OPEN SUBGROUPS AND THE CENTRE PROBLEM
FOR THE FOURIER ALGEBRA

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Abstract. Let \( A(G) \) be the Fourier algebra of a locally compact group and
\( UCB(\hat{G}) \) the \( C^\ast \)-algebra of uniformly continuous linear functionals on \( A(G) \).
We study how the centre problem for the algebra \( UCB(\hat{G}) \) (resp. \( A(G)'' \))
is related to the centre problem for the algebras \( UCB(\hat{H}) \) (resp. \( A(H)'' \)) of
\( \sigma \)-compact open subgroups \( H \) of \( G \). We extend some results of Lau-Losert
on the centres of \( UCB(\hat{G}) \) and \( A(G)'' \).

1. Introduction

Let \( A \) be a Banach algebra. As is well known, there exist two Banach algebra
multiplications on the second dual \( A'' \) of \( A \) such that each of them extends the
multiplication on \( A \) (cf. Arens [1]). We will always consider the first Arens multi-
plication on \( A'' \) throughout this paper. The dual of the space \( \text{span}(A^\ast A) \) equipped
with the multiplication induced by that on \( A'' \) is also a Banach algebra. In recent
years, the topological centre problem for the algebras \( A'' \) and \( \text{span}(A^\ast A)'' \),
in particular for \( A \) being some Banach algebras associated with a locally compact
group, has attracted some attention. Let \( A \) be either the group algebra \( L^1(G) \) or
the Fourier algebra \( A(G) \) of a locally compact group \( G \). Then the corresponding
algebras \( \text{span}(A^\ast A)'' \) are \( LUC(G)'' \) and \( UCB(\hat{G})'' \), respectively, where \( LUC(G) \)
is the space of bounded left uniformly continuous functions on \( G \) and \( UCB(\hat{G}) \) is
the space of uniformly continuous linear functionals on \( A(G) \).

Let \( Z_t(A'') \) (resp. \( Z_t(\text{span}(A^\ast A)'' \)) be the topological centre of \( A'' \) (resp. \( \text{span}(A^\ast A)'' \)). In [3], Grosser-Losert showed that \( Z_t(LUC(G)'' = M(G) \) if \( G \) is
abelian, where \( M(G) \) is the measure algebra of \( G \). Lau [15] extended this result
to all locally compact groups. For the group algebra \( L^1(G) \), Isik-Pym-Ülger [12]
proved that if \( G \) is compact, then \( Z_t(L^1(G)'' = L^1(G) \). This result has also been
extended to all locally compact groups by Lau-Losert [16].

When \( G \) is abelian with dual group \( \Gamma \), \( L^1(\Gamma) \cong A(G) \), \( LUC(\Gamma) \cong UCB(\hat{G}) \)
and \( M(\Gamma) \cong B_a(G) \) (the reduced Fourier-Stieltjes algebra of \( G \)). Therefore, if \( G \)
is abelian, then \( Z(UCB(\hat{G})'' = B_a(G) \) and \( Z(A(G)'' = A(G) \). It is natural to
consider when the above equalities hold for a non-abelian locally compact group.

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Lau-Losert [17] showed that if $G$ is second countable and $[G, G]$ is not open in $G$, where $[G, G]$ denotes the commutator subgroup of $G$, then

(i) $Z_t(UCB(\hat{G}))^* = B_{\hat{G}}(G)$;
(ii) $Z_t(A(G)^{**}) = A(G)$ if $G$ is assumed to be amenable.

Clearly, (i) holds for all discrete groups (cf. [13] Proposition 4.5 and [17] Theorem 5.8]). Lau-Losert [17] Theorem 6.5(i)] proved that $Z_t(A(G)^{**}) = A(G)$ is also true when $G$ is discrete and amenable. Moreover, Lau-Losert [18] proved that (i) and (ii) hold if $G$ is a countably infinite product of second countable locally compact groups $\{G_i\}_{i=0}^{\infty}$ with $G_i$ (i $\geq$ 1) compact and non-trivial. However, a special consequence of Losert [19] Theorem 3] says that $A(G) \neq Z_t(A(G)^**) \neq A(G)^{**}$ if $G$ is a discrete group containing the free group $F_r$ on $r$ generators $(2 \leq r < \infty)$. Very recently, Losert further showed that (ii) fails for $G = SU(3)$.

In this paper, we are concerned with locally compact groups with a large compact covering number. We study how the topological centre problem (i) for $UCB(\hat{G})^*$ is related to the same problem for the algebras $UCB(\hat{H})^*$ of open subgroups $H$ of $G$. We prove that $Z_t(UCB(\hat{G}))^* = B_{\hat{G}}(G)$ if and only if $Z_t(UCB(\hat{H})^* = B_{\hat{H}}(H)$ for all $\sigma$-compact open subgroups $H$ of $G$ (Theorem 3.4). We further investigate whether the parallel result holds for $(A(G)^{**}$ (cf. Theorem 3.9). As an application, we extend some results of Lau-Losert on $Z_t(UCB(\hat{G}))$ and $Z_t(A(G)^{**})$ to metrizable locally compact groups (cf. Theorem 3.14 and Theorem 3.16).

2. Preliminaries

Let $A$ be a Banach algebra. Then $A^*$ is a Banach $A$-bimodule under the actions

$$\langle x \cdot \phi, \psi \rangle = \langle x, \phi \psi \rangle \quad \text{and} \quad \langle \phi \cdot x, \psi \rangle = \langle x, \psi \phi \rangle \quad (x \in A^* \text{ and } \phi, \psi \in A).$$

Each of these two module actions naturally induces a Banach algebra multiplication on $A^{**}$ which extends that on $A$ (cf. Arens [11]). Let $\cdot$ and $\triangle$ denote the first and the second Arens multiplications on $A^{**}$, respectively. Evidently, for any fixed $m \in A^{**}$, the maps $n \mapsto n \cdot m$ and $n \mapsto m \triangle n$ are weak*-weak $^*$ continuous on $A^{**}$.

