COMMUTANTS OF CERTAIN ANALYTIC OPERATOR ALGEBRAS

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Abstract. We prove that algebraic commutants of maximal subdiagonal algebras and of analytic operator algebras determined by flows in a σ-finite von Neumann algebra are self-adjoint.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For a subset $E$ of $\mathcal{B}(\mathcal{H})$, we denote by $E'$ the algebraic commutant, that is,

$$E' = \{ X \in \mathcal{B}(\mathcal{H}) : AX = XA, \ ∀ A ∈ E \}.$$

If $T \in \mathcal{B}(\mathcal{H})$, we call $\{T\}'$ the algebraic commutant of $T$. The well-known theorem of Fuglede states that if $N$ is normal and $X$ commutes with $N$, so does $X^*$. That is, the algebraic commutant $\{N\}'$ of $N$ is self-adjoint. Note that $\{N\}'$ is the same as the commutant of the algebra generated by $N$ and $I$, which is non-self-adjoint in general. Thus it may be asked which subalgebras have a self-adjoint commutant. For example, if all elements in a subalgebra are normal or the algebra itself is self-adjoint, then its algebraic commutant is self-adjoint. In general, this problem is not particularly interesting. However special cases of this problem are interesting. F. Gilfeather and D.R. Larson in [6] showed that the algebraic commutant of a nest subalgebra of a von Neumann algebra is self-adjoint. We note that a nest subalgebra of a von Neumann algebra is a kind of analytic operator algebra. Thus it is interesting to consider this problem for general analytic operator algebras.

In [2], W. Arveson introduced the notion of subdiagonal algebras to give a unified theory of non-self-adjoint operator algebras, including the algebra of bounded analytic matrix-valued (or more generally, operator-valued) functions and nest subalgebras of von Neumann algebras.

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Let $\mathcal{M}$ be a $\sigma$-finite von Neumann algebra acting on $\mathcal{H}$. We denote by $\mathcal{M}_*$ the space of all $\sigma$-weakly continuous linear functionals of $\mathcal{M}$. For a von Neumann subalgebra $\mathfrak{D}$ of $\mathcal{M}$, let $\Phi$ be a faithful normal conditional expectation from $\mathcal{M}$ onto $\mathfrak{D}$. A subalgebra $\mathfrak{A}$ of $\mathcal{M}$, containing $\mathfrak{D}$, is called a subdiagonal algebra of $\mathcal{M}$ with respect to $\Phi$ if

(i) $\mathfrak{A} \cap \mathfrak{A}^* = \mathfrak{D}$,
(ii) $\Phi$ is multiplicative on $\mathfrak{A}$, and
(iii) $\mathfrak{A} + \mathfrak{A}^*$ is $\sigma$-weakly dense in $\mathcal{M}$.

The algebra $\mathfrak{D}$ is called the diagonal of $\mathfrak{A}$. Although subdiagonal algebras are not assumed to be $\sigma$-weakly closed in [2], the $\sigma$-weak closure of a subdiagonal algebra is again a subdiagonal algebra of $\mathcal{M}$ with respect to $\Phi$ (Remark 2.1.2 in [2]). Thus we assume that our subdiagonal algebras are always $\sigma$-weakly closed.

We say that $\mathfrak{A}$ is a maximal subdiagonal algebra in $\mathcal{M}$ with respect to $\Phi$ in case that $\mathfrak{A}$ is not properly contained in any other subalgebra of $\mathcal{M}$ which is subdiagonal with respect to $\Phi$. Put $\mathfrak{A}_0 = \{ X \in \mathfrak{A} : \Phi(X) = 0 \}$ and $\mathfrak{A}_n = \{ X \in \mathfrak{M} : \Phi(AXB) = \Phi(BXA) = 0, \forall A, B \in \mathfrak{A}_0 \}$. By Theorem 2.2.1 in [2], we recall that $\mathfrak{A}_n$ is a maximal subdiagonal algebra of $\mathcal{M}$ with respect to $\Phi$ containing $\mathfrak{A}$. If there is a faithful normal finite trace $\tau$ on $\mathcal{M}$ such that $\tau \circ \Phi = \tau$, we say that $\mathfrak{A}$ is finite subdiagonal.

