\phi_{N+1} = 1$, and we obtain in particular $\sum_{i=1}^{k'} \lambda'_i = \mu(\Omega)$ (with possibly a different number $1 \leq k' \leq N + 1$ of points x'_i and weights λ'_i).

In the setting of Theorem 1 assume that μ is a probability measure on \mathbb{R}^N . Then—by the previous consideration— $E = \int_{\mathbb{R}^N} x\mu(dx)$ lies in the convex hull $\text{conv}(A)$. This fact is well known in financial mathematics, since it means that

the price range of forward contracts is given by the relative interior of the convex hull of the no-arbitrage bounds of the (discounted) price process (see for instance [2], Th. 1.40).

The result is also well known in the field of geometry of the moment problem; see for instance [4]. As mentioned therein, the result for compactly supported measures essentially even goes back to F. Riesz; see [8].

Proof. We solve the problem with respect to $\phi_*\mu$ on \mathbb{R}^N and obtain $1 \leq k \leq N$, $y_1, \dots, y_k \in \phi(A)$ and $\lambda_1, \dots, \lambda_k > 0$ such that

$$\int_{\mathbb{R}^N} f(y)(\phi_*\mu)(dy) = \sum_{i=1}^k \lambda_i f(y_i)$$

for all polynomials f of degree 1. Thus we obtain points $\omega_1, \dots, \omega_k$ with $\phi(\omega_i) = y_i$ for $1 \leq i \leq k$, and furthermore

$$\int_{\mathbb{R}^N} f(y)(\phi_*\mu)(dy) = \int_{\Omega} (f \circ \phi)(\omega)\mu(d\omega)$$

by definition, hence the result. \square

In an adequate algebraic framework the previous Theorem 1 yields all cubature results in full generality, and even generalizes those results (see [1], [6] and [7] for related theory and interesting extensive references).

For this purpose we consider polynomials in N (commuting) variables e_1, \dots, e_N with degree function $\deg(e_i) := k_i$ for $1 \leq i \leq N$ and integers $k_i \geq 1$. Hence, we can associate a degree to monomials $e_{i_1} \cdots e_{i_l}$ with $(i_1, \dots, i_l) \in \{1, \dots, N\}^l$ for $l \geq 0$ (note that the monomial associated to the empty sequence is by convenience 1), namely

$$\deg(e_{i_1} \cdots e_{i_l}) = \sum_{r=1}^l k_{i_r}.$$

We denote by $\mathbb{A}_{\deg \leq m}^N$ the vector space of polynomials generated by monomials of degree less or equal to m , for some integer $m \geq 1$. We define a continuous map $\phi : \mathbb{R}^N \rightarrow \mathbb{A}_{\deg \leq m}^N$, via

$$\phi(x_1, \dots, x_N) = \sum_{l \geq 0} \sum_{(i_1, \dots, i_l) \in \{1, \dots, N\}^l, \sum_{r=1}^l k_{i_r} \leq m} x_{i_1} \cdots x_{i_l} e_{i_1} \cdots e_{i_l}.$$

Continuity is obvious, since we are given monomials in each coordinate. ϕ is even an embedding and a closed map.

The following example shows the relevant idea in coordinates, since for $N = 1$ and $\deg(e_1) = 1$ we obtain $\mathbb{A}_{\deg \leq m}^1 = \mathbb{R}^{m+1}$.

Example 1. Fix $m \geq 1$. Then $\phi(x) = (1, x, x^2, \dots, x^m)$ is a continuous map $\phi : \mathbb{R}^1 \rightarrow \mathbb{R}^{m+1}$. Given a positive Borel measure μ on \mathbb{R}^1 such that moments up to degree m exist, i.e.

$$\int_{\mathbb{R}} |x|^k \mu(dx) < \infty$$

for $0 \leq k \leq m$, then $\phi_*\mu$ admits moments up to degree 1. Hence we conclude that there exist $1 \leq k \leq m+1$, points x_1, \dots, x_k and weights $\lambda_1, \dots, \lambda_k > 0$ such that

$$\int_{\mathbb{R}^N} P(x)\mu(dx) = \sum_{i=1}^k \lambda_i P(x_i)$$

for all polynomials P of degree less than or equal to m .

