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THE FURSTENBERG LEMMA CHARACTERIZES AMENABILITY

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ABSTRACT. We characterize amenability in terms of the existence of equivariant assignments of measures for cocycles into the homeomorphism group of a *single* compact *metric* space.

1. INTRODUCTION

It is well known that every cocycle from an amenable equivalence relation to the homeomorphism group of a compact metric space or from an amenable action of a group to the homeomorphism group of a compact metric space admits a suitably equivariant assignment of measures. This theorem in some form dates back to [3], though an explicit discussion can be found in chapter 4 of [5].

In this note we obtain a converse using ideas suggested by the concept of *ultrahomogeneous structure* from [4].

It is important to note here that the metric space is fixed and *metrizable*. If we have the cocycle into the homeomorphism groups of different compact nonmetrizable spaces, then the existence of an equivariant assignment of measures becomes closer to the definition of §4.3 of [5] and the result is already known; see [1].

Theorem 1.1. *Let E be a countable, measurable equivalence relation on a standard Borel probability space (S, μ) and suppose that for every measurable cocycle*

$$\alpha : E \to \mathrm{Hom}(K)$$

from E to the homeomorphism group of a compact metric space we have a measurable assignment

$$x \mapsto \mu_x,$$
$$S \to M(K),$$

from S to the probability measures on K, which is equivariant in the sense that

$$\mu_z = \alpha(z, x) \cdot \mu_x.$$

Then E is amenable.

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Recall that a *cocycle* from E to a group Γ consists of a function $\alpha : E \to \Gamma$ with

$$\alpha(x_3, x_2)\alpha(x_2, x_1) = \alpha(x_3, x_1).$$

In the case that a group Λ acts on the space X a *cocycle* for the action is a function

$$\alpha : X \times \Lambda \to \Gamma$$

such that

$$\alpha(g, h \cdot x)\alpha(h, x) = \alpha(gh, x).$$

(Below we generally do not distinguish between something happening everywhere and almost everywere; the distinction will be irrelevant.)

We take the Connes-Feldman-Weiss definition of amenability for equivalence relations.

Definition. An equivalence relation E on (S, μ) is *amenable* if there is a linear function

$$P : L^\infty(E) \to L^\infty(S)$$

which is positive, sends the constant function 1 on E to the constant function 1 on S, and commutes with all projections and all morphisms included in the equivalence relation. Following [5], we say that the measurable action of a lcsc group Γ on a space X is *amenable* if the induced orbit equivalence relation is amenable and almost every point x has its stabilizer,

$$\Gamma_x = \{g \in \Gamma : g \cdot x = x\},$$

amenable.

Notation. For $\psi \in L^\infty(E)$ and $f : A \to B$ a measurable bijection included in the graph of E, we let ψ^f be defined by $\psi^f(f(x), y) = \psi(x, y)$ for $(f(x), y) \in E \cap B \times S$ and $\psi^f(z, y) = 0$ for z not in B. For $\psi \in L^\infty(S)$ we would let $\psi^f(f(x)) = \psi(x)$ and $\psi^f(z) = 0$ for z outside B.

With this notation we can rephrase the commutativity requirement for P: Given any such $f : A \to B$ a measurable bijection with $f(x)Ex$ all $x \in A$, and given an $\psi \in L^\infty(E)$,

$$P(\psi^f) = (P(\psi))^f.$$

The Connes-Feldman-Weiss definition of amenability is equivalent to the equivalence relation being measurably hyperfinite.

Theorem 1.2. *Let Γ be a countable group acting measurably on a standard Borel probability space (S, μ), and suppose that for every measurable cocycle*

$$\alpha : S \times \Gamma \to \mathrm{Hom}(K)$$

we have a measurable assignment

$$x \mapsto \mu_x,$$
$$S \to M(K),$$

from S to the probability measures on K, which is equivariant in the sense that

$$\mu_{g \cdot x} = \alpha(x, g) \cdot \mu_x.$$

Then the action is amenable.