On the other hand, let $\alpha = \{ \alpha_t \}_{t \in \mathbb{R}}$ be a flow of $\mathcal{R}$ on $\mathcal{M}$, i.e., $\{ \alpha_t \}_{t \in \mathbb{R}}$ is a one-parameter group of $\sigma$-automorphisms of $\mathcal{M}$ such that, for each $X \in \mathcal{M}$, $t \mapsto \alpha_t(X)$ is $\sigma$-weakly continuous. Write $H^\infty(\alpha) = \{ X \in \mathcal{M} : sp_\alpha(X) \subseteq [0, \infty) \}$, where $sp_\alpha(\cdot)$ is an Arveson spectrum (Section 3). Then $H^\infty(\alpha)$ is a $\sigma$-weakly closed subalgebra of $\mathcal{M}$ satisfying that $H^\infty(\alpha) + (H^\infty(\alpha))^*$ is $\sigma$-weakly dense in $\mathcal{M}$. The structure of $H^\infty(\alpha)$ was studied by several authors (cf. [3][13][14][16]). It is known that if there is a faithful normal conditional expectation from $\mathcal{M}$ onto $H^\infty(\alpha) \cap (H^\infty(\alpha))^*$, then $H^\infty(\alpha)$ is a maximal subdiagonal algebra of $\mathcal{M}$. Moreover, if $\mathcal{A}$ is a nest subalgebra of $\mathcal{M}$, then there is an inner flow $\alpha = \{ \alpha_t \}_{t \in \mathbb{R}}$, that is, $\alpha$ is implemented by a continuous unitary group $\{ U_t : t \in \mathbb{R} \} \subset \mathcal{M}$, such that $\mathcal{A} = H^\infty(\alpha)$ (cf. Theorem 4.2.3 in [14]).

In this note we prove that algebraic commutants of maximal subdiagonal algebras and of analytic operator algebras are self-adjoint.

2. THE COMMUTANT OF A MAXIMAL SUBDIAGONAL ALGEBRA

We consider the algebraic commutant of a maximal subdiagonal algebra $\mathfrak{A}$ with respect to $\Phi$. The following result was proved in [9]. For completeness, we give the proof here also.

**Lemma 1.** Let $\mathfrak{A}$ be a finite subdiagonal algebra with respect to $\Phi$ of $\mathcal{M}$. Then $\mathfrak{A}' = \mathcal{M}'$.

**Proof.** It is trivial that $\mathfrak{A}' \supseteq \mathcal{M}'$. Now let $X \in \mathfrak{A}'$ and $T \in \mathcal{M}$. Then for any $\epsilon > 0$, we have that $T^*T + \epsilon I$ is a positive invertible operator in $\mathcal{M}$. Note that $\mathfrak{A}$ is maximal subdiagonal (cf. [2]). By Theorem 4.2.1 in [2], there is an invertible operator $A$ in $\mathfrak{A}$ so that $T^*T + \epsilon I = A^*A$. Then

$$X^*(T^*T + \epsilon I)X = X^*A^*AX = A^*X^*XA \leq \|X\|^2A^*A = \|X\|^2(T^*T + \epsilon I).$$

It follows that $X^*T^*TX \leq \|X\|^2T^*T$ by letting $\epsilon \to 0$. In particular, $X^*EX \leq \|X\|^2E$ for every positive projection $E$ in $\mathcal{M}$. It follows that $(I - E)X^*EX(I - E) \leq$
0, which implies that $EX(I-E) = 0$. Thus $EX =XE$ for every projection $E \in \mathcal{M}$, which implies that $X \in \mathcal{M}'$. The proof is complete.

We next recall Haagerup’s reduction theory \cite{18}. Since $\mathcal{M}$ is $\sigma$-finite, there exists a faithful normal state $\varphi$ of $\mathcal{M}$ such that $\varphi \circ \Phi = \varphi$. Let $\sigma^\varphi = \{\sigma^\varphi_t\}_{t \in \mathbb{R}}$ be the modular automorphism group of $\mathcal{M}$ associated with $\varphi$. We know that $\mathcal{A}$ is $\{\sigma^\varphi_t\}_{t \in \mathbb{R}}$ invariant from Theorem 2.4 in \cite{22}. Let $G$ be the discrete subgroup $\bigcup_{n \geq 1} 2^{-n} \mathbb{Z}$ of $\mathbb{R}$. We consider the crossed product $\mathcal{M} \rtimes_{\sigma^\varphi} G$ with respect to $\sigma^\varphi$. Then we have that $\mathcal{M} \rtimes_{\sigma^\varphi} G$ is a von Neumann algebra on $l^2(G, \mathcal{H})$ generated by the operators $\pi(X), X \in \mathcal{M}$, and $\lambda(s), s \in G$, defined by the equations

\[
(\pi(X)\xi)(t) = \sigma^\varphi_t(X)\xi(t), \quad \xi \in l^2(G, \mathcal{H}), \quad t \in G,
\]

and

\[
(\lambda(s)\xi)(t) = \xi(t-s), \quad \xi \in l^2(G, \mathcal{H}), \quad t \in G.
\]