The first and the second topological centres of $A^{**}$ are defined as follows:

$$Z^1(A^{**}) = \{m \in A^{**} : \text{the map } n \mapsto n \cdot m \text{ is } w^*-w^* \text{ continuous on } A^{**}\},$$
$$Z^2(A^{**}) = \{m \in A^{**} : \text{the map } n \mapsto n \triangle m \text{ is } w^*-w^* \text{ continuous on } A^{**}\}.$$

It is readily seen that $A \subseteq Z^1(A^{**}) \cap Z^2(A^{**})$. $A$ is said to be Arens regular if $Z^1(A^{**}) = Z^2(A^{**}) = A^{**}$.

Let $X$ be a topologically left invariant subspace of $A^*$ (i.e., $X \cdot A \subseteq X$). For $m \in X^*$ and $x \in X$, one can define $m \cdot x \in A^*$ by

$$\langle m \cdot x, \phi \rangle = \langle m, x \cdot \phi \rangle \quad (\phi \in A).$$

$X$ is called topologically left introverted if $m \cdot x \in X$ for all $m \in X^*$ and $x \in X$. It can be seen that $\text{span}(A^*A)$ is topologically left introverted in $A^*$ (cf. the proof of Lau [13] Proposition 5.2]).

Let $X$ be a topologically left introverted subspace of $A^*$. Then $X^*$ becomes a Banach algebra under the multiplication defined by $\langle m \cdot n, x \rangle = \langle m, n \cdot x \rangle \quad (m, n \in X^* \text{ and } x \in X)$. It is evident that this multiplication on $X^*$ is induced by the first Arens multiplication on $A^{**}$. That is, if $m, n \in X^*$ and $\tilde{m}, \tilde{n} \in A^{**}$ are extensions of $m, n$, respectively, then $\tilde{m} \cdot \tilde{n} \in A^{**}$ is an extension of $m \cdot n$. Obviously, for any
fixed \( m \in X^* \), the map \( n \mapsto n \cdot m \) is weak*-weak* continuous on \( X^* \). The (left) topological centre of \( X^* \) is defined as

\[
Z_l(X^*) = \{ m \in X^* : \text{ the map } n \mapsto n \cdot m \text{ is } w^*-w^* \text{ continuous on } X^* \}.
\]

If \( A \) is a commutative Banach algebra, then \( Z_1(A^{**}) = Z_2(A^{**}) \) is just the algebraic centre \( Z(A^{**}) \) of \( A^{**} \) (equipped with either of the Arens multiplications). The following lemma is clearly true.

**Lemma 2.1.** Let \( A \) be a commutative Banach algebra and let \( X \) be a topologically introverted subspace of \( A^* \). Then \( Z_l(X^*) \) is the algebraic centre \( Z(X^*) \) of \( X^* \).

For a linear subspace \( Y \) of \( A^* \) containing \( A^*A \), \( y^* \in Y^* \) and \( f \in A^* \), let \( y^* \cdot f \in A^* \) be defined by \( \langle y^* \cdot f, a \rangle = \langle y^*, f \cdot a \rangle \) \((a \in A) \). If \( X \) is a subset of \( A^* \), \( X^\perp \) will denote the annihilator of \( X \) in \( A^{**} \). Here we collect some simple facts on \( A^*A \) and \( Z(A^{**}) \), which will be used in the sequel.

**Lemma 2.2.** Let \( A \) be a commutative Banach algebra. Then

(i) \( (A^*A)^\perp = \{ m \in A^{**} : n \cdot m = 0 \text{ for all } n \in A^* \} \).

(ii) \( (A^{**}A^*)^\perp = \{ m \in A^{**} : m \cdot n = n \cdot m = 0 \text{ for all } n \in A^* \} \).

(iii) \( Z(A^{**}) \cap (A^*A)^\perp = (A^{**}A^*)^\perp \).

(iv) \( \text{span}(A^*A)^* \cdot A^* = A^{**}A^* \) and \( Z(A^{**}) \cap (A^*A)^\perp = (\text{span}(A^*A))^* \cdot A^* \).

**Proof.** (i) is obviously true, and (ii) is included in \cite{[19]} Lemma 1. By (ii), we have \( (A^{**}A^*)^\perp \subseteq Z(A^{**}) \). Therefore, (iii) follows from (i) and (ii).

For (iv), let \( y^* \in [\text{span}(A^*A)]^* \). It is easy to see that if \( n \in A^{**} \) is an extension of \( y^* \), then \( y^* \cdot f = n \cdot f \) for all \( f \in A^* \). Therefore, \( [\text{span}(A^*A)]^* \cdot A^* = A^{**}A^* \) and hence, by (iii), \( Z(A^{**}) \cap (A^*A)^\perp = (A^{**}A^*)^\perp = ([\text{span}(A^*A)]^* \cdot A^*)^\perp \). \( \Box \)

Let \( G \) be a locally compact group. The Fourier-Stieltjes algebra \( B(G) \) is the linear span of positive definite continuous functions on \( G \) and can be identified with the dual of the group \( C^* \)-algebra \( C_s^*(G) \) of \( G \). With the dual norm and the pointwise multiplication, \( B(G) \) is a commutative Banach algebra. The reduced Fourier-Stieltjes algebra \( B_p(G) \) is the closure of \( B(G) \cap C_{00}(G) \) in the \( \ast \)-topology of \( B(G) \), where \( C_{00}(G) \) is the set of continuous functions on \( G \) with compact support. \( B_p(G) \) is a closed ideal in \( B(G) \) and is precisely the dual of the reduced group \( C^* \)-algebra \( C_s^*(G) \) of \( G \). As is known, \( B_p(G) = B(G) \) if and only if \( G \) is amenable.

