PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 10, October 2006, Pages 2943–2950
S 0002-9939(06)08333-X

BEURLING-NEVANLINNA INEQUALITY FOR SUBFUNCTIONS OF THE STATIONARY SCHRÖDINGER OPERATOR

ALEXANDER KHEYFITS

(Communicated by Juha M. Heinonen)

To Iossif V. Ostrovskii on the occasion of his 70th Anniversary

Abstract. The classical Beurling-Nevanlinna upper bound for subharmonic functions is extended to subsolutions of the stationary Schrödinger equation.

Let u be a subharmonic function in the disk $D_R = \{|z| < R\}$ in the complex plane. By the maximum principle, if

$$\limsup_{z\to\partial D_R} u(z) \le M, \tag{1}$$

then $u \le M$ in the whole D_R, and since a constant function $u(z) \equiv M$ is subharmonic, (1) cannot be improved. However, if some additional information is available, the latter inequality can be made more precise. Thus, improving on the preceding results of H. Milloux and E. Schmidt, A. Beurling [2] and R. Nevanlinna [10] proved that if in addition to (1)

$$\inf_{|z|=r} u(z) \le m,\ \forall r \in [0, R), \tag{2}$$

then everywhere in D_R

$$u(z) \le \delta_0(r)m + (1 - \delta_0(r))\, M, \tag{3}$$

where

$$\delta_0(r) = \frac{2}{\pi} \arcsin \frac{R - r}{R + r}.$$

Here we follow L. Hörmander [6, p. 194]; Beurling and Nevanlinna actually proved a more general result, assuming that (2) is known only on a subset of $[0, R)$. For the relevant references see W. Hayman [5, p. 289].

Our goal is to extend (3) to weak solutions of the stationary Schrödinger inequality $-L_c u \equiv \Delta u - c(z)u \ge 0$ in the disk D_R, Δ being the Laplace operator. We call these functions *subfunctions* or *$c-$subfunctions*, to be more precise. Solutions of the equation $\Delta u - c(z)u = 0$ are called *$c-$harmonic functions*. One can find the necessary properties of subfunctions in [8, 9] and the references therein.

We always assume that the potential $c(z)$ is a nonnegative function in the disk D_R and $c(z) \in L^2(D_{R-\epsilon})$ for every ϵ, $0 < \epsilon < R$. This assumption is sufficient

---

Received by the editors May 19, 2004 and, in revised form, April 26, 2005.
2000 *Mathematics Subject Classification.* Primary 31A05, 30C80, 35J10.
Key words and phrases. Beurling-Nevanlinna inequality, subharmonic functions associated with the stationary Schrödinger operator.

to derive the local properties of subfunctions. However, for our purposes we must control the behavior of the potential near the boundary. Thus, we assume that $c(z)$ has an appropriate radial majorant, that is, there exist a function $Q(r),\ 0 \le r < R$, and a number $\varepsilon > 0$ such that

$$0 \le c(z) \le Q(|z|) \in L^{2+\varepsilon}(0,R)\,,\ \forall z \in D_R.$$

The results are stated in terms of special solutions of the following ordinary differential equations:

$$f''(r) + \frac{1}{r} f'(r) - Q(r) f(r) = 0,\ 0 < r < R, \tag{4}$$

and

$$f''(r) + \frac{1}{r} f'(r) - \left(\frac{(2l+1)^2}{4r^2} + Q(r) \right) f(r) = 0,\ l = 0,1,2,\dots,\ 0 < r < R. \tag{5}$$

Under our assumptions each of these equations has a fundamental system of positive solutions, denote them $\{V_0, U_0\}$ for (4) and $\{V_{2l+1}, U_{2l+1}\}$ for (5), such that $U_0(0^+) = U_{2l+1}(0^+) = +\infty$, $U_0(R^-) = U_{2l+1}(R^-) = 0$, $V_{2l+1}(0^+) = 0$ for all $l \ge 0$ and $V_0(0^+) \ge 0$ (see, e.g., [4, Chap. 11]). The functions U_0, U_{2l+1} monotonically decay, while V_{2l+1} and V_0 monotonically increase (the latter may be constant) and are bounded when $r \to R$, that is, $V_0(R^-) < \infty$ and $V_{2l+1}(R^-) < \infty$. Now we can state our result. As always, $a^+ = \max\{a;\ 0\}$.