Note that $\pi$ is a normal faithful representation of $\mathcal{M}$ on $l^2(G, \mathcal{H})$. Let $\hat{\varphi}$ be the dual weight of $\varphi$ on $\mathcal{M} \rtimes_{\sigma^\varphi} G$. Then $\hat{\varphi}$ is again a faithful normal state on $\mathcal{M} \rtimes_{\sigma^\varphi} G$. Haagerup’s reduction theorem asserts that there is an increasing sequence $\{\mathcal{R}_n\}_{n \geq 1}$ of von Neumann subalgebras of $\mathcal{M} \rtimes_{\sigma^\varphi} G$ with the following properties:

(i) each $\mathcal{R}_n$ is finite;
(ii) $\bigcup_{n \geq 1} \mathcal{R}_n$ is $\sigma$-weakly dense in $\mathcal{M} \rtimes_{\sigma^\varphi} G$;
(iii) for each $n \geq 1$ there is a faithful normal conditional expectation $\mathcal{E}_n$ from $\mathcal{M} \rtimes_{\sigma^\varphi} G$ onto $\mathcal{R}_n$ such that $\hat{\varphi} \circ \mathcal{E}_n = \hat{\varphi}$. $\mathcal{E}_n \circ \mathcal{E}_{n+1} = \mathcal{E}_n$, $n \geq 1$, and

\[
\lim_{n \to \infty} \|\psi \circ \mathcal{E}_n - \psi\| = 0 \quad \text{for all } \psi \in (\mathcal{M} \rtimes_{\sigma^\varphi} G)^{\ast}.
\]

We refer the readers to \cite{18} and \cite{22} for more details.

We now can extend $\Phi$ to a normal faithful conditional expectation $\hat{\Phi}$ from $\mathcal{M} \rtimes_{\sigma^\varphi} G$ onto $\mathcal{D} \rtimes_{\sigma^\varphi} G$, which is naturally identified as a von Neumann subalgebra of $\mathcal{M} \rtimes_{\sigma^\varphi} G$.

Let $\mathfrak{A}$ be the $\sigma$-weakly closed subalgebra generated by $\{\pi(X) : X \in \mathfrak{A}\}$ and $\{\lambda(s) : s \in G\}$. Since $\mathfrak{A}$ is $\sigma^\varphi_t$ invariant by Theorem 2.4 in \cite{22}, $\mathfrak{A}$ is the $\sigma$-weak closure of the set of all linear combinations of $\lambda(s)\pi(X), s \in G, X \in \mathfrak{A}$. The following lemma was proved in \cite{22}.

**Lemma 2.** $\hat{\mathfrak{A}}$ is a maximal subdiagonal algebra with respect to $\hat{\Phi}$.

Let $\mathfrak{A}_n = \mathcal{R}_n \cap \hat{\mathfrak{A}} = \mathcal{E}_n(\hat{\mathfrak{A}})$. We have the following lemma (Lemma 2 in \cite{22}).

**Lemma 3.** $\mathfrak{A}_n$ is a finite subdiagonal algebra in $\mathcal{R}_n$ with respect to $\hat{\Phi}|\mathcal{R}_n$ and $\bigcup_{n \geq 1} \mathfrak{A}_n$ is $\sigma$-weakly dense in $\hat{\mathfrak{A}}$.

We now have the main theorem in this section.

**Theorem 1.** Let $\mathcal{M}$ be a $\sigma$-finite von Neumann algebra and let $\mathfrak{A}$ be a maximal subdiagonal algebra with respect to $\Phi$ of $\mathcal{M}$. Then the commutant $\mathfrak{A}'$ of $\mathfrak{A}$ is self-adjoint, that is, $\mathfrak{A}' = \mathcal{M}'$.

**Proof.** We first claim that $\hat{\mathfrak{A}}' = (\mathcal{M} \rtimes_{\sigma^\varphi} G)'$. In fact, let $X \in \hat{\mathfrak{A}}'$. Then $X \in \mathfrak{A}_n'$ for all $n \in \mathbb{N}$. By Lemmas 1 and 3, we have $X^* \in \mathfrak{A}_n'$. Note that for every $Y \in \mathfrak{A}$, we have $\mathcal{E}_n(Y) \in \mathfrak{A}_n$ for all $n \in \mathbb{N}$ and $Y = \lim \mathcal{E}_n(Y)$ $\sigma$-weakly from Haagerup’s theory. Then it follows that $X^*Y = YX^*$, which implies that $X^* \in \hat{\mathfrak{A}}'$. 