Theorem 2. Given $N \geq 1$ and degree function deg and $m \geq 1$, fix a finite, positive Borel measure μ on \mathbb{R}^N concentrated in $A \subset \mathbb{R}^N$, i.e. $\mu(\mathbb{R}^N \setminus A) = 0$, such that

$$\int_{\mathbb{R}^N} |x_{i_1} \cdots x_{i_l}| \mu(dx) < \infty$$

for $(i_1, \dots, i_l) \in \{1, \dots, N\}^l$ with $\sum_{r=1}^l k_{i_r} \leq m$. Then there exist an integer $1 \leq k \leq \dim \mathbb{A}_{\text{deg} \leq m}^N$, points $x_1, \dots, x_k \in A$ and weights $\lambda_1, \dots, \lambda_k > 0$ such that

$$\int_{\mathbb{R}^N} P(x)\mu(dx) = \sum_{i=1}^k \lambda_i P(x_i)$$

for $P \in \mathbb{A}_{\text{deg} \leq m}^N$.

Proof. The measure $\phi_*\mu$ admits first moments by assumption, hence we conclude by Corollary 2. \square

Remark 3. Tchakaloff's Theorem is a special case of the above theorem with $A = \text{supp } \mu$.

Remark 4. Fix a non-empty, closed set $K \subset \mathbb{R}^N$. We note that a finite sequence of real numbers $m_{i_1 \dots i_l}$ for $(i_1, \dots, i_l) \in \{1, \dots, N\}^l$ with $\sum_{r=1}^l k_{i_r} \leq m$ represents the sequence of moments of a Borel probability measure μ with support $\text{supp } \mu \subset K$, where moments of degree less than or equal to m exist, if and only if

$$\sum_{l \geq 0} \sum_{(i_1, \dots, i_l) \in \{1, \dots, N\}^l, \sum_{r=1}^l k_{i_r} \leq m} m_{i_1 \dots i_l} e_{i_1} \cdots e_{i_l} \in \text{conv } \phi(K).$$

The argument in one direction is that any element of $\text{conv } \phi(K)$ is represented as expectation with respect to some probability measure with support in K , for instance the given convex combination. The other direction is Tchakaloff's Theorem in the general form of Theorem 2. Consequently we have a precise geometric characterization of solvability of the Truncated Moment Problem for measures with support in K . Note that one can often describe $\text{conv } \phi(K)$ by finitely many inequalities.

REFERENCES

- [1] Raul E. Curto and Lawrence A. Fialkow, *A Duality Proof of Tchakaloff's Theorem*, J. Math. Anal. Appl. **269** (2), 519–532, 2002. MR1907129 (2003e:41047)
- [2] Hans Föllmer and Alexander Schied, *Stochastic Finance*, Walter de Gruyter, Studies in Mathematics **27**, Berlin, 2002. MR1925197 (2004h:91051)
- [3] Samuel Karlin and Lloyd S. Shapley, *Geometry of moment spaces*, Mem. Amer. Math. Soc. **12** (1953). MR0059329 (15:512c)
- [4] Johannes H. B. Kemperman, *On the sharpness of Tchebycheff type inequalities*, Indag. Math. **27**, 554–601, 1965. MR0190964 (32:8374)
- [5] Johannes H. B. Kemperman, *Geometry of the moment problem*, Proc. Symp. Appl. Math. **37** (1987), AMS, pp. 16–52. MR0921083 (89h:44004)
- [6] Mihai Putinar, *A Note on Tchakaloff's Theorem*, Proc. Amer. Math. Soc. **125** (8), 2409–2414, 1997. MR1389533 (97j:65045)

- [7] Bruce Reznick, *Sums of even powers of real linear forms*, Mem. Amer. Math. Soc., **96** (463), 1992. MR1096187 (93h:11043)
- [8] Frigyes Riesz, *Sur certains systèmes singuliers d'équations intégrales*, Ann. Sci. École Norm. Sup. (3), **15**, 69–81, 1911.
- [9] R. Tyrrell Rockafellar, *Convex Analysis*, Princeton University Press, 1972. MR0274683 (43:445)
- [10] V. Tchakaloff, *Formules de cubature mécanique à coefficients non négatifs*, Bull. Sci. Math. **81**, 123–134, 1957. MR0094632 (20:1145)

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