

## THE FIRST EIGENVALUE OF A CLOSED MANIFOLD WITH POSITIVE RICCI CURVATURE

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ABSTRACT. We give a new estimate on the lower bound for the first positive eigenvalue of the Laplacian on a closed manifold with positive Ricci curvature in terms of the lower bound of the Ricci curvature and the largest interior radius of the nodal domains of eigenfunctions of the eigenvalue.

### 1. INTRODUCTION

It has been proved by Li-Yau [5] and Zhong-Yang [12] that if  $M^n$  is a closed  $n$ -dimensional manifold with nonnegative Ricci curvature, then the first nonzero eigenvalue  $\lambda$  of the manifold has a lower bound

$$\lambda \geq \frac{\pi^2}{d^2}.$$

On the other hand, if the Ricci curvature  $\text{Ric}(M^n)$  has a positive lower bound  $(n-1)K$  for some positive constant  $K$ , the Lichnerowicz Theorem states that

$$(1.1) \quad \lambda \geq nK.$$

The Lichnerowicz-type estimate (1.1) is nice and optimal for positive  $K$ . But it gives no information when the Ricci curvature lower bound is zero. Therefore it is an interesting problem to find a unified lower bound of the first positive eigenvalue  $\lambda$  in terms of the lower bound  $(n-1)K$  of the Ricci curvature and other geometric quantities for a closed manifold with positive Ricci curvature so that the lower bound of  $\lambda$  does not vanish as the lower bound of the Ricci curvature does. P. Li conjectured that the lower bound for the first positive eigenvalue  $\lambda$  is  $(n-1)K + \pi^2/d^2$ . D. Yang [11] showed that the coefficient before the Ricci lower bound  $(n-1)K$  is  $1/4$ . Along the line of the Li Conjecture, we give a new estimate on the lower bound of the first positive eigenvalue of a closed Riemannian manifold with positive lower bound of Ricci curvature in terms of the lower bound of the Ricci curvature and the largest interior radius of the nodal domains of eigenfunctions of the eigenvalue. Instead of using Zhong-Yang's canonical function or the "midrange" of the normalized eigenfunction of the first positive eigenvalue in the proof, we use a function  $\xi$  that the author constructed in [8] for the construction of the suitable

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test function and use the structure of the nodal domains of the eigenfunction. We have the following result.

**Theorem 1.1.** *Let  $M$  be a closed  $n$ -dimensional Riemannian manifold with Ricci curvature  $\text{Ric}(M)$  bounded below by*

$$(1.2) \quad \text{Ric}(M) \geq (n-1)K > 0.$$

*Then the first positive eigenvalue  $\lambda$  of the Laplacian  $\Delta$  on  $M$  has the lower bound*

$$(1.3) \quad \lambda \geq \frac{1}{2}(n-1)K + \frac{\pi^2}{4r^2},$$

*where  $r$  is the largest interior radius of the nodal domains of eigenfunctions of the eigenvalue.*

We derive some preliminary estimates and conditions for test functions in the next section and construct the needed test function and prove the main result in the last section.

## 2. PRELIMINARY ESTIMATES

The classic Lichnerowicz Theorem [6] states that if  $M$  is a closed  $n$ -dimensional manifold whose Ricci curvature satisfies (1.2), then the first positive eigenvalue  $\lambda$  of the Laplacian on  $M$  has a lower bound in (1.1). For completeness and consistency, we use gradient estimate in [3]-[5] and [10] to derive the Lichnerowicz estimate.

**Lemma 2.1** (Lichnerowicz). *Under the conditions in Theorem 1.1, estimate (1.1) holds.*

*Proof.* Let  $v$  be a normalized eigenfunction of the first positive eigenvalue  $\lambda$  of the Laplacian  $\Delta$  on  $M$  such that

$$(2.1) \quad \sup_M v = 1 \quad \text{and} \quad \inf_M v = -k$$

with  $0 < k \leq 1$ . The function  $v$  satisfies the following equation:

$$(2.2) \quad \Delta v = -\lambda v \quad \text{in } M.$$

Take an orthonormal frame  $\{e_1, \dots, e_n\}$  of  $M$  about  $x_0 \in M$ . At  $x_0$  we have