Theorem 1. *Let a potential* $c(z),\ z \in D_R$, *satisfy our assumptions and let a* $c-$*subfunction* u *satisfy* (1) *and* (2). *Then*

$$u(z) \le \delta(r) m + (1 - \delta(r))\, M^+, \tag{6}$$

where

$$\delta(r) = \frac{V_0(r)}{V_0(R)} - \frac{4}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l V_{2l+1}(r)}{(2l+1) V_{2l+1}(R)}.$$

The statement is precise in the sense that M^+ *in* (6) *cannot be replaced by* M.

Moreover, inequality (3) *in general fails for* $c-$*subfunctions unless* $c = 0$ *almost everywhere.*

Remark 1. The appearance of M^+ in (6) is due to the fact that for the operators under consideration the maximum principle has a restricted validity—only a positive maximum cannot be attained inside the domain.

Proof. Since $(\Delta - Q)(u - M^+) = (\Delta - c)u + (c - Q)(u - M^+) + cM^+ \ge 0$, perhaps in the sense of distributions, $u(z) - M^+$ is a nonpositive $Q-$subfunction. By the Riesz-Herglotz integral representation [7],

$$u(z) - M^+ = -\int_{D_R} G(z,\zeta)\, d\mu(\zeta) - \int_{S_R} \frac{\partial G(z,\zeta)}{\partial n(\zeta)}\, d\sigma(\zeta),$$

where $S_R = \partial D_R$, $d\mu$ and $d\sigma$ are, respectively, the Riesz measure and the boundary measure of u, G is Green's function of the operator

$$L_Q = -\Delta + Q(|z|) I$$

in D_R with the Dirichlet boundary conditions, and $\partial G/\partial n$ is its derivative with respect to the inward normal to S_R. It is known (see, for instance, [9] and the references therein) that G exists, $G(z,\zeta) > 0$ and $\partial G/\partial n \ge 0$.

We follow the argument of [6, pp. 194-196]. However, unlike the case $c = 0$, we do not have an explicit integral representation of Green's function G, so the extension is not straightforward and we cannot expect in general to derive an explicit formula for the function $\delta(r)$.

Project all associated masses of u on the positive real axis along concentric circumferences. Define a measure $d\mu^*$ on $[0, R)$ such that

$$\int \varphi(r) d\mu^*(r) = \int \varphi(|z|) d\mu(z)$$

for any function $\varphi \in C_0([0, R))$, denote $\sigma^* = \int d\sigma \geq 0$, and consider a function

$$v(z) = -\int_{[0.R)} G(z, r) d\mu^*(r) - \sigma^* \left. \frac{\partial G(z, \zeta)}{\partial n(\zeta)} \right|_{\zeta = R}. \tag{7}$$

In the proof we need some inequalities between the functions u and v. To establish them, we estimate changes of Green's function G when z moves along circumferences $|z| = r,\ 0 < r < R$. To this end, the following definition [1] is helpful.

A function $b(r, \theta)$ is said to be symmetric on an annulus

$$A(r_1, r_2) = \left\{ (r, \theta) \middle| \, r_1 < r < r_2,\ 0 \leq \theta \leq 2\pi \right\}$$

if $b(r, \theta) = b(r, 2\pi - \theta),\ \forall r,\ r_1 < r < r_2$.

Lemma 1. *If ρ is real, then $G(re^{i\theta}, \rho)$ is symmetric on D_R. Next, for $0 < \theta < \pi$,*

$$\frac{\partial}{\partial \theta} G(re^{i\theta}, \rho) \leq 0,$$

and so $\frac{\partial}{\partial \theta} G(re^{i\theta}, \rho) \geq 0$ for $\pi < \theta < 2\pi$.