The Fourier algebra \( A(G) \) is the closed ideal in \( B(G) \) generated by \( B(G) \cap C_{00}(G) \). \( A(G) \) can be identified with the predual of the group von Neumann algebra \( VN(G) \) of \( G \). Naturally, \( VN(G) \) is a Banach \( B(G) \)-module under the action defined by \( \langle u \cdot T, v \rangle = \langle T, uv \rangle \) \((u \in B(G), v \in A(G) \text{ and } T \in VN(G)) \). See Eymard \cite{[2]} for more information on \( B(G), B_p(G), A(G) \), and \( VN(G) \).

The support of an operator \( T \) in \( VN(G) \) is defined by saying that \( x \in \text{ supp } T \) if and only if \( u \cdot T = 0 \) implies \( u(x) = 0 \) for all \( u \in A(G) \) (cf. Eymard \cite{[2]} and Herz \cite{[5]}). The space \( UCB(G) \) of uniformly continuous linear functionals on \( A(G) \) is the norm closure of \( A(G) \cdot VN(G) \) in \( VN(G) \). It is known that \( UCB(G) \) is a \( C^* \)-subalgebra of \( VN(G) \) and also a closed \( B(G) \)-submodule of \( VN(G) \) which coincides with the norm closure of \( \{T \in VN(G) : \text{ supp } T \text{ is compact} \} \) in \( VN(G) \) (cf. Granirer \cite{[3]}–\cite{[4]}).

We recall that \( UCB(G) \) is a topologically introverted subspace of \( VN(G) \). Thus, \( UCB(G)^* \) is a Banach algebra and \( Z_l(UCB(G)^*) \) is just the algebraic centre \( Z(UCB(G)^*) \) of \( UCB(G)^* \) (cf. Lemma 2.1).
Throughout this paper, $H$ will denote an open subgroup of $G$. Let $r_H : A(G) \to A(H)$ be the restriction map and $t_H : A(H) \to A(G)$ the trivial extension map (i.e., $(t_H u)(x) = 0$ for $x \in G - H$). The adjoint map $r_H^*$ is a $\ast$-isomorphism of $VN(H)$ onto the sub von Neumann algebra $VN_H(G)$ of $VN(G)$, where

$$VN_H(G) = \{ T \in VN(G) : \text{supp } T \subseteq H \}$$

(cf. Eymard [2, Proposition 3.21]). Also, $t_H^*$ is an algebraic homomorphism of $A(G)^\ast$ onto $A(H)^\ast$ and $t_H^\ast$ is an algebraic isomorphism of $A(H)^\ast$ into $A(G)^\ast$.

It is known that $r_H^\ast(UCB(H)) \subseteq UCB(\hat{G})$ and $t_H^\ast(UCB(\hat{G})) = UCB(H)$ (cf. Granirer [3]). We let $\Phi = r_H^\ast|_{UCB(\hat{H})} : UCB(H) \to UCB(\hat{G})$ and $\Psi = t_H^\ast|_{UCB(\hat{G})} : UCB(\hat{G}) \to UCB(\hat{H})$. Then $\Psi \circ \Phi = id$ and $\Phi$ is a $\ast$-isomorphism of $UCB(\hat{H})$ onto $UCB(\hat{G}) \cap VN_H(G)$. Furthermore, $\Phi^\ast$ is an algebraic homomorphism of $UCB(\hat{G})^\ast$ onto $UCB(\hat{H})^\ast$, and $\Psi^\ast$ is an algebraic isomorphism of $UCB(\hat{H})^\ast$ into $UCB(\hat{G})^\ast$.

A direct computation shows the following result on the images of the centres under the maps $\Phi^\ast$ and $\Psi^\ast$.

**Lemma 2.3.** Let $G$ be a locally compact group and let $H$ be an open subgroup of $G$. Then

(i) $\Psi^\ast[Z(UCB(\hat{H})^\ast)] \subseteq Z(UCB(\hat{G})^\ast)$.

(ii) $\Phi^\ast[Z(UCB(\hat{G})^\ast)] = Z(UCB(H)^\ast)$.

3. The Centres of $UCB(\hat{G})^\ast$ and $A(G)^\ast$

Let $G$ be a locally compact group. In [17], Lau-Losert defined an isometric embedding $\pi = \pi_G : B_\rho(G) \to UCB(\hat{G})^\ast$ satisfying

$$\langle \pi(\varphi), u \cdot T \rangle = \langle T, \varphi u \rangle \quad \text{for } \varphi \in B_\rho(G), u \in A(G) \text{ and } T \in VN(G),$$

i.e., $\pi$ is the natural extension of the isometric embedding of $A(G)$ into $UCB(\hat{G})^\ast$ (see Lau [14] for the amenable case). $\pi : B_\rho(G) \to UCB(\hat{G})^\ast$ is also an algebraic isomorphism, i.e., $\pi(\varphi \psi) = \pi(\varphi) \cdot \pi(\psi)$ ($\varphi, \psi \in B_\rho(G)$). Furthermore,

$$\varphi \cdot T = \pi(\varphi) \cdot T \text{ for all } \varphi \in B_\rho(G) \text{ and } T \in VN(G),$$

where $\varphi \cdot T$ is the $B_\rho(G)$-module product and $\pi(\varphi) \cdot T$ is as defined in Section 2. In the following, $B_\rho(G)$ will be identified with the closed subalgebra $\pi(B_\rho(G))$ of $UCB(\hat{G})^\ast$. Lau-Losert [17 Proposition 4.5] showed that $B_\rho(G) \subseteq Z(UCB(G)^\ast)$.