Now let $X \in \mathfrak{A}$. We define an operator $\hat{X}$ on $\ell^2(G, \mathcal{H})$ by

$$(\hat{X}\xi)(s) = X\xi(s), \ s \in G, \xi \in \ell^2(G, \mathcal{H}).$$

Then we have that $\hat{X} \in \mathfrak{A}'$. In fact, for $\xi \in \ell^2(G, \mathcal{H})$, $t, s \in \mathbb{R}$ and $Y \in \mathfrak{A}$,

$$(\hat{X}\pi(Y)\xi)(t) = X(\pi(Y)\xi)(t) = X\sigma_t^\mathcal{E}(Y)\xi(t)$$

$$= \sigma_t^\mathcal{E}(Y)X\xi(t) = \sigma_t^\mathcal{E}(Y)((\hat{X}\xi)(t))$$

and

$$(\hat{X}\lambda_s\xi)(t) = X((\lambda_s\xi)(t)) = X\xi(t - s)$$

$$= (\hat{X}\xi)(t - s) = (\lambda_s(\hat{X}\xi))(t)$$

It follows that $(\hat{X})^* \in \mathfrak{A}'$, which implies that $(\hat{X})^*\pi(Y) = \pi(Y)(\hat{X})^*$ for all $Y \in \mathfrak{A}$. Note that $(\hat{X})^* = (X^*)^*$. Then $(X^*)^*\pi(Y) = \pi(Y)(X^*)^*$ for all $Y \in \mathfrak{A}$. In particular, $X^*Y = YX^*$. Thus we have $X^* \in \mathfrak{A}'$. Note that $\mathfrak{A} + \mathfrak{A}'$ is $\sigma$-weakly dense in $\mathcal{M}$. It then follows that $X \in \mathcal{M}'$. Hence $\mathfrak{A}' = \mathcal{M}'$. The proof is complete. \hfill $\square$  

3. The commutant of an analytic operator algebra

In this section we consider the algebraic commutant of an analytic operator algebra determined by a flow on $\mathcal{M}$. We need Arveson’s theory of spectral subspaces and, so we recall the definitions here. Let $\alpha = \{\alpha_t\}_{t \in \mathbb{R}}$ be a flow on $\mathcal{M}$, i.e. a $\sigma$-weakly continuous one parameter group of $^*$-automorphisms of $\mathcal{M}$. For each element $X \in \mathcal{M}$ and a function $f \in L^1(\mathbb{R})$, we define the convolution $f \ast X$ by

$$f \ast X = \int_{-\infty}^{+\infty} f(t)\alpha_t(X)dt.$$  

For $f \in L^1(\mathbb{R})$, let $Z(f) = \{t \in \mathbb{R} : \hat{f}(t) = 0\}$, where $\hat{f}(t) = \int_{-\infty}^{+\infty} e^{-ist}f(s)ds$ is the Fourier transform of $f$. For $X \in \mathcal{M}$, we define the Arveson spectrum of $X$ with respect to the flow $\alpha$ to be the set

$$\bigcap\{Z(f) : f \ast X = 0\}$$

and denote it by $sp_{\alpha}(X)$. For any subset $S$ of $\mathbb{R}$ we define the spectral subspace $M^\alpha(S)$ to be the $\sigma$-weak closure of the set $\{X \in \mathcal{M} : sp_{\alpha}(X) \subset S\}$. We refer the readers to [1] [2] [13] [14] for the elementary properties of spectra and spectral subspaces. Put $H^\alpha(\mathbb{R}) = M^\alpha([0, \infty))$ and $H^\alpha(\mathbb{R}) = M^\alpha((0, \infty))$. It is known that $H^\alpha(\mathbb{R})$ is a two-sided ideal of $H^\alpha(\mathbb{R})$. Let $\mathfrak{D} = H^\alpha(\mathbb{R}) \cap (H^\alpha(\mathbb{R}))^*$ be the fixed point subalgebra of $\alpha$. We recall that $\mathcal{M}$ is said to be $\mathbb{R}$-finite relative to $\alpha$ if there is a separating family of $\alpha$-invariant normal states on $\mathcal{M}$. At the opposite extreme, we say that $\mathcal{M}$ is completely non-$\mathbb{R}$-finite relative to $\alpha$ in case there are no invariant normal states.

**Lemma 4.** If $\mathcal{M}$ is completely non-$\mathbb{R}$-finite relative to $\alpha$, then $H^\alpha(\mathbb{R}) = H_0^\alpha(\mathbb{R})$. 


Proof. Since \( \mathcal{M} \) is \( \sigma \)-finite, without loss of generality by choosing an appropriate representation for \( \mathcal{M} \), we may assume that \( \mathcal{M} \) has a cyclic and separating vector in \( \mathcal{H} \).