Proof. The symmetry of G follows immediately, since $Q(r)$ is a radial potential. The formula

$$(\Delta - Q(r)) \left(\frac{\partial b}{\partial \theta} \right) = \frac{\partial}{\partial \theta} (\Delta b - Q(r) b)$$

is readily verified for smooth functions $b(r, \theta)$ on the semi-annulus

$$A^+(r_1, r_2) = \left\{ (r, \theta) \middle| \, 0 \leq r_1 < r < r_2,\ 0 < \theta < \pi \right\}.$$

Thus, since $G(z, \zeta)$ is $Q-$harmonic for $z \neq \zeta$, the function b defined by

$$b(r, \theta) = \frac{\partial}{\partial \theta} G((r, \theta), \rho)$$

is also $Q-$harmonic in the punctured disk $D_R \setminus \{(\rho, 0)\}$.

Moreover, $b(r, \theta) \to 0$ as $(r, \theta) \to (R, \theta)$, because $Q(r) \in L^{2+\varepsilon}(0, R)$ and all points of S_R are regular for both L_c and L_Q. Also $\lim b(r, \theta) = 0$ as $\theta \to 0$ and $\theta \to \pi,\ r \neq \rho$, by virtue of the symmetry of G. In addition, $\limsup_{z \to \rho} b(z) \leq 0$.

Indeed, the equation $-\Delta u + Q(|z|) u(z) = f(z)$ can be rewritten in a standard way as an integral equation

$$u(z) = \int_{D_R} g(z, \zeta) f(\zeta) d\zeta + \int_{D_R} g(z, \zeta) u(\zeta) d\zeta, \tag{8}$$

where g is Green's function of the Laplace operator $-\Delta$ in the disk D_R with zero boundary values. Iterating (8), we can derive Green's function G of the operator L_Q and study its properties; see [9] for details. In particular, the following representation holds true:

$$G(z,\rho)=\frac{1}{2\pi}\ln\frac{1}{|z-\rho|}+a\int_{D_R}\ln\frac{1}{|z-t|}Q(|t|)\ln\frac{1}{|t-\rho|}dt+h(z,\rho), \tag{9}$$

where a is a constant and h is a function with bounded first partial derivatives, perhaps in the sense of distributions. Now, since $Q\in L^{2+\varepsilon}$, we easily conclude that the integral in (9) also has bounded first derivatives.

Thus, $b(r,\theta)<\epsilon,\ \epsilon>0$, in some small vicinity of the point $(\rho,0)$, and also b is $Q-$harmonic outside this vicinity. By the maximum principle, $\frac{\partial G}{\partial\theta}=b(z)\le 0$ in the semi-annulus A^+. ♢

Corollary 1. *Lemma* 1 *implies immediately that*

$$G(-|z|,\rho)\le G(z,\rho)\le G(|z|,\rho). \tag{10}$$

♢

To handle the second term, $\partial G/\partial n$, in the right-hand side of (7), we note that since Green's function vanishes at regular boundary points and is symmetric, then (10) with $\rho\to R$ implies

$$P(-|z|,R)\le P(z,Re^{i\theta})\le P(|z|,R),$$

where $P(z,Re^{i\theta})=\frac{\partial}{\partial n}G(z,\zeta)|_{\zeta=Re^{i\theta}}$ is the $Q-$harmonic Poisson kernel for the disk D_R. Thus,

$$u(z)-M^+\le v(-r),\ r=|z|,$$

for $|z|<R$, where the function v is given by (7). Moreover,

$$v(r)=-\int_{[0,R)}G(r,\rho)d\mu^*(\rho)-\sigma^*P(r,R)$$
$$\le-\int_{[0,R)}G(z,\rho)d\mu^*(\rho)-\sigma^*P(z,Re^{i\theta})=u(z)-M^+,$$

hence

$$v(r)\le\inf_{|z|=r}u(z)-M^+\le m-M^+,\ 0\le r<R.$$

To find an analog of the function δ_0 in (3), we first solve the following Dirichlet problem in the slit disk $D_R^-=D_R\setminus\{0\le r<R\}$:

$$\begin{cases} L_Qw(z)=0 & \text{for } z\in D_R^-,\\ w(Re^{i\theta})=1 & \text{if } \theta\neq 0,\\ w(r,0^+)=w(r,2\pi^-)=0 & \text{if } 0<r<R.\end{cases} \tag{11}$$

Separating variables by substituting $w(r,\theta)=f(r)\Theta(\theta)$, we deduce for the angular component the equation $\Theta''(\theta)+\lambda\Theta(\theta)=0,\ 0<\theta<2\pi$, so $\Theta_\lambda(\theta)=A_\lambda\sin(\sqrt{\lambda}\theta)+B_\lambda\cos(\sqrt{\lambda}\theta)$. Due to the nonnegativity of the potential Q, the spectrum of the operator L_Q is nonnegative, thus $\lambda\ge 0$.