Note that $A(G) \subseteq B_\rho(G) \subseteq UCB(\hat{G})^\ast$. Applying Lemma 2.2 to $A(G)$, we obviously have the following

**Corollary 3.1.** Let $G$ be a locally compact group. Then

(i) $[VN(G)^\ast VN(G)]^\perp = [UCB(\hat{G})^\ast \cdot VN(G)]^\perp \subseteq [B_\rho(G) \cdot VN(G)]^\perp \subseteq UCB(\hat{G})^\perp$.

(ii) $[VN(G)^\ast VN(G)]^\perp = Z(A(G)^\ast) \cap [B_\rho(G) \cdot VN(G)]^\perp = Z(A(G)^\ast) \cap UCB(\hat{G})^\perp$.

(iii) $B_\rho(G) = UCB(\hat{G})^\ast$ and $VN(G)^\ast VN(G) = B_\rho(G) \cdot VN(G)$ if $G$ is discrete.

Let $\mathcal{H}_0$ be the collection of all $\sigma$-compact open subgroups of $G$.

**Lemma 3.2.** Let $G$ be a locally compact group and let $\theta$ be a function on $G$ such that $\theta|_H \in B_\rho(H)$ for all $H \in \mathcal{H}_0$. Then $\theta \in B_\rho(G)$. 

Proof. First, we observe that there exists a constant \( M > 0 \) such that \( \|\theta_H\|_{B_{\rho}(H)} \leq M \) for all \( H \in \mathcal{H}_0 \). Otherwise, for each positive integer \( n \), there exists an \( H_n \in \mathcal{H}_0 \) such that \( \|\theta_{H_n}\|_{B_{\rho}(H_n)} \geq n \). Let \( H \) be the subgroup of \( G \) generated by \( H_n \) (\( n = 1, 2, \cdots \)). Then \( H \in \mathcal{H}_0 \) and \( \|\theta_H\|_{B_{\rho}(H)} \geq \|\theta_{H_n}\|_{B_{\rho}(H_n)} \geq n \) for all \( n \), which is a contradiction.

Next, we show that \( \theta \in B_{\rho}(G) \). Note that \( G = \bigcup_{H \in \mathcal{H}_0} H \) and \( \| \cdot \|_\infty \leq \| \cdot \|_{\rho} \).

Thus, \( \theta \) is a bounded continuous function on \( G \). Let \( f \in L^1(G) \). Then there exists an \( H \in \mathcal{H}_0 \) such that \( f = 0 \) on \( G - H \). So,

\[
| \int_G f(x)\theta(x)\,dx | = | \int_H f(x)\theta_H(x)\,dx | \leq \|\theta_H\|_{B_{\rho}(H)}\|f\|_{C^*_\theta(H)} \leq M\|f\|_{C^*_\theta(G)}.
\]

Therefore, \( \sup \{ | \int_G f(x)\theta(x)\,dx | : f \in L^1(G) \text{ and } \|f\|_{C^*_\theta(G)} \leq 1 \} \leq M < \infty \).

According to [2, Proposition 2.1], \( \theta \in B_{\rho}(G) \). \( \Box \)

For any open subgroup \( H \) of \( G \), let \( \lambda_H \) be the left regular representation of \( H \) and let \( m_H \) denote \( \Phi^*(m) \) for \( m \in UCB(\hat{H})^* \), where \( \Phi : UCB(\hat{H}) \rightarrow UCB(\hat{G}) \) is the *-isomorphism as defined in Section 2. It is easy to see that \( \Phi(\lambda_H(h)) = \lambda_G(h) \) for all \( h \in H \). Also, for all \( x \in G \) and \( \varphi \in B_{\rho}(G) \), we have \( \langle \pi(\varphi), \lambda_G(x) \rangle = \varphi(x) \).

Lemma 3.3. Let \( G \) be a locally compact group and let \( m \in UCB(\hat{G})^* \) be such that \( m_H \in B_{\rho}(H) \) for all \( H \in \mathcal{H}_0 \). Then \( m \in B_{\rho}(G) \).

Proof. By the assumption, for each \( H \in \mathcal{H}_0 \), there exists a \( \theta_H \in B_{\rho}(H) \) such that \( m_H = \pi_H(\theta_H) \), where \( \pi_H : B_{\rho}(H) \hookrightarrow UCB(\hat{H})^* \) is the isometric embedding.

For \( H \in \mathcal{H}_0 \) and \( x \in H \), we have

\[
\langle m, \lambda_G(x) \rangle = \langle m, \Phi(\lambda_H(x)) \rangle = \langle m_H, \lambda_H(x) \rangle = \langle \pi_H(\theta_H), \lambda_H(x) \rangle = \theta_H(x).
\]

Thus, \( \theta_H(x) = \langle m, \lambda_G(x) \rangle \) for all \( H \in \mathcal{H}_0 \) and \( x \in H \). Let \( \theta(x) = \langle m, \lambda_G(x) \rangle \) (\( x \in G \)). Then \( \theta|_H = \theta_H \in B_{\rho}(H) \) for all \( H \in \mathcal{H}_0 \). By Lemma 3.2, \( \theta \in B_{\rho}(G) \).