$$\nabla_{e_j}(|\nabla v|^2)(x_0) = \sum_{i=1}^n 2v_i v_{ij}$$

and

$$\begin{aligned} \Delta(|\nabla v|^2)(x_0) &= 2 \sum_{i,j=1}^n v_{ij} v_{ij} + 2 \sum_{i,j=1}^n v_i v_{ijj} \\ &= 2 \sum_{i,j=1}^n v_{ij} v_{ij} + 2 \sum_{i,j=1}^n v_i v_{jji} + 2 \sum_{i,j=1}^n R_{ij} v_i v_j \\ &= 2 \sum_{i,j=1}^n v_{ij} v_{ij} + 2 \nabla v \nabla(\Delta v) + 2 \text{Ric}(\nabla v, \nabla v) \\ &\geq 2 \sum_{i=1}^n v_{ii}^2 + 2 \nabla v \nabla(\Delta v) + 2(n-1)K |\nabla v|^2 \\ &\geq \frac{2}{n} (\Delta v)^2 - 2\lambda |\nabla v|^2 + 2(n-1)K |\nabla v|^2. \end{aligned}$$

Thus at all points  $x \in M$ ,

$$(2.3) \quad \frac{1}{2}\Delta(|\nabla v|^2) \geq \frac{1}{n}\lambda^2 v^2 + [(n-1)K - \lambda]|\nabla v|^2.$$

On the other hand, after multiplying (2.2) by  $v$  and integrating both sides over  $M$ , we have

$$\int_M \lambda v^2 dx = - \int_M v \Delta v dx = \int_M |\nabla v|^2 dx.$$

Integrating (2.3) over  $M$  and using the above equality, we get

$$(2.4) \quad 0 \geq \int_M (nK - \lambda) \frac{n-1}{n} \lambda v^2 dx.$$

Therefore (1.1) holds. □

**Lemma 2.2.** *Let  $v$  be, as above, the normalized eigenfunction for the first positive eigenvalue  $\lambda$ . Then  $v$  satisfies the following:*

$$(2.5) \quad \frac{|\nabla v|^2}{b^2 - v^2} \leq \lambda,$$

where  $b > 1$  is an arbitrary constant.

*Proof.* Consider the function

$$(2.6) \quad P(x) = |\nabla v|^2 + Av^2,$$

where  $A = \lambda(1 + \epsilon)$  for small  $\epsilon > 0$ . Function  $P$  must achieve its maximum at some point  $x_0 \in M$ . We claim that  $\nabla v(x_0) = 0$ .

If on the contrary  $\nabla v(x_0) \neq 0$ , then we can rotate the local orthonormal about  $x_0$  such that

$$|v_1(x_0)| = |\nabla v(x_0)| \neq 0 \quad \text{and} \quad v_i(x_0) = 0, \quad i \geq 2.$$

Since  $P$  achieves its maximum at  $x_0$ , we have

$$\nabla P(x_0) = 0 \quad \text{and} \quad \Delta P(x_0) \leq 0.$$

That is, at  $x_0$  we have

$$(2.7) \quad \begin{aligned} 0 &= \frac{1}{2}\nabla_i P = \sum_{j=1}^n v_j v_{ji} + Av v_i, \\ v_{11} &= -Av \quad \text{and} \quad v_{1i} = 0, \quad i \geq 2, \end{aligned}$$

and

$$\begin{aligned} 0 &\geq \frac{1}{2}\Delta P(x_0) = \sum_{i,j=1}^n (v_{ji}v_{ji} + v_j v_{jii} + Av_i v_i + Av v_{ii}) \\ &= \sum_{i,j=1}^n (v_{ji}^2 + v_j (v_{ii})_j + R_{ji} v_j v_i + Av_{ii}^2 + Av v_{ii}) \\ &= \sum_{i,j=1}^n v_{ji}^2 + \nabla v \nabla (\Delta v) + \text{Ric}(\nabla v, \nabla v) + A|\nabla v|^2 + Av \Delta v \\ &\geq v_{11}^2 + \nabla v \nabla (\Delta v) + (n-1)K|\nabla v|^2 + A|\nabla v|^2 + Av \Delta v \\ &= (-Av)^2 - \lambda|\nabla v|^2 + (n-1)K|\nabla v|^2 + A|\nabla v|^2 - \lambda Av^2 \\ &= (A - \lambda + (n-1)K)|\nabla v|^2 + Av^2(A - \lambda), \end{aligned}$$