An equation for the radial component of w is (cf. (5))

$$f''+\frac{1}{r}f'-\left(\frac{\lambda}{r^2}+Q(r)\right)f(r)=0,\ 0<r<R. \tag{12}$$

It is known (see, e.g., [4, Chap. 11]) that (12) has a fundamental system of positive solutions, $V_\lambda(r)$ and $U_\lambda(r)$, such that $U_\lambda(0^+)=\infty$, $U_\lambda(r)$ monotonically decreases to zero as $0<r\to R$, while $V_\lambda(0^+)\ge 0$, and $V_\lambda(r)$ does not decrease as $0<r\to R$.

Moreover, if $\lambda > 0$ or $Q(r)$ is positive on a set of positive measure, then $V_\lambda(0^+) = 0$ and $V_\lambda(r)$ monotonically increases as $0 < r \to R$. Thus, the general solution of (12) is

$$f_\lambda(r) = C_\lambda V_\lambda(r) + D_\lambda U_\lambda(r).$$

In the proof we do not need the general solution of the Dirichlet problem (11), so we fix $D_\lambda = 0$ and $C_\lambda = 1$, and consider a radial solution $f_\lambda(r) = V_\lambda(r)$, leading to a solution of (11),

$$w_\lambda(r,\theta) = \left\{A_\lambda \sin(\sqrt{\lambda}\theta) + B_\lambda \cos(\sqrt{\lambda}\theta)\right\} V_\lambda(r).$$

Considering the boundary conditions at the slit, we get $B_\lambda = 0$ since $V_\lambda(r) \neq 0$ for $r > 0$, and $A_\lambda \sin(2\pi\sqrt{\lambda}) + B_\lambda \cos(2\pi\sqrt{\lambda}) = 0$, implying $\sin(2\pi\sqrt{\lambda}) = 0$. Thus, $\lambda = \lambda_k = \frac{k^2}{4},\ k = 0, 1, 2, \ldots,$

$$w_\lambda(r,\theta) = w_k(r,\theta) = A_k V_k(r) \sin\left(\frac{k}{2}\theta\right),\ k = 0, 1, 2, \ldots,$$

and $w(r,\theta) = \sum_{k=0}^{\infty} A_k V_k(r) \sin\left(\frac{k}{2}\theta\right)$. Next, the boundary condition at $r = R$ gives

$$w(R,\theta) = \sum_{k=0}^{\infty} A_k V_k(R) \sin\left(\frac{k}{2}\theta\right) = 1,\ 0 < \theta < 2\pi.$$

From here,

$$A_k = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \frac{4}{\pi k V_k(R)} & \text{if } k = 2l+1 \text{ is odd,} \end{cases}$$

and we have constructed a solution of (11),

$$w(z) = \frac{4}{\pi} \sum_{l=0}^{\infty} \frac{V_{2l+1}(r)}{(2l+1)V_{2l+1}(R)} \sin\left((l + \frac{1}{2})\theta\right),$$

which we sought.

Since constants are not $c-$harmonic functions (unless $c \equiv 0$), to get a solution of (11) with zero boundary values at S_R, we consider a radial positive $Q-$harmonic function $w_1(z) = \frac{V_0(r)}{V_0(R)},\ 0 \le r = |z| \le R$, in the slit disk D_R^-, where V_0 is a growing solution of equation (4). It is obvious that $w_1(R) = 1,\ 0 \le \theta \le 2\pi$, while $w_1(0) = V_0(0)/V_0(R) \ge 0$, and $0 < w_1(r) < 1$ for $0 < r < R$. By the maximum principle, $w(z) \le w_1(z)$ in $\overline{D_R^-}$, so that the function

$$W(z) = w(z) - w_1(z) = \frac{4}{\pi} \sum_{l=0}^{\infty} \frac{V_{2l+1}(r)}{(2l+1)V_{2l+1}(R)} \sin\left((l + \frac{1}{2})\theta\right) - \frac{V_0(r)}{V_0(R)}$$

is a negative $Q-$harmonic function in D_R^-, $W(R,\theta) = 0$ for $0 < \theta < 2\pi$, and $W(r, 0^+) = W(r, 2\pi^-) = -\frac{V_0(r)}{V_0(R)} > -1$ for $0 < r < R$.

