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ENTROPY FOR AUTOMORPHISMS OF FREE GROUPS

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Abstract. Let σ be the automorphism of the free group F_∞ which is arising from a permutation of the free generators of F_∞. The σ naturally induces the automorphism $\hat{\sigma}$ of the reduced C^*-algebra $C_r^*(F_\infty)$, and also the automorphism $\bar{\hat{\sigma}}$ of the group factor $L(F_\infty)$. We show that the Brown-Germain entropy $ha(\sigma)$ is zero. This implies that the Brown-Voiculescu topological entropy $ht(\hat{\sigma})$, the Connes-Narnhofer-Thirring dynamical entropy $h_\phi(\hat{\sigma})$ and the Connes-Størmer entropy $H(\bar{\hat{\sigma}})$ are all zero.

1. Introduction

Two notions of entropy for automorphisms were extended to non-commutative framework in the theory of operator algebras from the ergodic theory.

The dynamical entropy $H(\alpha)$ was introduced by Connes and Størmer in [6] as an extended version of the Kolmogorov and Sinai invariant for an automorphism α of a finite von Neumann algebra M. The value depends on a given normal faithful tracial state τ of M which is preserved by the automorphism α.

Replacing the trace τ to an α-invariant state ϕ, Connes-Narnhofer-Thirring defined in [7] the entropy $h_\phi(\alpha)$ for an automorphism α of a unital C^*-algebra A as an extension of $H(\alpha)$. They proved that if M is the von Neumann algebra generated by the image of A under the GNS representation of ϕ, and if $\bar{\alpha}$ and $\bar{\phi}$ are the canonical extensions of α and ϕ, then $h_\phi(\alpha) = h_{\bar{\phi}}(\bar{\alpha})$. Moreover if ϕ is a tracial state, then $h_{\bar{\phi}}(\bar{\alpha}) = H(\bar{\alpha})$.

The topological entropy $ht(\alpha)$ is defined for an automorphism α of unital C^*-algebras which have the approximation property. The definition is based on the approximation property. It was invented by Voiculescu [15] for nuclear C^*-algebras, and extended by Brown [2] to exact C^*-algebras.

These two kinds of "entropy" always satisfy that $h_\phi(\alpha) \leq ht(\alpha)$ by Dykema [8].

Discrete groups are one of the most basic objects producing many very interesting examples of operator algebras, that is, the reduced C^*-algebra $C_r^*(G)$ and the von Neumann algebra $L(G)$, which are generated by the left regular representation of a discrete group G. A discrete group G is called exact if $C_r^*(G)$ is exact ([10]), and it was shown by Ozawa [11] that G is exact if and only if G admits an amenable action on a compact space (see [1] for the notion of amenability). Based on this

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fact, Brown-Germain [4] defined the entropy $ha(\alpha)$ for an automorphism α of an exact discrete group.

Let α be an automorphism of a discrete group G. Then α induces the automorphism $\hat{\alpha}$ of $C^*_r(G)$, and $\hat{\alpha}$ is also extended to the automorphism $\bar{\hat{\alpha}}$ of the group von Neumann algebra $L(G)$.

In this paper we restrict our attention to free groups. Let F_∞ be the free group with free generators indexed by the integers $\mathbb{Z}$, and assume that σ is the automorphism of F_∞ which corresponds to the shift $n \to n+1$ ($n \in \mathbb{Z}$).

As the first typical example of entropy about a highly non-commutative dynamical system, Størmer showed in [12] that $H(\bar{\hat{\sigma}}) = 0$ for the free shift $\bar{\hat{\sigma}}$ on the type II_1 factor $L(F_\infty)$. Later he also proved in [13] that $h_\phi(\hat{\sigma}) = 0$ with respect to the unique $\hat{\sigma}$-invariant state ϕ of $C^*_r(F_\infty)$.

We also see the same phenomenon for the topological entropy in [3] and [8], that is, $ht(\hat{\sigma}) = 0$.

The purpose of this paper is to show that $ha(\sigma) = 0$. This implies all the above results because $ht(\hat{\alpha}) \leq ha(\alpha)$ for all automorphisms α of a group by [4].