where we have used (2.7) and (1.2). Therefore at  $x_0$ ,

$$(2.8) \quad 0 \geq (A - \lambda)|\nabla v|^2 + A(A - \lambda)v^2,$$

that is,

$$|\nabla v(x_0)|^2 + \lambda(1 + \epsilon)v(x_0)^2 \leq 0.$$

Thus  $\nabla v(x_0) = 0$ . This contradicts  $\nabla v(x_0) \neq 0$ . So the above claim is true.

Therefore we have  $\nabla v(x_0) = 0$ ,

$$P(x_0) = |\nabla v(x_0)|^2 + Av(x_0)^2 = Av(x_0)^2 \leq A,$$

and at all  $x \in M$

$$|\nabla v(x)|^2 + Av(x)^2 = P(x) \leq P(x_0) \leq A.$$

Letting  $\epsilon \rightarrow 0$  in the above inequality, the estimate (2.5) follows.  $\square$

We want to improve the upper bound in (2.5) further and proceed in the following way.

Define a function  $Z$  on  $[-\sin^{-1}(k/b), \sin^{-1}(1/b)]$  by

$$Z(t) = \max_{x \in M, t = \sin^{-1}(v(x)/b)} \frac{|\nabla v|^2}{b^2 - v^2} / \lambda.$$

From (2.5) we have

$$(2.9) \quad Z(t) \leq 1 \quad \text{on } [-\sin^{-1}(k/b), \sin^{-1}(1/b)].$$

For convenience, in this paper we let

$$(2.10) \quad \alpha = \frac{1}{2}(n-1)K \quad \text{and} \quad \delta = \alpha/\lambda.$$

By (1.1) we have

$$(2.11) \quad \delta \leq \frac{n-1}{2n}.$$

We have the following conditions on the test function  $Z$ .

**Theorem 2.3.** *If the function  $z : [-\sin^{-1}(k/b), \sin^{-1}(1/b)] \mapsto \mathbf{R}^1$  satisfies the following:*

- (1)  $z(t) \geq Z(t)$ ,  $t \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)]$ ,
- (2) *there exists some  $x_0 \in M$  such that  $z(t_0) = Z(t_0)$  at point  $t_0 = \sin^{-1}(v(x_0)/b)$ ,*
- (3)  $z(t_0) > 0$ , *and*
- (4)  $z'(t_0) \sin t_0 \geq 0$ ,

*then we have*

$$(2.12) \quad 0 \leq \frac{1}{2}z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 - z(t_0) + 1 - 2\delta \cos^2 t_0.$$

*Proof.* Define

$$J(x) = \left\{ \frac{|\nabla v|^2}{b^2 - v^2} - \lambda z \right\} \cos^2 t,$$

where  $t = \sin^{-1}(v(x)/b)$ . Then

$$J(x) \leq 0 \quad \text{for } x \in M \quad \text{and} \quad J(x_0) = 0.$$

If  $\nabla v(x_0) = 0$ , then

$$0 = J(x_0) = -\lambda z \cos^2 t.$$

This contradicts condition (3) in the theorem. Therefore

$$\nabla v(x_0) \neq 0.$$

The Maximum Principle implies that

$$(2.13) \quad \nabla J(x_0) = 0 \quad \text{and} \quad \Delta J(x_0) \leq 0.$$

$J(x)$  can be rewritten as

$$J(x) = \frac{1}{b^2} |\nabla v|^2 - \lambda z \cos^2 t.$$

Thus (2.13) is equivalent to

$$(2.14) \quad \frac{2}{b^2} \sum_i v_i v_{ij} \Big|_{x_0} = \lambda \cos t [z' \cos t - 2z \sin t] t_j \Big|_{x_0}$$

and

$$(2.15) \quad 0 \geq \frac{2}{b^2} \sum_{i,j} v_{ij}^2 + \frac{2}{b^2} \sum_{i,j} v_i v_{ijj} - \lambda(z'' |\nabla t|^2 + z' \Delta t) \cos^2 t \\ + 4\lambda z' \cos t \sin t |\nabla t|^2 - \lambda z \Delta \cos^2 t \Big|_{x_0}.$$