As a working definition, let us simply say that an action is *amenable* if the resulting equivalence relation is amenable and the stabilizers of a.e. point are amenable.

2. Proof

We need a number of specific definitions to support the construction ahead.

Definition. A *partition* of an infinite set X consists of a set $\{P_1, P_2, ..., P_\ell\}$ such that P_i is disjoint from P_j for $i \neq j$ and

$$\bigcup_{i \leq \ell} P_i = X.$$

Given a finite sequence $\vec{f} = (f_1, f_2, ..., f_n)$ of functions from X to $\{0,1\}$, we say that $\mathcal{P} = \{P_1, P_2, ..., P_\ell\}$ is the *partition generated by* $\vec{f}$ if each P_i equals

$$\{a \in X : f_1(a) = k_1^i, f_2(a) = k_2^i, ..., f_n(a) = k_n^i\}$$

for some choice of $k_1^i, ..., k_n^i \in \{0,1\}$.

Remark. The trivial partition $\mathcal{P} = \{X\}$ is generated by the empty sequence.

Definition. Let X be a countably infinite set. A collection

$$\mathcal{F} \subset \{0,1\}^X$$

is said to be *random* if whenever

(i) $\mathcal{P} = \{P_1, P_2, ..., P_\ell\}$ is a partition generated by some finite sequence $\vec{f}$ from $\mathcal{F}$, and
(ii) we have cardinals $\kappa_i, \lambda_i \in \{0, 1, 2, ...\aleph_0\}$, each $i \leq n$, with $\kappa_i + \lambda_i$ always equal to $|P_i|$, and
(iii) at each i we have finite subsets $\vec{u}_i \subset P_i$, $\vec{v}_i \subset P_i$, each of cardinality at most κ_i and λ_i respectively,

there is some $g \in \mathcal{F}$, not appearing on the $\vec{f}$ sequence, with

$$|\{n \in P_i : g(n) = 1\}| = \kappa_i,$$
$$|\{n \in P_i : g(n) = 0\}| = \lambda_i,$$

and g assuming the value 1 on each element of $\vec{u}_i$ and the value 0 on each element of $\vec{v}_i$.

Remarks. (i) Here $|Y|$ denotes the cardinality of Y.

(ii) If one of κ_i, λ_i equals $\aleph_0$, then $\kappa_i + \lambda_i = \aleph_0$. ($\aleph_0 + 1 = \aleph_0 + 2 = ... = \aleph_0 + \aleph_0 = \aleph_0$.)

(iii) The idea of the definition of a random collection $\mathcal{F}$ is this: Given any finite sequence $\vec{f}$ from $\mathcal{F}$ and finite $\vec{a}$ from X, we can find a new function $g \in \mathcal{F}$ which behaves relative to $\vec{f}$ and $\vec{a}$ in any previously prescribed manner. Anything which can happen does.

Lemma 2.1. *If X is a countably infinite set, and $\mathcal{F}_0 \subset \{0,1\}^X$ is a countable collection of functions, then we can find a countable random collection $\mathcal{F} \subset \{0,1\}^X$ which includes $\mathcal{F}_0$.*

Proof. We build $\mathcal{F}$ in stages. We first take some countable $\mathcal{F}_1 \supset \mathcal{F}_0$ which includes enough functions to satisfy the definition of random for any choice of parameters from $\mathcal{F}_0$. We then repeat, replacing $\mathcal{F}_0$ by $\mathcal{F}_1$ and obtain $\mathcal{F}_2 \supset \mathcal{F}_1$. In this way we obtain an ascending chain

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset ... \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} \subset ...$$

and finish with

$$\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n.$$

□

Lemma 2.2. *If X_1, X_2 are countably infinite sets, $\mathcal{F}_1 \subset \{0,1\}^{X_1}, \mathcal{F}_2 \subset \{0,1\}^{X_2}$ both random, then there is a bijection*

$$\pi : X_1 \to X_2$$

such that the induced function

$$\varphi : \{0,1\}^{X_1} \to \{0,1\}^{X_2}$$

defined by

$$\varphi(f)(a) = f(\pi^{-1}(a))$$

provides a bijection between $\mathcal{F}_1$ and $\mathcal{F}_2$.