The key point of our proof is that the automorphism does not change the length of all reduced words.

2. PRELIMINARIES

In this section, we summarize notations, terminologies and basic facts on the entropy $ha(\alpha)$ in [4] for an automorphism α of an exact discrete group G.

2.1. **Amenable action.** Let G be a discrete group and let α^G be the canonical action of G on $l^\infty(G)$. That is, $\alpha^G : G \to \mathrm{Aut}(l^\infty(G))$ is a homomorphism given by

$$(\alpha^G_g(x))(h) = x(g^{-1}h), \quad \text{for all} \quad x \in l^\infty(G),\ g, h \in G.$$

Let $l^1(G, l^\infty(G))$ be the closure of the linear space of finitely supported functions $T : G \to l^\infty(G)$ with respect to the norm

$$||T||_1 = ||\sum_g |Tg|\ ||_{l^\infty(G)}.$$

The action α^G is said to be *amenable* if there exist functions $T_n \in l^1(G, l^\infty(G))$ such that T_n is non-negative (i.e. $T_n g \geq 0$, for all $g \in G$), finitely supported, $\sum_g T_n g = 1_{l^\infty(G)}$ and $||s.T_n - T_n||_1 \to 0$ for all $s \in G$, where

$$(s.T)g = \alpha^G_s(Ts^{-1}g).$$

2.2. **Entropy** $ha(\alpha)$**.** Assume that the action α^G is amenable. Let α be an automorphism of G. By [4, Proposition 2.6], the definition of $ha(\alpha)$ is as follows:

Given a finite subset $\omega \subset G$, and $\delta > 0$, let

$$ra(\omega, \delta) = \inf\{|\ supp(T)\ | : \ ||s.T - T||_1 < \delta, \text{ for all } s \in \omega\},$$

where infimum is taken over all finitely supported non-negative functions $T : G \to l^\infty(G)$ which satisfy that $\sum_g Tg = 1$. Here $|\ supp(T)\ |$ denotes the cardinality of the support of T. Let

$$ha(\alpha, \omega, \delta) = \limsup_{n\to\infty} \frac{\log(ra(\omega \cup \alpha(\omega) \cup \cdots \alpha^{n-1}(\omega), \delta))}{n}$$

and

$$ha(\alpha,\omega) = \sup_{\delta>0} ha(\alpha,\omega,\delta).$$

Then

$$ha(\alpha) = \sup_{\omega} ha(\alpha,\omega).$$

If $\omega_1 \subset \omega_2 \subset \cdots$ are finite sets with the property that $G = \bigcup_{i\in\mathbb{N}} \bigcup_{n\in\mathbb{Z}} \alpha^n(\omega_i)$, then $ha(\alpha) = \sup_i ha(\alpha,\omega_i)$ by [4, Proposition 5.4].

3. Results

Let I be a set, and let F_I be the free group with generators which is indexed by the set I. Let $\{s_i \ ; \ i \in I\}$ be a set of free generators of F_I, and let $S = \{s_i, s_i^{-1} \ ; \ i \in I\}$. Each element $x \in F_I$ has a unique expression as a finite product $x = x_1 x_2 \cdots x_k$ with $x_i \in S$ $(i = 1, 2, \cdots, k)$ and $x_{i+1}^{-1} \neq x_i$ for all $i \leq k-1$. The expression is called the reduced word in the letters S, and the number k is called the length of the reduced word x, which is denoted by $\mid x \mid$. We denote by e the empty word with the length 0, which is the identity of F_I. For a subset $J \subset I$, we let $S_J = \{s_i, s_i^{-1} \ ; \ i \in J\}$. We denote by $W_n(J)$ the set of the reduced words x in the letters S_J with $\mid x \mid = n$. For a reduced word $x = x_1 x_2 \cdots$ (the length does not need to be finite), we let $x_{[i,j]} = x_i x_{i+1} \cdots x_j$, where $1 \leq i \leq j$.