Rotate the frame so that  $v_1(x_0) \neq 0$  and  $v_i(x_0) = 0$  for  $i \geq 2$ . Then (2.14) implies

$$(2.16) \quad v_{11} \Big|_{x_0} = \frac{\lambda b}{2} (z' \cos t - 2z \sin t) \Big|_{x_0} \quad \text{and} \quad v_{1i} \Big|_{x_0} = 0 \text{ for } i \geq 2.$$

Now we have

$$\begin{aligned} |\nabla v|^2 \Big|_{x_0} &= \lambda b^2 z \cos^2 t \Big|_{x_0}, \\ |\nabla t|^2 \Big|_{x_0} &= \frac{|\nabla v|^2}{b^2 - v^2} = \lambda z \Big|_{x_0}, \\ \frac{\Delta v}{b} \Big|_{x_0} &= \Delta \sin t = \cos t \Delta t - \sin t |\nabla t|^2 \Big|_{x_0}, \\ \Delta t \Big|_{x_0} &= \frac{1}{\cos t} (\sin t |\nabla t|^2 + \frac{\Delta v}{b}) \\ &= \frac{1}{\cos t} [\lambda z \sin t - \frac{\lambda}{b} v] \Big|_{x_0}, \quad \text{and} \\ \Delta \cos^2 t \Big|_{x_0} &= \Delta \left( 1 - \frac{v^2}{b^2} \right) = -\frac{2}{b^2} |\nabla v|^2 - \frac{2}{b^2} v \Delta v \\ &= -2\lambda z \cos^2 t + \frac{2}{b^2} \lambda v^2 \Big|_{x_0}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{2}{b^2} \sum_{i,j} v_{ij}^2 \Big|_{x_0} &\geq \frac{2}{b^2} v_{11}^2 \\ &= \frac{\lambda^2}{2} (z')^2 \cos^2 t - 2\lambda^2 z z' \cos t \sin t + 2\lambda^2 z^2 \sin^2 t \Big|_{x_0}, \end{aligned}$$

$$\begin{aligned}
\frac{2}{b^2} \sum_{i,j} v_i v_{ijj} \Big|_{x_0} &= \frac{2}{b^2} (\nabla v \nabla(\Delta v) + \text{Ric}(\nabla v, \nabla v)) \\
&\geq \frac{2}{b^2} (\nabla v \nabla(\Delta v) + (n-1)K|\nabla v|^2) \\
&= -2\lambda^2 z \cos^2 t + 4\alpha\lambda z \cos^2 t \Big|_{x_0}, \\
&\quad -\lambda(z''|\nabla t|^2 + z'\Delta t) \cos^2 t \Big|_{x_0} \\
&= -\lambda^2 z z'' \cos^2 t - \lambda^2 z z' \cos t \sin t \\
&\quad + \frac{1}{b} \lambda^2 z' v \cos t \Big|_{x_0}
\end{aligned}$$

and

$$\begin{aligned}
&4\lambda z' \cos t \sin t |\nabla t|^2 - \lambda z \Delta \cos^2 t \Big|_{x_0} \\
&= 4\lambda^2 z z' \cos t \sin t + 2\lambda^2 z^2 \cos^2 t - \frac{2}{b} \lambda^2 z v \sin t \Big|_{x_0}.
\end{aligned}$$