Proof. We build π in a sequence of stages, with only a finite amount of π established at each stage. We simultaneously build φ finite step by finite step. We guarantee an isomorphism at the very end stage by a "back and forth argument", in the sense of §3.2 of [4].

Let $(f_i)_{i\in\mathbb{N}}$, $(g_i)_{i\in\mathbb{N}}$ enumerate $\mathcal{F}_1$, $\mathcal{F}_2$ respectively, and let $(a_i)_{i\in\mathbb{N}}$, $(b_i)_{i\in\mathbb{N}}$ enumerate X_1, X_2 respectively. Using the fact that X_2 is random and considering the trivial partition $\{X_2\}$, we can find some $g \in \mathcal{F}_2$ such that

$$|\{b \in X_2 : g(b) = 1\}| = |\{a \in X_1 : f_1(a) = 1\}|,$$
$$|\{b \in X_2 : g(b) = 0\}| = |\{a \in X_1 : f_1(a) = 0\}|.$$

Let $\varphi(f_1) = g$, choose $b' \in X_2$ with $g(b') = f_1(a_1)$ and let $\pi(a_1) = b'$.

Now consider g_1. We want to find suitable $f \in \mathcal{F}_1$ with $\varphi(f) = g_1$. First let us assume that $g_1 \neq g$. Then by considering the randomness of $\mathcal{F}_1$ we can find some $f \in \mathcal{F}_1$ with

$$f(a_1) = g_1(b')$$

and

$$|\{b \in X_2 : g(b) = \ell_1, g_1(b) = \ell_2\}| = |\{a \in X_1 : f_1(a) = \ell_1, f(a) = \ell_2\}|,$$

for any choice of $\ell_1, \ell_2 \in \{0,1\}$. We then let $\varphi(f) = g_1$, and go on to consider how to extend π by looking at b_1. If $b_1 = b'$, then $\pi(a_1) = b_1$; otherwise we choose some $a' \neq a_1$ with

$$f(a') = g_1(b_1),$$
$$f_1(a') = g(b_1),$$

and let $\pi(a') = b_1$. (Note: We can find such an a' since $|\{a : f(a) = g_1(b_1), f_1(a) = g(b_1)\}|$ equals $|\{a : g_1(a) = g_1(b_1), g(a) = g(b_1)\}|$.)

Having done all this under the assumption that g_1 does not equal g, we need to also consider the case $g_1 = g$. But here there is really nothing to do, since we have already decided that $\varphi(f) = g$, and we simply go on to find some suitable choice of a' with $\pi(a') = b_1$ as above.

Now we go back to the X_1 side and try to find a suitable value for $\varphi(f_2)$.

Again there is a split in cases. If $f_2 \in \{f_1, f\}$, then the value of $\varphi(f_2)$ is already determined, and the whole process is kind of trivial.

So suppose instead $f_2 \neq f, f_1$. Then we can choose $g' \in \mathcal{F}_2$, $g' \neq \{g_1, g\}$ so that at each choice of ℓ_1, ℓ_2, ℓ_3 we have

$$|\{b \in X_2 : g(b) = \ell_1, g_1(b) = \ell_2, g'(b) = \ell_3\}| \\ = |\{a \in X_1 : f_1(a) = \ell_1, f(a) = \ell_2, f_2(a) = \ell_3\}|,$$

and

$$g'(b_1) = f_2(a'),$$
$$g'(b') = f_2(a_1).$$

We again choose a suitable value of $\pi(a_2)$ so that

$$g(\pi(a_2)) = f_1(a_2),$$
$$g_1(\pi(a_2)) = f(a_2),$$
$$g'(\pi(a_2)) = f_2(a_2).$$