3.1. **Example.** Let $J \subset I$ be a finite subset, and let r be a positive integer. We define the map $T_{J,r} : F_I \to l^\infty(F_I)$ as follows:

We fix a reduced word with infinite length $w = w_1 w_2 \cdots$, which satisfies that

$$w_i \in S_J \quad \text{and} \quad w_i \notin \{w_1, w_2, \cdots, w_{i-1}\} \ (\mathrm{mod}\ 2 \mid J \mid), \ \text{for all } i.$$

Case 1. Assume that $z \in F_I$ is either the empty word e or the first letter of z is not contained in S_J, i.e., $z = z_1 z_2 \cdots z_d$ with $z_1 \notin S_J$. Then we let

$$Y(J, r \ ; z) = \{w_{[1,\ 2r]},\ w_{[2,\ 2r]},\ \cdots,\ w_{[r,\ 2r]}\},$$

and we let

$$T_{J,r} y(z) = \begin{cases} r^{-1}, & \text{if } y \in Y(J, r \ ; z), \\ 0, & \text{otherwise.} \end{cases}$$

Case 2. Assume that a reduced word $z \in F_I$ satisfies that $z = gz'$, where g is a reduced word in S_J and z' is a reduced word such as in Case 1. We let $g = g_1 g_2 \cdots g_d$ $(g_i \in S_J)$. From the ordered set $\{g_1,\ g_{[1,2]},\ \cdots,\ g,\ gw_{[1,\ 2r]}\ gw_{[2,\ 2r]},\ \cdots\}$, we choose the set $Y(J, r \ ; z)$ of the first r different reduced words. That means,

2-1) If $r \leq d$, then

$$Y(J, r \ ; z) = \{g_1,\ g_{[1,2]},\ \cdots,\ g_{[1,r]}\}.$$

2-2) If $d < r$, then

$$Y(J, r \ ; z) = \{g_1,\ g_{[1,2]},\ \cdots,\ g,\ gw_{[1,\ 2r]},\ gw_{[2,\ 2r]},\ \cdots,\ gw_{[r-d,\ 2r]}\}.$$

We let

$$T_{J,r} y(z) = \begin{cases} r^{-1}, & \text{if } y \in Y(J, r \ ; z), \\ 0, & \text{otherwise.} \end{cases}$$

3.2. **Amenability of α^{F_I}.** The following shows that the α^{F_I} is an amenable action of F_I on $l^\infty(F_I)$.

Lemma. *Let $J \subset I$ be a finite subset. Let $x \in F_J$, and let r be a positive integer with $|x| < r$. Then the map $T_{J,r}$ is non-negative,*

$$\sum_{y \in F_I} T_{J,r} y = 1_{l^\infty(F_I)},$$

$$\mid supp(T_{J,r}) \mid =\mid J \mid \frac{(2 \mid J \mid -1)^r - 1}{\mid J \mid -1}$$

and

$$\| T_{J,r} - x.T_{J,r} \|_1 \leq \frac{2 \mid x \mid}{r}.$$

Proof. It is obvious that $T_{J,r}$ is non-negative and $\sum_{y \in F_I} T_{J,r} y = 1$.

Since $\mid W_n(J) \mid = 2 \mid J \mid (2 \mid J \mid -1)^{n-1}$ for all $n \in$N and since

$$supp(T_{J,r}) = \bigcup_{n=1}^{r} W_n(J),$$

we have that

$$\mid supp(T_{J,r}) \mid =\mid J \mid \frac{(2 \mid J \mid -1)^r - 1}{\mid J \mid -1}.$$

To prove that $\| T_{J,r} - x.T_{J,r} \|_1 \leq \frac{2|x|}{r}$, we show that $Y(J,r;z) \cap xY(J,r;x^{-1}z)$ contains at least $(r - |x|)$ reduced words for all $z \in F_I$.

Let $m = |x|$, and let $x = x_1 \cdots x_m$, where $x_i \in S_J$ for all i.

Case 1) Assume that $z \in F_I$ is such a reduced word as in Case 1 in Example 3.1. Since $m < r$, we have

$$Y(J,r\ ;x^{-1}z) = \{x_m^{-1}, \cdots, x^{-1},\ x^{-1}w_{[1,2r]},\ \cdots,\ x^{-1}w_{[r-m,\ 2r]}\}$$

and

$$\{w_{[1,\ 2r]},\ w_{[2,\ 2r]},\ \cdots,\ w_{[r-m,\ 2r]}\} \subseteq Y(J,r;z) \cap xY(J,r;x^{-1}z).$$

Case 2) Assume that $z = gz'$, where $g = g_1 g_2 \cdots g_d$ $(g_i \in S_J)$, and z' is such a reduced word as in Case 1 in Example 3.1.