To show that \( m = \pi(\theta) \), let \( T \in UCB(\hat{G}) \). Note that \( \{ R \in VN(G) : \text{supp } R \text{ is compact} \} \) is norm dense in \( UCB(\hat{G}) \). So, we assume that \( \text{supp } T \) is compact. Then \( \text{supp } T \subseteq H \) for some \( H \in \mathcal{H}_0 \), i.e., \( T \in VN_H(G) \). Since \( UCB(G) \cap VN_H(G) = \Phi[UCB(\hat{H})] \), there exists an \( S \in UCB(\hat{H}) \) such that \( T = \Phi(S) \). Thus,

\[
\langle m, T \rangle = \langle \Phi^*(m), S \rangle = \langle m_H, S \rangle = \langle \pi_H(\theta_H), S \rangle = \langle \pi_H(\theta_H), S \rangle.
\]

On the other hand, note that \( \Phi(v \cdot P) = t_H(v) \cdot r_H^*(P) \) for all \( v \in A(H) \) and \( P \in VN(H) \). Hence, we have

\[
\Phi^*(\pi(\theta)), v \cdot P = \langle \pi(\theta), t_H(v) \cdot r_H^*(P) \rangle = \langle r_H^*(P), t_H(v)\theta \rangle = \langle P, \theta|_H v \rangle = \langle \pi_H(\theta|_H), v \cdot P \rangle
\]

for all \( v \in A(H) \) and \( P \in VN(H) \), i.e., \( \pi_H(\theta|_H) = \Phi^*(\pi(\theta)) \). It follows that

\[
\langle m, T \rangle = \langle \pi_H(\theta|_H), S \rangle = \langle \Phi^*(\pi(\theta)), S \rangle = \langle \pi(\theta), \Phi(S) \rangle = \langle \pi(\theta), T \rangle.
\]

Therefore, \( m = \pi(\theta) \in B_{\rho}(G) \). \( \Box \)

Immediately, we have the following

Theorem 3.4. Let \( G \) be a locally compact group and let \( \mathcal{H}_0 \) be the collection of \( \sigma \)-compact open subgroups of \( G \). Then \( Z(UCB(\hat{G})^*) = B_{\rho}(G) \) if and only if \( Z(UCB(\hat{H})^*) = B_{\rho}(H) \) for all \( H \in \mathcal{H}_0 \).
Proof. Assume that $Z(UCB(\hat{G}^*)) = B_p(G)$. Let $H$ be any open subgroup of $G$ and let $\Phi^*: UCB(\hat{G}^*) \rightarrow UCB(\hat{H}^*)$ be the algebraic homomorphism as defined in Section 2. It is readily seen that for all $\varphi \in B_p(G)$, $\Phi^*(\varphi)|_H \in B_p(H)$.

By Lemma 2.3(ii), we have $Z(UCB(\hat{H}^*)) = \Phi^*(Z(UCB(\hat{G}^*))) = \Phi^*(B_p(G)) \subseteq B_p(H)$, i.e., $Z(UCB(\hat{H}^*)) = B_p(H)$.

Conversely, suppose $Z(UCB(\hat{H}^*)) = B_p(H)$ for all $H \in \mathcal{H}_0$. To get the non-trivial inclusion $Z(UCB(\hat{G}^*)) \subseteq B_p(G)$, let $m \in Z(UCB(\hat{G}^*))$. Then, for all $H \in \mathcal{H}_0$, by Lemma 2.3(ii), $m_H = \Phi^*(m) \in Z(UCB(\hat{H}^*)) = B_p(H)$. It follows from Lemma 3.3 that $m \in B_p(G)$. \hfill \Box

Remark 3.5. (I) Theorem 3.4 would be trivial if we had $\bigcup_{H \in \mathcal{H}_0} Z(UCB(\hat{H}^*)) = Z(UCB(\hat{G}^*))$ (cf. Lemma 2.3(i)). However, even though $\bigcup_{H \in \mathcal{H}_0} UCB(\hat{H}) = UCB(\hat{G})$, $\bigcup_{H \in \mathcal{H}_0} Z(UCB(\hat{H}^*))$ is in general a proper subspace of $Z(UCB(\hat{G}^*))$ (e.g., it is the case when $G$ is abelian but non-$\sigma$-compact).

(II) We note that Lemma 3.3 (and hence Theorem 3.4) stays true if $\mathcal{H}_0$ is replaced by the class $\mathcal{H}_c$ of compactly generated open subgroups of $G$. In fact, that $\theta$ is in $B_p(G)$ follows from the boundedness of the family $\{\|\theta_H\|_{B_p(H)}: H \in \mathcal{H}_c\}$ and the same argument as in the second paragraph of the proof of Lemma 3.2.

Now, we turn our attention to the Fourier algebra $A(G)$. We investigate whether parallel results hold for $A(G)$ and $Z(A(G)^*)$.

Note that $\|u\|_{B_p(H)} \leq \|u^o\|_{B_p(G)}$ for $u \in B_p(H)$, where $H$ is any open subgroup of $G$ and $u^o$ is the trivial extension of $u$ to $G$ (cf. [8 Lemma 3.1]). So, one can see that Lemma 3.2 remains valid if $B_p(H)$ and $B_p(G)$ are replaced by $B(H)$ and $B(G)$, respectively. The assertion also holds if $B_p(H)$ and $B_p(G)$ are replaced by $A(H)$ and $A(G)$, respectively. However, the proof is different from the proof of Lemma 3.2. For the reader’s convenience and our later use, we include the following lemma with a very short proof as suggested by the referee.