With only further notational complications, we keep going in this way for infinitely many steps, back and forth between the two sides, thereby enforcing an isomorphism. □

Definition. A measurable equivalence relation E on a standard Borel probability space (S, μ) is said to have the *Furstenberg property* if for every measurable cocycle

$$\alpha : E \to \mathrm{Hom}(K)$$

from E to the homeomorphism group of a compact metric space we have a measurable assignment

$$x \mapsto \mu_x,$$
$$S \to M(K),$$

from S to the probability measures on K, which is equivariant in the sense that

$$\mu_z = \alpha(z, x) \cdot \mu_x.$$

A measurable action of a countable group Γ on standard Borel probability space (S, μ) is said to have the *Furstenberg property* if for every measurable cocycle

$$\alpha : S \times \Gamma \to \mathrm{Hom}(K)$$

we have a measurable assignment

$$x \mapsto \mu_x,$$
$$S \to M(K),$$

from S to the probability measures on K, which is equivariant in the sense that

$$\mu_{g \cdot x} = \alpha(x, g) \cdot \mu_x.$$

Assume from now on that E on (S, μ) is an equivalence relation with
(i) (S, μ) a standard Borel probability space;
(ii) every E equivalence class is countably infinite;
(iii) E is measurable (e.g. in the sense of being a measurable subset of (S^2, μ^2));
(iv) E has the Furstenberg property.

We will work towards showing that E is amenable. After all that is said and finished, we will indicate how the argument modifies for the case of group actions.

Note that the collection of means (i.e. positive, linear, norm one, unit respecting)

$$P : L^\infty(E) \to L^\infty(S)$$

is compact in the topology given by the functions

$$P \mapsto \int_A P(\psi) d\mu$$

($\psi \in L^\infty(E), A \subset X$ measurable). (The point here is that we can represent P by looking at its restriction to the unit ball $(L^\infty(E))_1$. Then each $P(\psi)$ lands in $(L^\infty(S))_1$ which is compact in the weak-$*$ topology.) Inside this collection, for any given $\psi \in L^\infty(E)$, $f \subset E$, the ones which appropriately commute, in the sense $P(\psi^f) = (P(\psi))^f$, form a closed subset.

For purely notational convenience, let us fix G as a group of measurable bijections of S whose orbit equivalence relation equals E. Appealing to the compactness of the space of means, it suffices to show that given a countable collection $\mathcal{F}_0 \subset L^\infty(E)$ consisting of functions assuming only the values 0 and 1 and closed under the G action, we may find

$$P : \mathcal{F}_0 \to L^\infty(S)$$

such that:

(i) $P(\psi^f) = (P(\psi))^f$ for all $\psi \in \mathcal{F}_0$ and $f \in G$;
(ii) if $A \subset S$ measurable and $\psi \in \mathcal{F}_0$ has $\psi(x, y) = 0$ for all $x \in A$, then likewise $P(\psi)(x) = 0$ for all $x \in A$;
(iii) if $\psi \leq \psi'$ both in $\mathcal{F}_0$, then $P(\psi) \leq P(\psi')$;
(iv) if $\psi + \psi' = \psi''$ for all in $\mathcal{F}_0$, then $P(\psi'') = P(\psi') + P(\psi)$;
(v) $P(1) = 1$.

For each $\psi \in \mathcal{F}_0$ and $x \in S$ we define $\psi_x \in \ell^\infty([x]_E)$ by $\psi_x(y) = \psi(x, y)$. After possibly expanding the set $\mathcal{F}_0$ we may assume that at each $x \in S$, the collection

$$\{\psi_x : \psi \in \mathcal{F}_0\}$$

is random (in our earlier sense) for $\{0, 1\}^{[x]_E}$.