If \( H^\infty(\alpha) \neq H^\infty_0(\alpha) \), then there is an element \( f \in \mathcal{M} \) such that \( f(A) = 0 \) for all \( A \in H^\infty_0(\alpha) \) and \( f(T) \neq 0 \) for some \( T \in H^\infty(\alpha) \). Since \( \mathcal{M} \) has a separating vector in \( \mathcal{H} \), there are vectors \( x, y \in \mathcal{H} \) such that \( f(A) = (Ax, y) \) for all \( A \in \mathcal{M} \) by Proposition 7.4.5 and Corollary 7.3.3 in [12]. Let \( \mathfrak{M} = \{Ax : A \in H^\infty(\alpha)\} \) (resp. \( \mathfrak{M}_0 = \{Ax : A \in H^\infty_0(\alpha)\} \) be the closed subspace generated by \( \{Ax : A \in H^\infty(\alpha)\} \) (resp. \( \{Ax : A \in H^\infty_0(\alpha)\} \) of \( \mathcal{H} \). Then both \( \mathfrak{M} \) and \( \mathfrak{M}_0 \) are invariant subspaces for \( H^\infty(\alpha) \). We have \( \mathfrak{M}_0 \subsetneq \mathfrak{M} \) since \( (Tx, y) \neq 0 \) for some \( T \in H^\infty(\alpha) \) and \( (Tx, y) = 0 \) for all \( T \in H^\infty_0(\alpha) \). It is trivial that \( H^\infty_0(\alpha) \mathfrak{M} \subseteq \mathfrak{M}_0 \) since \( H^\infty_0(\alpha) \) is a two-sided ideal of \( H^\infty(\alpha) \). On the other hand, since \( \mathcal{M} \) is completely non-\( \mathbb{R} \)-finite relative to \( \alpha \), by Corollary 5.7 in [14], \( \mathfrak{M} \) is completely normalized in the sense of Definition 5.1 in [14], that is,

\[
\mathfrak{M} = \bigcup_{s<0} [M^\alpha([s, \infty)) \mathfrak{M}] = \bigcup_{s>0} [M^\alpha([s, \infty)) \mathfrak{M}].
\]

Note that \( M^\alpha([s, \infty)) \subseteq H^\infty_0(\alpha) \) for all \( s > 0 \). We then have that \( [M^\alpha([s, \infty)) \mathfrak{M}] \subseteq \mathfrak{M}_0 \) for all \( s > 0 \) which implies that \( \mathfrak{M} \subseteq \mathfrak{M}_0 \). This is a contradiction. Hence \( H^\infty(\alpha) = H^\infty_0(\alpha) \). The proof is complete.

We recall that the crossed product \( \mathcal{M} \rtimes_\alpha \mathbb{R} \) determined by \( \mathcal{M} \) and \( \alpha \) is the von Neumann algebra on the Hilbert space \( L^2(\mathbb{R}, \mathcal{H}) \) generated by the operators \( \pi(X) \), \( X \in \mathcal{M} \), and \( \lambda(s) \), \( s \in \mathbb{R} \), defined by the equations

\[
(\pi(X)f)(t) = \alpha_t(X)f(t), \quad f \in L^2(\mathbb{R}, \mathcal{H}), \quad t \in \mathbb{R},
\]

and

\[
(\lambda(s)f)(t) = f(t-s), \quad f \in L^2(\mathbb{R}, \mathcal{H}), \quad t \in \mathbb{R}.
\]

It is clear that \( \pi(\alpha_t(X)) = \lambda(t)\pi(X)\lambda(t)^* \) for all \( X \in \mathcal{M} \) and \( t \in \mathbb{R} \). For any \( Y \in \mathcal{M} \rtimes_\alpha \mathbb{R} \), we define \( \beta_t(Y) = \lambda(t)Y\lambda(t)^* \), \( \forall t \in \mathbb{R} \). Then \( \beta = \{\beta_t \}_{t \in \mathbb{R}} \) is an inner flow on \( \mathcal{M} \rtimes_\alpha \mathbb{R} \). We know that \( H^\infty(\beta) \) is a nest subalgebra in \( \mathcal{M} \rtimes_\alpha \mathbb{R} \) by Theorem 4.2.3 in [14].

Let \( \mathcal{A} \) be the \( \sigma \)-weakly closed subalgebra of \( \mathcal{M} \rtimes_\alpha \mathbb{R} \) generated by \( \{\pi(X) : X \in H^\infty(\alpha)\} \) \( \cup \{\lambda(t) : t \in \mathbb{R}\} \). Since \( \pi(\alpha_t(X)) = \beta_t(\pi(X)) \) for \( X \in \mathcal{M} \) and \( t \in \mathbb{R} \), \( \mathcal{A} \) is a subalgebra of \( H^\infty(\beta) \). It is noted that \( \mathcal{A} + \mathcal{A}^* \) is \( \sigma \)-weakly dense in \( \mathcal{M} \rtimes_\alpha \mathbb{R} \) since \( H^\infty(\alpha) + H^\infty(\alpha)^* \) is \( \sigma \)-weakly dense in \( \mathcal{M} \) by Theorem 3.15 in [14].