Lemma 3.6. Let $G$ be a locally compact group and let $u$ be a function on $G$ such that $u|_H \in A(H)$ for all $H \in \mathcal{H}_0$. Then $u \in A(G)$.

Proof. Let $H_0$ be a fixed $\sigma$-compact open subgroup of $G$. Note that $1_{xH_0} \in B(G)$ for all $x \in G$. So, for any $x \in G$ and $H \in \mathcal{H}_0$ with $xH_0 \subseteq H$, by the assumption, $u \cdot 1_{xH_0} \in t_H(A(H)) \subseteq A(G)$, where $t_H : A(H) \rightarrow A(G)$ is the trivial extension map.

We also note that, for any countable subset $D$ of $G$, $DH_0 \subseteq H$ for some $H \in \mathcal{H}_0$. Since $A(G) \cap C_00(G)$ is norm dense in $A(G)$ and $\|1_{xH_0}\|_{B_p(G)} = 1$, it is readily seen that $\|u \cdot 1_{xH_0}\|_{A(G)} > 0$ can hold for countably many cosets $xH_0$ only. Therefore, there exists an $H \in \mathcal{H}_0$ such that $\text{supp } u \subseteq H$ and hence $u \in t_H(A(H)) \subseteq A(G)$. \hfill \Box

Remark 3.7. Let $B_p(G) = A(G) \oplus B_p^a(G)$ be the Lebesgue decomposition (see Kaniuth-Lau-Schlichting [13 Corollary 2.5]). Then each $u \in B_p(G)$ can be written as $u = u^o + u^a$ with $u^o \in A(G)$ and $u^a \in B_p^a(G)$. It follows from Lemma 3.6 that if $u \in B_p^a(G)$ and $u \neq 0$, then there exists an $H \in \mathcal{H}_0$ such that $(u|_H)^* \neq 0$. Our original proof of Lemma 3.6 was derived from the above assertion.
Note that \( \bigcup_{H \in \mathcal{H}_0} VN_H(G) \| = \bigcup_{H \in \mathcal{H}_0} VN_H(G) \). As is shown in [11] Proposition 7.3, we can have \( \bigcup_{H \in \mathcal{H}_0} VN_H(G) \neq VN(G) \) when \( G \) is non-\( \sigma \)-compact (and even abelian). Thus, we do not have the corresponding result for \( A(G)^* \) parallel to Lemma 3.3, i.e., for an \( m \in A(G)^* \), we can have \( m \notin A(G) \) even \( r_H^*(m) \in A(H) \) for all \( H \in \mathcal{H}_0 \). The following proposition tells us that for such elements \( m \in A(G)^* \), we should take \( UCB(\hat{G})^\perp \) (the annihilator of \( UCB(\hat{G}) \) in \( VN(G)^* \)) into account. Note that \( UCB(\hat{G}) = \bigcup_{H \in \mathcal{H}_0} r_H^*(UCB(\hat{H})) \). Therefore, for any \( m \in A(G)^* \),

\[
m \in UCB(\hat{G})^\perp \text{ iff } r_H^*(m) \in UCB(\hat{H})^\perp \text{ for all } H \in \mathcal{H}_0.
\]

Clearly, \( A(G) \cap UCB(\hat{G})^\perp = \{0\} \). We show next how \( A(G) \oplus UCB(\hat{G})^\perp \) is related to the family \( \{A(H) \oplus UCB(\hat{H})^\perp : H \in \mathcal{H}_0\} \).

**Proposition 3.8.** Let \( G \) be a locally compact group and let \( m \in A(G)^* \). Then

\[
m \in A(G) \oplus UCB(\hat{G})^\perp \text{ iff } r_H^*(m) \in A(H) \oplus UCB(\hat{H})^\perp \text{ for all } H \in \mathcal{H}_0.
\]

**Proof.** Obviously, if \( m \in A(G) \oplus UCB(\hat{G})^\perp \), then \( r_H^*(m) \in A(H) \oplus UCB(\hat{H})^\perp \) for all open subgroups \( H \) of \( G \). Conversely, suppose \( r_H^*(m) \in A(H) \oplus UCB(\hat{H})^\perp \) for all \( H \in \mathcal{H}_0 \). Then, for each \( H \in \mathcal{H}_0 \), there exists \( u_H \in A(H) \) such that \( r_H^*(m) - u_H \in UCB(\hat{H})^\perp \). Let \( u(x) = \langle m, \lambda_G(x) \rangle \) \((x \in G)\). It is evident that \( u_H \in A(H) \) for all \( H \in \mathcal{H}_0 \). By Lemma 3.6, \( u \in A(G) \). Since

\[
r_H^*(m - u) = r_H^*(m) - r_H(u) = r_H^*(m) - u_H \in UCB(\hat{H})^\perp
\]

for all \( H \in \mathcal{H}_0 \), \( m - u \in UCB(\hat{G})^\perp \). So, \( m = u + (m - u) \in A(G) \oplus UCB(\hat{G})^\perp \). \( \square \)

Corollary 3.1 together with the equality \( A(G) \cap UCB(\hat{G})^\perp = \{0\} \) implies that if \( Z(A(G)^*) = A(G) \), then \( \text{span}[VN(G)^*VN(G)] \) is norm dense in \( VN(G) \).

The counterexample \( G = SU(3) \) by Losert shows that the converse of the above assertion is not true. In general, for non-\( \sigma \)-compact and non-metrizable locally compact groups (cf. Theorem 3.12), we have the following

**Theorem 3.9.** Let \( G \) be a locally compact group and let \( \mathcal{H}_0 \) be the collection of \( \sigma \)-compact open subgroups of \( G \). Then the following statements are equivalent:

(i) \( Z(A(G)^*) = A(G) \).