As we have proved, the function v, defined by (7), satisfies $v(z) \le 0$ in D_R, and $v(r)/(M^+ - m) \le -1,\ 0 < r < R$. By the maximum principle,

$$\frac{v(z)}{M^+ - m} \le W(z),\ z \in \overline{D_R^-},$$

hence

$$u(z) - M^+ \leq v(-r) \leq (M^+ - m)W(-r)$$
$$= (M^+ - m)\left\{\frac{4}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l V_{2l+1}(r)}{(2l+1)V_{2l+1}(R)} - \frac{V_0(r)}{V_0(R)}\right\}$$

or

$$u(z) \leq \delta(r)m + (1-\delta(r))M^+,$$

where

$$\delta(r) = -W(r,\pi) = \frac{V_0(r)}{V_0(R)} - \frac{4}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l V_{2l+1}(r)}{(2l+1)V_{2l+1}(R)} > 0.$$

The latter series converges by the Abel test [3, p. 307], since $\left\{\frac{V_{2l+1}(r)}{V_{2l+1}(R)}\right\}$ is monotone in l for any r, $0 \leq r < R$, [4, Chap. 11, Sect. 6, Cor. 6.5].

Now consider a negative subfunction $-w_1(z)$. It satisfies (1)-(2) with $M = 0$, $m = -1$, and if it were possible to replace M^+ in (6) with M, we would have $-\frac{V_0(r)}{V_0(R)} \leq -1$, $0 \leq r < R$, contrary to the definition of V_0.

Moreover, if we could substitute a simpler weight δ_0 (see (3)) in (6) instead of the general weight δ, we would get from (6) for the same function $-w_1(z)$,

$$\frac{2}{\pi}\arcsin\frac{R-r}{R+r} \leq \frac{V_0(r)}{V_0(R)},$$

an obviously contradictory inequality as $r \to 0^-$, unless V_0 is constant, which holds only in the classical case $c = 0$. The proof of Theorem 1 is complete. ♢

Corollary 2. *Under the conditions of Theorem* 1*,*

$$\sup_{0\leq r\leq R}\inf_{|z|=r} u(z) \geq M^+(R) - \frac{M^+(R) - M(r)}{\delta(r)}$$
$$\geq M(r) - \inf_{0\leq r<R}\frac{M^+(R)-M(r)}{\delta(r)}.$$

Proof. Since $u(z) \leq \delta(r)m + (1-\delta(r))M^+$, $r = |z|$, we have $M(r) = M_u(r) \leq \delta(r)m + (1-\delta(r))M^+$. Thus

$$m \geq \frac{M(r) - (1-\delta(r))M^+(R)}{\delta(r)} = M^+(R) - \frac{M^+(R) - M(r)}{\delta(r)},$$

and the conclusion follows. ♢

Corollary 3. *Let u and v be $c-$subfunctions in D_R and $M(R) = M_u(R) < \infty$. Then*

$$v(0) \leq \sup_{|z|<R}(u(z)+v(z)) - M^+(R) + \inf_{0\leq r\leq R}\frac{M^+(R)-M(r)}{\delta(r)}.$$

Proof. Indeed,

$$v(0) \leq \sup_{|z|\leq r} v(z) \leq \sup_{|z|\leq r}(u+v) - \inf_{|z|=r} u \leq \sup_{|z|<R}(u(z)+v(z)) - \inf_{|z|=r} u(z),$$

and it suffices now to apply Corollary 2.

We consider two examples. First, let $Q(r) = qr^{-2}$, where q is a positive constant. In this case, $V_0(r) = A_0 r^{\sqrt{q}}$, $V_{2l+1}(r) = A_l r^{\sqrt{(l+1/2)^2+q}}$, and

$$\delta(r) = \left(\frac{r}{R}\right)^{\sqrt{q}} - \frac{4}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{r}{R}\right)^{\sqrt{(l+1/2)^2+q}}.$$

It should be mentioned that $\delta(R) = \delta_0(R) = 0$. However, $\delta(0) = 0 \neq \delta_0(0) = 1$, unless the potential vanishes almost everywhere on $(0, R)$; in the latter case $\delta(0) = \delta_0(0) = 1$.