**Lemma 5.** \( H^\infty_0(\beta) \subset \mathcal{A} \).

**Proof.** We know that \( H^\infty_0(\beta) \) is the \( \sigma \)-weak closure of the set \( \{X \in \mathcal{M} \rtimes_\alpha \mathbb{R} : sp_\beta(X) \text{ is compact in } (0, +\infty)\} \) by Lemma 2.8 in [10]. Let \( X \in \mathcal{M} \rtimes_\alpha \mathbb{R} \) be such that \( sp_\beta(X) \) is compact in \( (0, +\infty) \). Choose \( f \in L^1(\mathbb{R}) \) with compactly supported Fourier transform such that support \( suppf \) of \( f \) is in \( (0, \infty) \) and such that \( f \beta X = X \).

Note that since \( \mathcal{A} + \mathcal{A}^* \) is \( \sigma \)-weakly dense in \( \mathcal{M} \rtimes_\alpha \mathbb{R} \), there are nets \( \{A_i\}, \{B_i\} \) in \( \mathcal{A} \) such that \( lim_{i}(A_i + B_i^*) = X \) \( \sigma \)-weakly. It follows that \( lim_{i} f \beta A_i + f \beta B_i^* = f \beta X = X \). However, we have \( f \beta B_i^* = 0 \) since \( suppf \cap sp_\beta(B_i^*) = \emptyset \).

Note that \( \mathcal{A} \) is \( \{\beta_t\}_{t \in \mathbb{R}} \) invariant, it follows that \( f \beta A_i \in \mathcal{A} \) and then \( X \in \mathcal{A} \). The proof is complete. \( \square \)
The next result might be known, but we were unable to find a reference for it.

**Lemma 6.** If $M$ is $\mathbb{R}$-finite relative to $\alpha$, then $M \rtimes_\alpha \mathbb{R}$ is $\mathbb{R}$-finite relative to $\beta$.

**Proof.** By considering a covariant representation of the pair $(M, \alpha)$ in the sense of Definition 2.5 and Proposition 2.6 in [14], we may assume that $\alpha = \{\alpha_t\}_{t \in \mathbb{R}}$ is implemented by a continuous unitary representation $t \rightarrow U_t$ of $\mathbb{R}$ on $\mathcal{H}$, that is, $\alpha_t(X) = U_tXU_t^*$ for all $x \in M$. Then by Definition 13.2.6 in [12], the crossed product $M \rtimes_\alpha \mathbb{R}$ is a von Neumann algebra on $L^2(\mathbb{R}, \mathcal{H}) = \mathcal{H} \otimes L^2(\mathbb{R})$ generated by $\{A \otimes I : A \in M\}$ and $\{U_t \otimes l_s : t, s \in \mathbb{R}\}$, where $l_s$ is the shift operator on $L^2(\mathbb{R})$ defined by $(l_s f)(s) = f(s - t)$, $t, s \in \mathbb{R}$, $f \in L^2(\mathbb{R})$. It is known that $\beta$ is implemented by $\{U_t \otimes l_s \}_{t, s} \in \mathbb{R}$.

Since $L^2(\mathbb{R})$ is separable, there is an orthonormal basis $\{e_n : n = 1, 2, \cdots\}$ of $L^2(\mathbb{R})$. If we define $\mu(T) = \sum_{n=1}^{\infty} \frac{1}{n}(Te_n, e_n)$, $\forall T \in \mathcal{B}(L^2(\mathbb{R}))$, then $\mu$ is a faithful normal state on $\mathcal{B}(L^2(\mathbb{R}))$. Let $\phi$ be a faithful normal state of $M$ such that $\phi \circ \alpha_t = \phi$ for all $t \in \mathbb{R}$. It follows from Proposition 11.2.7 in [12] that $\phi \otimes \mu$ is a normal state on $M \otimes \mathcal{B}(L^2(\mathbb{R})) \supset M \rtimes_\alpha \mathbb{R}$. We claim that $\phi \otimes \mu$ is faithful. In fact, we may identify $M \otimes \mathcal{B}(L^2(\mathbb{R}))$ with $\{(A_{ij}) \in \mathcal{B}(H \otimes L^2(\mathbb{R})) : A_{ij} \in M\}$ by Remark 11.2.3 in [12]. For any $(A_{ij}) \in M \otimes \mathcal{B}(L^2(\mathbb{R}))$, we define $\psi((A_{ij})) = \sum_{n=1}^{\infty} \frac{1}{n} \phi(A_{nn})$. Then $\psi$ is a faithful normal state on $M \otimes \mathcal{B}(L^2(\mathbb{R}))$. Since for any $A \in M$ and $T \in \mathcal{B}(L^2(\mathbb{R}))$, $A \otimes T$ identifies with $(t_{ij} A)$, where $t_{ij} = (T e_i, e_j)$ for all $i, j = 1, 2, \cdots$, $\phi \otimes \mu(A \otimes T) = \phi(A) \mu(T) = \psi((t_{ij} A))$. It follows that $\phi \otimes \mu = \psi$ and then $\phi \otimes \mu$ is faithful.