(ii) \( \text{span}[VN(G)^*VN(G)] = VN(G) \) and \( Z(A(H)^*) = A(H) \) for all \( H \in \mathcal{H}_0 \).

**Proof.** It is easy to see that if \( Z(A(G)^*) = A(G) \), then \( Z(A(H)^*) = A(H) \) for all open subgroups \( H \) of \( G \) (cf. the proof of [11] Proposition 8.2). So, we only need to prove that (ii) \( \Rightarrow \) (i). Assume that \( \text{span}[VN(G)^*VN(G)] \) is norm dense in \( VN(G) \) and \( Z(A(H)^*) = A(H) \) for all \( H \in \mathcal{H}_0 \). Let \( m \in Z(A(G)^*) \). Then \( r_H^*(m) \in Z(A(H)^*) = A(H) \) for all \( H \in \mathcal{H}_0 \). By Proposition 3.8, there exists \( u \in A(G) \) such that \( m - u \in UCB(\hat{G})^\perp \). Note that we also have \( m - u \in Z(A(G)^*) \). It follows from Corollary 3.1(ii) that \( m - u \in Z(A(G)^*) \cap UCB(\hat{G})^\perp = [VN(G)^*VN(G)]^\perp = \{0\} \), i.e., \( m = u \in A(G) \). \( \square \)

If \( G \) is amenable, then \( B_p(G) \cdot VN(G) = VN(G) \) and hence, by Corollary 3.1(i), \( [VN(G)^*VN(G)]^\perp \subseteq [B_p(G) \cdot VN(G)]^\perp = \{0\} \), i.e., \( \text{span}[VN(G)^*VN(G)] \) is norm dense in \( VN(G) \). Immediately, we have the following corollary.
Corollary 3.10. Let $G$ be an amenable locally compact group. Then $Z(A(G)^{**}) = A(G)$ if and only if $Z(A(H)^{**}) = A(H)$ for all $H \in \mathcal{H}_0$.

We should point out that Corollary 3.10 can also be derived from Lau-Losert [17 Lemma 6.3]. For a discrete group $G$ containing the free group $F_r$ on $r$ generators $(2 \leq r < \infty)$, more than obtaining $Z(A(G)^{**}) \neq A(G)$, Losert [19] actually showed that $span[VN(G)^*VN(G)] = span[B_\rho(G) \cdot VN(G)]$ is not norm dense in $VN(G)$ (cf. [19 Corollary 1]). We do not know if this is true for all non-amenable locally compact groups. Opposite to the case $span[VN(G)^*VN(G)] = VN(G)$, Ülger [20] Theorem 3.3] implies that if $G$ is discrete, then $span[VN(G)^*VN(G)] = UCB(\hat{G})$ if and only if $A(G)$ is Arens regular.

As is shown in [11], the assertion in Corollary 3.10 remains valid for all locally compact groups $G$ if $\mathcal{H}_0$ is replaced by the following family $\mathcal{H}$:

$\mathcal{H} = \{H : H$ is an open subgroup of $G$ and $\kappa(H) \leq \chi(G) \cdot R_0\}$,

where $\kappa(H)$ is the compact covering number of $H$ and $\chi(G)$ is the character of $G$ (i.e., the least cardinality of an open basis at the identity of $G$). In fact, we have the following analogue of Lemma 3.3 with $\mathcal{H}_0$ replaced by $\mathcal{H}$. The proof included here is different from that given in [11], where a higher level Mazur property of $A(G)$ is used.

Lemma 3.11 ([11 Proposition 8.2]). Let $G$ be a locally compact group and let $m \in A(G)^{**}$ be such that $r^*_H(m) \in A(H)$ for all $H \in \mathcal{H}$. Then $m \in A(G)$.

Proof. Following the proof of Proposition 3.8 and noting that $\mathcal{H}_0 \subseteq \mathcal{H}$, we see that there exists $u \in A(G)$ such that $r^*_H(m) = r_H(u)$ for all $H \in \mathcal{H}$. We only need to show that $\langle m, T \rangle = \langle u, T \rangle$ for all $T \in VN(G)$. Let $T \in VN(G)$. Then there exists an $H \in \mathcal{H}$ such that $supp T \subseteq H$ (cf. [9 Proposition 4.1]), i.e., $T \in VN_H(G)$. Since $VN_H(G) = r_H^*(VN(H))$, $T = r_H^*(T_1)$ for some $T_1 \in VN(G)$. It follows that $\langle m, T \rangle = \langle r^*_H(m), T_1 \rangle = \langle r_H(u), T_1 \rangle = \langle u, r_H^*(T_1) \rangle = \langle u, T \rangle$.

Therefore, $m = u \in A(G)$.

Consequently, we arrive at the following

Theorem 3.12 ([11 Theorem 8.3]). Let $G$ be a locally compact group and let $\mathcal{H}$ be the collection of open subgroups of $G$ as defined above. Then $Z(A(G)^{**}) = A(G)$ if and only if $Z(A(H)^{**}) = A(H)$ for all $H \in \mathcal{H}$.

In particular, if $G$ is a metrizable locally compact group, then $Z(A(G)^{**}) = A(G)$ if and only if $Z(A(H)^{**}) = A(H)$ for all $H \in \mathcal{H}_0$.

Remark 3.13. (I) By [9 Proposition 4.1], we have $span[VN(G)^*VN(G)] = VN(G)$ if and only if $span[VN(H)^*VN(H)] = VN(H)$ for all $H \in \mathcal{H}$. Thus, Theorem 3.12 also follows from Theorem 3.9.