Putting these results into (2.15) we get

$$\begin{aligned}
(2.17) \quad 0 &\geq -\lambda^2 z z'' \cos^2 t + \frac{\lambda^2}{2} (z')^2 \cos^2 t + \lambda^2 z' \cos t (z \sin t + \sin t) \\
&\quad + 2\lambda^2 z^2 - 2\lambda^2 z + 4\alpha\lambda z \cos^2 t \Big|_{x_0},
\end{aligned}$$

where we used (2.16). Now

$$(2.18) \quad z(t_0) > 0,$$

by condition (3) in the theorem. Dividing two sides of (2.17) by  $2\lambda^2 z \Big|_{x_0}$ , we have

$$\begin{aligned}
0 &\geq -\frac{1}{2} z''(t_0) \cos^2 t_0 + \frac{1}{2} z'(t_0) \cos t_0 \left( \sin t_0 + \frac{\sin t_0}{z(t_0)} \right) + z(t_0) \\
&\quad - 1 + 2\delta \cos^2 t_0 + \frac{1}{4z(t_0)} (z'(t_0))^2 \cos^2 t_0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(2.19) \quad 0 &\geq -\frac{1}{2} z''(t_0) \cos^2 t_0 + z'(t_0) \cos t_0 \sin t_0 + z(t_0) - 1 + 2\delta \cos^2 t_0 \\
&\quad + \frac{1}{4z(t_0)} (z'(t_0))^2 \cos^2 t_0 + \frac{1}{2} z'(t_0) \sin t_0 \cos t_0 \left[ \frac{1}{z(t_0)} - 1 \right].
\end{aligned}$$

Conditions (1), (2) and (4) in the theorem imply that  $0 < z(t_0) = Z(t_0) \leq 1$  and  $z'(t_0) \sin t_0 \geq 0$ . Thus the last two terms in (2.19) are nonnegative and (2.12) follows.  $\square$

### 3. PROOF OF THE MAIN RESULT

*Proof of Theorem 1.1.* Let

$$(3.1) \quad z(t) = 1 + \delta \xi(t),$$

where  $\xi$  is the function defined by (3.9) in Lemma 3.1. We claim that

$$(3.2) \quad Z(t) \leq z(t) \quad \text{on } [-\sin^{-1}(k/b), \sin^{-1}(1/b)].$$

Lemma 3.1 implies that for  $t \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)]$  we have the following:

$$(3.3) \quad \frac{1}{2}z'' \cos^2 t - z' \cos t \sin t - z = -1 + 2\delta \cos^2 t,$$

$$(3.4) \quad z'(t) \sin t \geq 0,$$

$$(3.5) \quad 0 < 1 - \left(\frac{\pi^2}{4} - 1\right) \frac{n-1}{2n} \leq 1 - \left(\frac{\pi^2}{4} - 1\right)\delta = z(0) \leq z(t), \quad \text{and}$$

$$(3.6) \quad z(t) \leq z\left(\frac{\pi}{2}\right) = 1.$$

Let  $P \in \mathbf{R}^1$  and  $t_0 \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)]$  such that

$$P = \max_{t \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)]} (Z(t) - z(t)) = Z(t_0) - z(t_0).$$

Thus

$$Z(t) \leq z(t) + P \quad \text{for } t \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)] \quad \text{and} \quad Z(t_0) = z(t_0) + P.$$

Suppose that  $P > 0$ . Then  $z + P$  satisfies the conditions in Theorem 2.3. (2.12) implies

$$\begin{aligned} z(t_0) + P &= Z(t_0) \\ &\leq \frac{1}{2}(z + P)''(t_0) \cos^2 t_0 - (z + P)'(t_0) \cos t_0 \sin t_0 + 1 - 2\delta \cos^2 t_0 \\ &= \frac{1}{2}z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 + 1 - 2\delta \cos^2 t_0 \\ &= z(t_0). \end{aligned}$$