We note that if $A \in M$, then $\phi \otimes \mu(\beta_s(A \otimes I)) = \phi \otimes \mu(A \otimes I)$ for all $s \in \mathbb{R}$. On the other hand, $\beta_t(U_s \otimes l_s) = U_s \otimes l_s$; then $\phi \otimes \mu(\beta_t(U_s \otimes l_s)) = \phi \otimes \mu(U_s \otimes l_s)$, $s, t \in \mathbb{R}$. We now have that $\phi \otimes \mu$ is a faithful normal state on $M \rtimes_\alpha \mathbb{R}$ such that $\phi \otimes \mu \circ \beta_s = \phi \otimes \mu$ for all $s \in \mathbb{R}$. It follows that $M \rtimes_\alpha \mathbb{R}$ is $\mathbb{R}$-finite relative to $\beta$. The proof is complete. \hfill \Box

**Lemma 7.** $H^\infty(\beta) = A$.

**Proof.** We first assume that $M$ is $\mathbb{R}$-finite relative to $\{\alpha_t\}_{t \in \mathbb{R}}$. Then there is a faithful normal expectation $\Phi$ from $M$ onto $\mathcal{D}$. Now there is a faithful normal expectation $\Psi$ from $M \rtimes_\alpha \mathbb{R}$ onto $\mathcal{N}$ such that $\Psi(\pi(A)) = \pi(\Phi(A))$ for all $A \in M$ by Lemma 6, where $\mathcal{N}$ is the fixed point algebra of $\beta$. We have that $\mathcal{N}$ is generated by $\{\pi(A) : A \in \mathcal{D}\}$ and $\{\lambda(t) : t \in \mathbb{R}\}$. In fact, take any $D \in \mathcal{N}$. There is a net $A_i = \sum_{j=1}^{n_i} \pi(X_j^i) \lambda(t_j^i)$ such that $\lim A_i = D$ $\sigma$-weakly, where $X_j^i \in M$ and $t_j^i \in \mathbb{R}$. We then have

$$D = \Psi(D) = \lim_i \Psi(A_i) = \lim_i \sum_{j=1}^{n_i} \Psi(\pi(X_j^i)) \lambda(t_j^i) = \lim \sum_{j=1}^{n_i} \pi(\Phi(X_j^i)) \lambda(t_j^i).$$

It now follows that $\mathcal{N} = H^\infty(\beta) \cap H^\infty(\beta)^* = A \cap A^*$. We know that $H^\infty(\beta) \subset A$ by Lemma 5. Thus $H^\infty(\beta) = \mathcal{N} = H^\infty(\beta)^* \subset A$ and therefore $H^\infty(\beta) = A$.

For the general case, there is a projection $E$ in the center of $\mathcal{D}$ such that $E M E$ is $\mathbb{R}$-finite, while $(I - E) M (I - E)$ is completely non-$\mathbb{R}$-finite relative to $\alpha$ from Remark

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3.4 in [14]. We note that in this case \(\pi(E)(\mathcal{M} \rtimes_{\alpha} \mathbb{R})\pi(E)\) is \(\mathbb{R}\)-finite by Lemma 6, and 
\((I-\pi(E))(\mathcal{M} \rtimes_{\alpha} \mathbb{R})(I-\pi(E))\) is completely non-\(\mathbb{R}\)-finite relative to \(\beta\). By considering 
\(\alpha\) restricted on \(E ME\), we have 
\(\pi(E)H^\infty(\beta)\pi(E) \subset \pi(E)A \pi(E) \subset \mathcal{A}\) by Lemma 6 again. On the other hand, if we consider \(\alpha\) on 
\((I-E)\mathcal{M}(I-E)\), then we have 
\((I-\pi(E))H^\infty(\beta)(I-\pi(E)) = (I-\pi(E))H^\infty_{0^\beta}(\beta)(I-\pi(E)) \subset H^\infty_{0^\beta}(\beta)\). In particular, 
\((I-\pi(E)) \in H^\infty_{0^\beta}(\beta)\) and then 
\((I-\pi(E))H^\infty(\beta) + H^\infty(\beta)(I-\pi(E)) \subset H^\infty_{0^\beta}(\beta) \subset \mathcal{A}\). Thus we have 
\(H^\infty(\beta) \subset \mathcal{A}\), and the proof is complete.