2-1) Assume that $r \leq\mid x^{-1}g \mid$.

2-1-1) If $x_1 \neq g_1$, then

$$Y(J,r\ ;x^{-1}z) = \{x_m^{-1}, \cdots, x^{-1},\ x^{-1}g_1,\ \cdots,\ x^{-1}g_{[1,\ r-m]}\}$$

and

$$\{g_1,\ g_{[1,2]},\ \cdots,\ g_{[1,\ r-m]}\} \subseteq Y(J,r;z) \cap xY(J,r;x^{-1}z).$$

2-1-2) If there exists an i with $x_k = g_k$ for all k $(1 \leq k \leq i)$ and $x_{i+1} \neq g_{i+1}$ (x_{i+1} may be the identity), then

$$Y(J,r\ ;x^{-1}z) = \{x_m^{-1}, \cdots,\ x_{[i+1,\ m]}^{-1}, x^{-1}g_{[1,\ i+1]},\ \cdots,\ x^{-1}g_{[1,\ r-m+2i]}\}.$$

2-1-2-1) If $r \leq d$, then

$$\{g_{[1,i+1]},\ \cdots,\ g_{[1,\ \min(r,\ r-m+2i)]}\} \subseteq Y(J,r;z) \cap xY(J,r;x^{-1}z).$$

2-1-2-2) If $d < r$, then

$$\{g_{[1,i+1]},\ \cdots,\ g_{[1,\ r-m+2i]}\} \subseteq Y(J,r;z) \cap xY(J,r;x^{-1}z).$$

2-2) Assume that $\mid x^{-1}g \mid < r$.

2-2-1) If $x_1 \neq g_1$, then

$$Y(J,r\ ;x^{-1}z) = \{x_m^{-1}, \cdots, x^{-1},\ x^{-1}g_1,\ \cdots, x^{-1}g,\\ x^{-1}gw_{[1,\ 2r]}, \cdots, x^{-1}gw_{[r-m-d,\ 2r]}\}$$

and

$$\{g_1,\ g_{[1,2]},\ \cdots,\ g,\ gw_{[1,\ 2r]}, \cdots, gw_{[r-m-d,\ 2r]}\} \subseteq Y(J,r;z) \cap xY(J,r;x^{-1}z).$$

2-2-2) If there exists an i with $x_k = g_k$ for all $k (1 \leq k \leq i)$ and $x_{i+1} \neq g_{i+1}$ (x_{i+1} may be the identity), then

$$Y(J,r\ ;x^{-1}z) = \{x_m^{-1}, \cdots,\ x_{[i+1,\ m]}^{-1},\ x^{-1}g_{[1,\ i+1]},\ \cdots,\ x^{-1}g,\\ x^{-1}gw_{[1,\ 2r]}, \cdots, x^{-1}gw_{[r-m-d+2i,\ 2r]}\}.$$

2-2-2-1) If $r \leq d$, then

$$\{g_{[1,\ i+1]},\ \cdots,\ g_{[1,\ r]}\} \subseteq Y(J,r;z) \cap xY(J,r;x^{-1}z).$$

2-2-2-2) If $d < r$, then

$$\{g_{[1,\ i+1]},\ \cdots, g,\ gw_{[1,\ 2r]}, \cdots, gw_{[\min(r-d,\ r-m-d+2i),\ 2r]}\}\\ \subseteq Y(J,r;z) \cap xY(J,r;x^{-1}z).$$

Thus $Y(J,r;z) \cap xY(J,r;x^{-1}z)$ always contains at least $(r - |x|)$ reduced words for all $z \in F_I$.