Now fix some arbitrary countably infinite set X and let $\mathcal{F}$ be a collection of random functions on X. Let K be the space of all means *just on* $\mathcal{F}$; that is to say, all

$$m : \mathcal{F} \to [0, 1]$$

such that:

(a) if $\psi \leq \psi'$ both in $\mathcal{F}$, then $m(\psi) \leq m(\psi')$;
(b) if $\psi + \psi' = \psi''$ for all in $\mathcal{F}$, then $m(\psi'') = m(\psi') + m(\psi)$;
(c) $m(1) = 1$.

Since $\mathcal{F}$ is countable, K is compact, metric.

At each $x \in S$ we may in a measurable manner assign a bijection

$$\pi_x : [x]_E \to X$$

such that the induced map

$$\rho_x : \ell^\infty([x]_E) \to \ell^\infty(X)$$

defined by

$$(\rho_x(\psi))(a) = \psi(\pi_x^{-1}(a))$$

provides a bijection of $\{\psi_x : \psi \in \mathcal{F}_0\}$ with $\mathcal{F}$. (The existence of some such assignment, without worrying about measurability, follows by Lemma 2.2. Then to see it can be found measurably, we can either appeal to the Jankov von Neumann uniformization theorem for measurable selectors, or go back into the proof of Lemma 2.2 and see that it is so uniform as to actually allow a *Borel* choice of $x \mapsto \rho_x$.)

We can then define a measurable cocyle

$$\alpha : E \to \mathrm{Hom}(K)$$

by

$$(\alpha(x,z))(m)(\psi) = m(\rho_z(\rho_x^{-1}(\psi))).$$

We apply the assumption of Furstenberg's property, and get a measurable assignment

$$S \to M(K),$$
$$x \mapsto \mu_x$$

with

$$\mu_x = \alpha(x,z) \cdot \mu_z.$$

We then let

$$(P(\psi))(x) = \int_K m(\rho_x(\psi_x))d\mu_x(m).$$

We want to check that the induced

$$P : L^\infty(E) \to L^\infty(S)$$

satisfies (i)-(v) above, and then we will be done. (ii)-(v) follow at once from the definition, so we only need to deal with (i).

So fix some $f : S \to S$ in Γ and $\psi \in \mathcal{F}_0$. Then

$$\begin{aligned}(P(\psi^f))(x) &= \int_K m\rho_x((\psi^f)_x)d\mu_x(m) \\ &= \int_K m\rho_x(\psi_{f^{-1}(x)})d\mu_x(m),\end{aligned}$$

since

$$(\rho_x((\psi^f)_x))\pi_x(y) = (\psi^f)_x(y) = \psi^f(x,y)$$
$$= \psi(f^{-1}(x),y) = \psi_{f^{-1}(x)}(y) = \rho_x(\psi_{f^{-1}(x)})\pi_x(y).$$

On the other hand,

$$(P(\psi))^f(x) = (P(\psi))(f^{-1}(x))$$
$$= \int_K m\rho_{f^{-1}(x)}((\psi)_{f^{-1}(x)})d\mu_{f^{-1}(x)}(m),$$

which in turn, by our assumption of α-equivariance, equals

$$\int_K m\rho_{f^{-1}(x)}((\psi)_{f^{-1}(x)})d\alpha(f^{-1}(x),x) \cdot \mu_x(m),$$

which by the definition of the cocycle unwinds as

$$\int_K (\rho_x(\rho_{f^{-1}(x)}^{-1}(\rho_{f^{-1}(x)}(\psi_{f^{-1}(x)}))))d\mu_x(m) = \int_K (\rho_x(\psi_{f^{-1}(x)}))d\mu_x(m),$$

as above and as required.

Thus we have established the amenability of E assuming the existence of equivariant cocycles for all the appropriate cocycles. It now remains to indicate the modifications that will enable the same result to be proved for group actions.