**Theorem 2.** The commutant of \(H^\infty(\alpha)\) is self-adjoint, that is, 
\((H^\infty(\alpha))' = \mathcal{M}'\).

**Proof.** Let \(\beta = \{\beta_t\}_{t \in \mathbb{R}}\) be as above. We recall that \(H^\infty(\beta)\) is a nest subalgebra of 
\(\mathcal{M} \rtimes_{\alpha} \mathbb{R}\). Then the commutant of \(H^\infty(\beta)\) is self-adjoint by Theorem 2.5 in [6]. Let 
\(\mathcal{X} \in (H^\infty(\alpha))'\). Define an operator \(\tilde{X}\) on \(L^2(\mathbb{R}, \mathcal{H})\) by 
\[(\tilde{X}\xi)(t) = X\xi(t), \quad \forall \xi \in L^2(\mathbb{R}, \mathcal{H}).\]

Then it is trivial that \(\tilde{X}\) is bounded. We claim that \(\tilde{X} \in (H^\infty(\beta))'\). By Lemma 7, it is sufficient to show that \(\tilde{X} \in \mathcal{A}'\). For any \(Y \in H^\infty(\alpha)\) we have 
\[(\tilde{X}\pi(Y)\xi)(t) = X\alpha_{-t}(Y)\xi(t) = \alpha_{-t}(Y)X\xi(t) = \alpha_{-t}(Y)(\tilde{X}\xi)(t) = (\pi(Y)\tilde{X}\xi)(t).\]

Then \(\tilde{X}\pi(Y) = \pi(Y)\tilde{X}\). On the other hand, for any \(s \in \mathbb{R}\), 
\[(\tilde{X}\lambda(s)\xi)(t) = (\tilde{X}\xi)(t-s) = \lambda(s)(\tilde{X}\xi)(t) = (\lambda(s)\tilde{X}\xi)(t).\]

It follows that \(\lambda(s)\tilde{X} = \tilde{X}\lambda(s)\) for any \(s \in \mathbb{R}\). We thus have \(\tilde{X} \in (H^\infty(\beta))'\). By Theorem 2.5 in [6], \((\tilde{X})^* \in (H^\infty(\beta))'\). In particular, \((\tilde{X})^*\) commutes with \(\pi(Y)\) for any \(Y \in H^\infty(\alpha)\). Note that \((\tilde{X})^* = (X^*)\). Given \(u \in \mathcal{H}\), let \(f\) be a continuous function in \(L^2(\mathbb{R})\) such that \(f(0) = 1\) and let \(\xi(t) = f(t)u\). Then \(\xi \in L^2(\mathbb{R}, \mathcal{H})\) and 
\[(\pi(Y)\xi)(t) = \pi(Y)(\lambda(s)\xi)(t) = \lambda(s)(\pi(Y)\xi)(t) = \pi(Y)(\lambda(s)\xi)(t) = (\pi(Y)\xi)(t-s) = (\tilde{X}\xi)(t-s) = (\tilde{X}\lambda(s)\xi)(t) = (\tilde{X}\xi)(t-s) = \tilde{X}\lambda(s)\xi(t-s).

When \(t = 0\), we have \(X^*Yu = YX^*u\) for all \(u \in \mathcal{H}\). Hence \(X^* \in (H^\infty(\alpha))'\) and 
\((H^\infty(\alpha))' = \mathcal{M}'\). The proof is complete.

**Remark 1.** We know that if \(\mathcal{A}\) is either a subdiagonal algebra or an analytic operator algebra of \(\mathcal{M}\), then we have that \(\mathcal{A} + \mathcal{A}^*\) is \(\sigma\)-weakly dense in \(\mathcal{M}\). However, if we assume that a subalgebra only satisfies this condition, it may not follow that the algebraic commutant is self-adjoint. For example, from Corollary 1.4 in [1], we know that there is a subalgebra \(\mathcal{A}\) of \(\mathcal{B}(\mathcal{H})\) such that \(\mathcal{A} + \mathcal{A}^*\) is \(\sigma\)-weakly dense in \(\mathcal{B}(\mathcal{H})\) and such that \(\mathcal{A}\) is similar to a proper von Neumann subalgebra of \(\mathcal{B}(\mathcal{H})\). It easily follows that the algebraic commutant of \(\mathcal{A}\) is not self-adjoint.

**Remark 2.** We considered two classes of non-self-adjoint operator algebras, subdiagonal algebras and analytic operator algebras determined by flows in von Neumann algebras. If \(\mathcal{H}\) is finite dimensional, we know that these two classes of operator algebras are nest subalgebras of von Neumann algebras (cf. Theorem 2.1 in [11]). However, if \(\mathcal{H}\) is infinite dimensional, these two classes are different. The analytic operator algebra \(H^\infty(\alpha)\) determined by a flow \(\alpha\) is a maximal subdiagonal algebra if and only if the flow \(\alpha\) is \(\mathbb{R}\)-finite. There are subdiagonal algebras which are not
analytic operator algebras determined by any flows. We refer the readers to see some examples given in [8].

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