Denote the symmetric difference of two sets Y_1 and Y_2 by $Y_1 \bigtriangledown Y_2$. Since

$$\{y \in F_I : T_{J,r}x^{-1}y(x^{-1}z) - T_{J,r}y(z) \neq 0\} \subseteq xY(J,r;x^{-1}z) \bigtriangledown Y(J,r;z),$$

we have by the definition of $T_{J,r}$

$$\begin{aligned} &\sum_{y \in F_I} \mid T_{J,r}x^{-1}y(x^{-1}z) - T_{J,r}y(z) \mid \\ &\leq \frac{|xY(J,r\ ;x^{-1}z) \bigtriangledown Y(J,r\ ;z)|}{r} \\ &\leq \frac{2|x|}{r}. \end{aligned}$$

Hence

$$\| T_{J,r} - x.T_{J,r} \|_1 = \sup_{z \in F_I} \sum_{y \in F_I} \mid T_{J,r}y(z) - T_{J,r}x^{-1}y(x^{-1}z) \mid \leq \frac{2|x|}{r}.$$

□

3.3. **Entropy of automorphisms by permutations.** The following theorem gives as the special case that the free shift automorphism of the free group F_∞ has zero entropy.

Theorem. *Let $\sigma \in Aut(F_I)$ be the automorphism arising from a permutation σ of the index set I. Then $ha(\sigma) = 0$.*

Proof. Given a finite subset $J \subset I$ and a positive integer m, we let

$$\omega(J,m) = \bigcup_{i=0}^{m} W_i(J).$$

Then $\bigcup_{J \subset I} \bigcup_{m=0}^{\infty} \omega(J,m) = F_I$.

Hence it is sufficient to show, for all finite subsets $J \subset I$ and $m \in \mathbb{N}$, that $ha(\sigma, \omega(J,m), \delta) = 0$ for any $\delta > 0$.

Let $\Im(J,n) = \bigcup_{i=0}^{n-1} \sigma^i(J)$. Then

$$\bigcup_{i=0}^{n-1} \sigma^i(\omega(J,m)) \ \subseteq\ \omega(\Im(J,n), m).$$

Given $\delta > 0$, we choose $r \in \mathbb{N}$ with $2m < \delta r$.

Then by the Lemma, $T_{\Im(J,n),r}$ satisfies that

$$\| T_{\Im(J,n),r} - x.T_{\Im(J,n),r} \|_1 \leq \frac{2|x|}{r} \leq \frac{2m}{r} < \delta, \quad \text{for all} \quad x \in \omega(\Im(J,n), m).$$

Since $| \Im(J,n) | \leq n \mid J \mid$, it follows again by the Lemma that

$$ra(\bigcup_{i=0}^{n-1} \sigma^i(\omega(J,m)), \delta) \leq | \, supp(T_{\Im(J,n),r}) \, | \leq n \mid J \mid \frac{(2n \mid J \mid -1)^r - 1}{n \mid J \mid -1}.$$

Hence

$$ha(\sigma, \omega(J,m), \delta) \leq \limsup_{n\to\infty} \frac{1}{n} \log(\ n \mid J \mid \frac{(2n \mid J \mid -1)^r - 1}{n \mid J \mid -1} \ ) = 0.$$

□

3.4. **Applications to entropy in the operator algebras.** As applications of the above theorem to the theory of operator algebras, we have the following:

Corollary. *Let $\sigma \in Aut(F_I)$ be the automorphism arising from a permutation σ of the index set I. Then*

$$ha(\sigma) = ht(\hat\sigma) = ht_\phi(\hat\sigma) = h_\phi(\hat\sigma) = H(\bar{\hat\sigma}) = 0,$$

where ϕ is a $\hat\sigma$ invariant state of $C_r^*(F_I)$.

Proof. Let α be an automorphism of an exact discrete group G. It always holds that $ha(\alpha) \geq ht(\hat\alpha)$ by [4, Proposition 3.3]. If β is an automorphism of an exact C^*-algebra, then $ht(\beta) \geq h_\phi(\beta)$ by [8, Proposition 9], and also for the modified dynamical entropy $ht_\phi(\beta)$ of $ht(\beta)$, we have that $ht_\phi(\beta) \leq ht(\beta)$ by [5] (cf. [9]). Furthermore, if ϕ is a tracial state, then $h_\phi(\beta) = H(\bar\beta)$ by [7]. □

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