This contradicts the assumption  $P > 0$ . Thus  $P \leq 0$  and (3.2) holds. That means

$$(3.7) \quad \sqrt{\lambda} \geq \frac{|\nabla t|}{\sqrt{z(t)}}.$$

Note that the eigenfunction  $v$  of the first nonzero eigenvalue has exactly two nodal domains  $D^+ = \{x : v(x) > 0\}$  and  $D^- = \{x : v(x) < 0\}$  and that the nodal set  $v^{-1}(0)$  is compact (see [1] and [2]). Take  $q_1$  on  $M$  such that  $v(q_1) = 1 = \sup_M v$  and  $q_2 \in v^{-1}(0)$  such that distance  $d(q_1, q_2) = \text{distance } d(q_1, v^{-1}(0))$ . Let  $L$  be the minimum geodesic segment joining  $q_1$  and  $q_2$ . We integrate both sides of (3.7) along  $L$  and change variable, and let  $b \rightarrow 1$ . Let  $r_+, r_-$  be the radius of the largest interior ball in  $D^+, D^-$  respectively, that is,

$$r_+ = \max_{x \in D^+} \text{dist}(x, v^{-1}(0))$$

and

$$r_- = \max_{x \in D^-} \text{dist}(x, v^{-1}(0)).$$

Set  $r = \max\{r_+, r_-\}$ . Then

$$(3.8) \quad r\sqrt{\lambda} \geq \int_L \frac{|\nabla t|}{\sqrt{z(t)}} dl = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{z(t)}} dt \geq \frac{\left(\int_0^{\pi/2} dt\right)^{\frac{3}{2}}}{\left(\int_0^{\pi/2} z(t) dt\right)^{\frac{1}{2}}} \geq \left(\frac{(\frac{\pi}{2})^3}{\int_0^{\pi/2} z(t) dt}\right)^{\frac{1}{2}}.$$

Square the two sides. Then

$$\lambda \geq \frac{\pi^3}{8r^2 \int_0^{\pi/2} z(t) dt}.$$

Now

$$\int_0^{\frac{\pi}{2}} z(t) dt = \int_0^{\frac{\pi}{2}} [1 + \delta\xi(t)] dt = \frac{\pi}{2}(1 - \delta),$$

by (3.12) in Lemma 3.1. That is,

$$\lambda \geq \frac{\pi^2}{4r^2(1 - \delta)} \quad \text{and} \quad \lambda \geq \frac{1}{2}(n - 1)K + \frac{\pi^2}{4r^2}.$$

This completes the proof.  $\square$

We now present a lemma that is used in the proof of Theorem 1.1.

**Lemma 3.1.** *Let*

$$(3.9) \quad \xi(t) = \frac{\cos^2 t + 2t \sin t \cos t + t^2 - \frac{\pi^2}{4}}{\cos^2 t} \quad \text{on} \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

*Then the function  $\xi$  satisfies the following:*

$$(3.10) \quad \frac{1}{2}\xi'' \cos^2 t - \xi' \cos t \sin t - \xi = 2 \cos^2 t \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

$$(3.11) \quad \xi' \cos t - 2\xi \sin t = 4t \cos t,$$

$$(3.12) \quad \int_0^{\frac{\pi}{2}} \xi(t) dt = -\frac{\pi}{2},$$

$$1 - \frac{\pi^2}{4} = \xi(0) \leq \xi(t) \leq \xi\left(\pm\frac{\pi}{2}\right) = 0 \quad \text{on} \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

$$\xi' \text{ is increasing on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and } \xi'\left(\pm\frac{\pi}{2}\right) = \pm\frac{2\pi}{3},$$

$$\xi'(t) < 0 \text{ on } \left(-\frac{\pi}{2}, 0\right) \text{ and } \xi'(t) > 0 \text{ on } \left(0, \frac{\pi}{2}\right),$$

$$\xi''\left(\pm\frac{\pi}{2}\right) = 2, \quad \xi''(0) = 2\left(3 - \frac{\pi^2}{4}\right) \text{ and } \xi''(t) > 0 \text{ on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

$$\left(\frac{\xi'(t)}{t}\right)' > 0 \text{ on } (0, \pi/2) \text{ and } 2\left(3 - \frac{\pi^2}{4}\right) \leq \frac{\xi'(t)}{t} \leq \frac{4}{3} \text{ on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

$$\xi'''\left(\frac{\pi}{2}\right) = \frac{8\pi}{15}, \xi'''(t) < 0 \text{ on } \left(-\frac{\pi}{2}, 0\right) \text{ and } \xi'''(t) > 0 \text{ on } \left(0, \frac{\pi}{2}\right).$$