(II) Note that $[B_\rho(G) \cdot VN(G)]^\perp \subseteq UCB(\hat{G})^\perp$. Comparing with Proposition 3.8, it is natural to ask whether for all $m \in A(G)^{**}$, we have

$\langle m, T \rangle = \langle r^*_H(m), T_1 \rangle$ if and only if $r^*_H(m) \in A(H) \oplus [B_\rho(H) \cdot VN(H)]^\perp$

for all $H \in \mathcal{H}_0$.

By Proposition 3.8 and Corollary 3.1(ii), it is readily seen that (1) holds for all $m \in Z(A(G)^{**})$. However, (1) does not hold in general for all $m \in A(G)^{**}$ (see
the paragraph following Remark 3.7). On the other hand, owing to [9 Proposition 4.1], we have $B_\rho(G) \cdot VN(G) = \bigcup_{H \in \mathcal{H}} B_\rho(H) \cdot VN(H)$. Therefore, (1) remains valid if $\mathcal{H}_0$ is replaced by $\mathcal{H}$, i.e., for all $m \in A(G)^*$, we have

(2) \[ m \in A(G) \oplus [B_\rho(G) \cdot VN(G)]^\perp \iff \mathcal{H}_0^*(m) \in A(H) \oplus [B_\rho(H) \cdot VN(H)]^\perp \quad \text{for all } H \in \mathcal{H}. \]

The above arguments are also valid when $B_\rho(G)$ (resp. $B_\rho(H)$) is replaced by $VN(G)^*$ (resp. $VN(H)^*$).

Lau-Losert [17, Theorem 5.8] showed that if $G$ is second countable and $[G,G]$ is not open in $G$, then $Z(UCB(\hat{G})^*) = B_\rho(G)$. They proved in the same paper that if $G$ is amenable, and $Z(UCB(\hat{G})^*) = B_\rho(G)$, then $Z(A(G)^**) = A(G)$ (cf. [17, Theorem 6.4]). Therefore, $Z(A(G)^**) = A(G)$ if $G$ is second countable and amenable, and $[G,G]$ is not open in $G$ ([17, Theorem 6.5(iii)]). Applying Theorem 3.4, we have the following minor extension of [17, Theorem 5.8 and Theorem 6.5(iii)].

**Theorem 3.14.** Let $G$ be a metrizable locally compact group such that $[G,G]$ is not open in $G$. Then

(i) $Z(UCB(\hat{G})^*) = B_\rho(G)$.

(ii) $Z(A(G)^**) = A(G)$ if $G$ is amenable.

**Proof.** (i) Let $H$ be any $\sigma$-compact open subgroup of $G$. Then $H$ is a second countable locally compact group. Note that $[H, H] \subset [G,G]$, and $[G,G]$ is not open in $G$. By [17, Theorem 5.8], $Z(UCB(\hat{H})^*) = B_\rho(H)$.

So, by Theorem 3.4, $Z(UCB(\hat{G})^*) = B_\rho(G)$.

(ii) It follows from (i) and [17, Theorem 6.4].

**Corollary 3.15.** Let $G = \prod_n G_n$ be a finite or countable product of metrizable locally compact groups such that $G_n$ is compact for all but finitely many $n$. Assume that either $[G_1, G_1]$ is not open in $G_1$ or $G_1$ is abelian and non-discrete. Then

(i) $Z(UCB(\hat{G})^*) = B_\rho(G)$.

(ii) $Z(A(G)^**) = A(G)$ if all $G_n$ are amenable.

**Proof.** Clearly, $G$ is a metrizable locally compact group. According to Theorem 3.14, we only need to show that $[G,G]$ is not open in $G$.

Let $q : G \to G_1$ be the canonical projection. Then $q([G,G]) \subset [G_1, G_1]$ and hence $q([G,G]) \subset [G_1, G_1]$ and hence $q([G,G]) \subset [G_1, G_1]$. Since $q : G \to G_1$ is an open map, $[G,G]$ is not open in $G$ if $[G_1, G_1]$ is not open in $G_1$.

Assume now $G_1$ is abelian and non-discrete. Let $G' = \prod_{n \neq 1} G_n$. Then $[G,G] \subset G'$. Since $G_1 \cong G'/G'$ is non-discrete, $G'$ is not open in $G$ and hence $[G,G]$ is not open in $G$.

A close inspection of the proof of Lemma 3.2, Lemma 3.3 and Theorem 3.4 shows that Lau-Losert [18, Theorem 4.2] is still true if the group $G_0$ is assumed to be metrizable (but may not be second countable). More precisely, for the group $G = G_0 \times \prod_{i=1}^\infty G_i$ as in [18, Theorem 4.2] with $G_0$ only metrizable, our Lemma 3.2,
Lemma 3.3 and hence Theorem 3.4 remain valid if $H_0$ is replaced by the family

$$\{H \times \prod_{i=1}^{\infty} G_i \mid H \text{ is a } \sigma\text{-compact open subgroup of } G_0\}.$$  

Therefore, slightly extending Lau-Losert [18, Theorem 4.2], we have the following

**Theorem 3.16.** Let $G = G_0 \times \prod_{i=1}^{\infty} G_i$, where each $G_i$ $(i \geq 0)$ is a metrizable locally compact group and $G_i$ is compact and non-trivial for $i \geq 1$. Then

(i) $Z(UCB(\hat{G})^*) = B_p(G)$.

(ii) $Z(A(G)^{**}) = A(G)$ if $G_0$ is amenable.

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**References**


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