From now on assume that Γ is a countable group acting measurably on a standard Borel (S, μ), and the action of Γ has the Furstenberg property. Given the earlier result, we know that the orbit equivalence relation E_Γ is amenable, and it suffices to show that a.e. the stabilizers are amenable. For this purpose it suffices to show that if Γ_0 is a finitely generated subgroup of Γ and there is a non-null set of $x \in S$ whose stabilizer includes Γ_0, then Γ_0 is non-null.

So fix such a Γ_0. Let $\{f_i : i \in \mathbb{N}\}$ be a countable collection of functions

$$\Gamma_0 \to \{0, 1\}$$

closed under Γ_0-translation. By compactness again, it suffices to show that there is a Γ_0-invariant mean defined just on $\{f_i : i \in \mathbb{N}\}$.

Let $(\Gamma_0\gamma_i)_{i\in\Lambda}$ enumerate without repititions the left cosets of Γ_0. At each $j \in \mathbb{N}$ define

$$\hat{f}_j : \Gamma \to \{0, 1\}$$

by

$$\hat{f}_j(\gamma\gamma_i) = f_j(\gamma)$$

for $\gamma \in \Gamma_0$. In other words, we multiply out f_j across the various cosets using our representatives $(\gamma_i)_{i\in\Lambda}$.

Now take a countable collection

$$\mathcal{F} \subset L^\infty(S \times \Gamma)$$

such that

(i) each $h \in \mathcal{F}$ assumes only the values 0 and 1;
(ii) if $x \in S$, $j \in \mathbb{N}$, then there is $h \in \mathcal{F}$ with

$$h_x = \hat{f}_j$$

(where as usual we define h_x by $h_x(\gamma) = h(x, \gamma)$);
(iii) at each x, $\{h_x : h \in \mathcal{F}\}$ is a random collection of functions from Γ to $\{0, 1\}$;
(iv) if $\gamma \in \Gamma$, $h \in \mathcal{F}$, then

$$\gamma \cdot h \in L^\infty(S \times \Gamma)$$

(where as usual $\gamma \cdot h$ is defined by $(\gamma \cdot h)(y, \gamma_1) = h(\gamma^{-1} \cdot y, \gamma^{-1}\gamma_1)$).

We again let X be some countably infinite space and let $\mathcal{F}$ be a random collection of functions from X to $\{0, 1\}$. At each $x \in S$ we again measurably assign a corresponding bijection

$$\pi_x : \Gamma \to X$$

so that the induced

$$\rho_x : \ell^\infty(\Gamma) \to \ell^\infty(X)$$

provides a bijection. We again let K be the space of means on $\mathcal{F}$.

Given $x \in S, \gamma \in \Gamma$, we obtain a continuous

$$\alpha(x, \gamma) : K \to K$$

defined by

$$(\alpha(x, \gamma))(m)(\psi) = m(\rho_x(\gamma^{-1} \cdot (\rho^{-1}_{\gamma\cdot x}(\psi)))),$$

where Γ acts on $\ell^\infty(\Gamma)$ by left translation in the usual way—that is to say, $(\gamma \cdot \varphi)(\gamma') = \varphi(\gamma^{-1}\gamma')$. It is routinely seen that this provides a measurable cocycle

$$\alpha : X \times \Gamma \to \mathrm{Hom}(K).$$

We apply the assumption of the Furstenberg property and get

$$S \to M(K),$$
$$x \mapsto \mu_x$$

with

$$\mu_{g\cdot x} = \alpha(x, g) \cdot \mu_x.$$

At each x we can pull back and obtain a mean ν_x defined on $\{\varphi_x : \varphi \in \mathcal{F}\}$ by

$$\nu_x(\varphi) = \int_K m(\rho_x(\varphi)) d\mu_x(m).$$

It is then immediate that if Γ_0 is included in the stabilizer of x, then ν_x is Γ_0-invariant. From this we obtain a Γ_0-invariant mean on $\{f_i : i \in \mathbb{N}\}$ with

$$\nu(f_j) = \nu_x(\hat{f}_j).$$

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