*Proof.* For convenience, let  $q(t) = \xi'(t)$ , i.e.,

$$(3.13) \quad q(t) = \xi'(t) = \frac{2(2t \cos t + t^2 \sin t + \cos^2 t \sin t - \frac{\pi^2}{4} \sin t)}{\cos^3 t}.$$

Equation (3.10) and the values  $\xi(\pm\frac{\pi}{2}) = 0$ ,  $\xi(0) = 1 - \frac{\pi^2}{4}$  and  $\xi'(\pm\frac{\pi}{2}) = \pm\frac{2\pi}{3}$  can be verified directly from (3.9) and (3.13). The values of  $\xi''$  at 0 and  $\pm\frac{\pi}{2}$  can be computed via (3.10). By (3.11),  $(\xi(t) \cos^2 t)' = 4t \cos^2 t$ . Therefore

$\xi(t) \cos^2 t = \int_{\frac{\pi}{2}}^t 4s \cos^2 s ds$ , and

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \xi(t) dt &= 2 \int_0^{\frac{\pi}{2}} \xi(t) dt = -8 \int_0^{\frac{\pi}{2}} \left( \frac{1}{\cos^2(t)} \int_t^{\frac{\pi}{2}} s \cos^2 s ds \right) dt \\ &= -8 \int_0^{\frac{\pi}{2}} \left( \int_0^s \frac{1}{\cos^2(t)} dt \right) s \cos^2 s ds = -8 \int_0^{\frac{\pi}{2}} s \cos s \sin s ds = -\pi. \end{aligned}$$

It is easy to see that  $q$  and  $q'$  satisfy the following equations:

$$(3.14) \quad \frac{1}{2}q'' \cos t - 2q' \sin t - 2q \cos t = -4 \sin t$$

and

$$(3.15) \quad \frac{\cos^2 t}{2(1 + \cos^2 t)}(q')'' - \frac{2 \cos t \sin t}{1 + \cos^2 t}(q')' - 2(q') = -\frac{4}{1 + \cos^2 t}.$$

The last equation implies that  $q' = \xi''$  cannot achieve its nonpositive local minimum at a point in  $(-\frac{\pi}{2}, \frac{\pi}{2})$ . On the other hand,  $\xi''(\pm\frac{\pi}{2}) = 2$ , by equation (3.10),  $\xi(\pm\frac{\pi}{2}) = 0$  and  $\xi'(\pm\frac{\pi}{2}) = \pm\frac{2\pi}{3}$ . Therefore  $\xi''(t) > 0$  on  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  and  $\xi'$  is increasing. Since  $\xi'(t) = 0$ , we have  $\xi'(t) < 0$  on  $(-\frac{\pi}{2}, 0)$  and  $\xi'(t) > 0$  on  $(0, \frac{\pi}{2})$ . Similarly, from the equation

$$(3.16) \quad \begin{aligned} & \frac{\cos^2 t}{2(1 + \cos^2 t)}(q'')'' - \frac{\cos t \sin t(3 + 2 \cos^2 t)}{(1 + \cos^2 t)^2}(q'')' - \frac{2(5 \cos^2 t + \cos^4 t)}{(1 + \cos^2 t)^2}(q'') \\ & = -\frac{8 \cos t \sin t}{(1 + \cos^2 t)^2} \end{aligned}$$

we get the results in the last line of the lemma.

Set  $h(t) = \xi''(t)t - \xi'(t)$ . Then  $h(0) = 0$  and  $h'(t) = \xi'''(t)t > 0$  in  $(0, \frac{\pi}{2})$ . Therefore  $(\frac{\xi'(t)}{t})' = \frac{h(t)}{t^2} > 0$  in  $(0, \frac{\pi}{2})$ . Note that  $\frac{\xi'(-t)}{-t} = \frac{\xi'(t)}{t}$ ,  $\frac{\xi'(t)}{t}|_{t=0} = \xi''(0) = 2(3 - \frac{\pi^2}{4})$  and  $\frac{\xi'(t)}{t}|_{t=\pi/2} = \frac{4}{3}$ . This completes the proof of the lemma.  $\square$

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