As the second example, we consider a constant potential $c(z) \equiv Q(r) = q \geq 0$. Now equation (12) reduces to the Bessel differential equation, $V_0(r) = AJ_0(i\sqrt{q}r)$, $V_{2l+1}(r) = A_l J_{l+1/2}(i\sqrt{q}r)$, where J_ν is a Bessel function of the first kind, and

$$\delta(r) = \frac{J_0(i\sqrt{q}r)}{J_0(i\sqrt{q}R)} - \frac{4}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l J_{l+1/2}(i\sqrt{q}r)}{(2l+1)J_{l+1/2}(i\sqrt{q}R)}.$$

In particular, if $q = 0$, then $V_{2l+1}(r) = A_l r^{l+1/2}$,

$$w(z) = \frac{4}{\pi}\sum_{l=0}^{\infty}\frac{(r/R)^{2l+1}}{l+1/2}\sin\left((l+1/2)\theta\right) = \frac{2}{\pi}\arctan\frac{2\sqrt{r/R}\sin\frac{\theta}{2}}{1-r/R},$$

so

$$w(r,\pi) = \frac{4}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l (r/R)^{l+1/2}}{2l+1} = \frac{2}{\pi}\arctan\frac{2\sqrt{r/R}}{1-r/R}.$$

In this case we can set $w_1(z) = 1$, so

$$\delta(r) = \delta_0(r) = 1 - \frac{2}{\pi}\arctan\frac{2\sqrt{r/R}}{1-r/R} = \frac{2}{\pi}\arcsin\frac{R-r}{R+r},$$

and we arrive at the classical inequality (3) [6, p. 194]. ♢

ACKNOWLEDGEMENT

The author is grateful to Professor L. Hörmander and to a referee for useful advice. This work was partially supported by a grant from the City University of New York.

REFERENCES

[1] Baernstein II, A. and Taylor, B. A., Spherical rearrangements, subharmonic functions, and *−functions in n−space. *Duke Math. J.* **43**(1976), 245-268. MR0402083 (53:5906)

[2] Beurling, A., *Études sur un problème de majoration.* Thèse, Uppsala, 1933.

[3] Bonic, R. A., Hajian, G. V., Cranford, E., and Krantz, S., *Freshman Calculus.* D. C. Heath and Co. Lexington, MA, 1971.

[4] Hartman, P., *Ordinary Differential Equations.* John Wiley & Sons, New York - London - Sydney, 1964. MR0171038 (30:1270)

[5] Hayman, W. K., *Subharmonic Functions.* Vol. 2. Academic Press, London - San Diego - New York - Berkeley - Boston - Sydney - Tokyo - Toronto, 1989. MR1049148 (91f:31001)

[6] Hörmander, L. *Notions of Convexity.* Birkhäuser, Boston - Basel - Berlin, 1994. MR1301332 (95k:00002)

[7] Kheyfits, A., The Riesz-Herglotz formula for generalized harmonic functions and their boundary behavior. *Soviet Math. Dokl.* **44**(1992), 688-691. MR1153552

[8] Levin, B. Ya. and Kheyfits, A., Asymptotic behavior of subfunctions of the Schrödinger operator in an n−dimensional cone. *Soviet Math. Dokl.* **38**(1989), 109-112. MR0968495 (91h:35099)

[9] Levin, B. Ya. and Kheyfits, A., *Asymptotic behavior of subfunctions of the stationary Schrödinger operator.* Preprint, http://arXiv/abs/math/021132896, 2002, 96 pp.

[10] Nevanlinna, R., Über eine Minimumaufgabe in der Theorie der konformen Abbildung. *Ges. Wiss. Göttingen, Math. Phys. Kl.* **37**(1933), 103-115.

GRADUATE SCHOOL AND BRONX COMMUNITY COLLEGE OF THE CITY UNIVERSITY OF NEW YORK, BRONX, NEW YORK 10453

E-mail address: `akheyfits@gc.cuny.edu`