PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 10, October 2006, Pages 3085–3095
S 0002-9939(06)08334-1
Article electronically published on May 5, 2006

OPEN SUBGROUPS AND THE CENTRE PROBLEM FOR THE FOURIER ALGEBRA

ZHIGUO HU

(Communicated by N. Tomczak-Jaegermann)

Abstract. Let $A(G)$ be the Fourier algebra of a locally compact group and $UCB(\hat{G})$ the C^*-algebra of uniformly continuous linear functionals on $A(G)$. We study how the centre problem for the algebra $UCB(\hat{G})^*$ (resp. $A(G)^{**}$) is related to the centre problem for the algebras $UCB(\hat{H})^*$ (resp. $A(H)^{**}$) of σ-compact open subgroups H of G. We extend some results of Lau-Losert on the centres of $UCB(\hat{G})^*$ and $A(G)^{**}$.

1. Introduction

Let A be a Banach algebra. As is well known, there exist two Banach algebra multiplications on the second dual A^{**} of A such that each of them extends the multiplication on A (cf. Arens [1]). We will always consider the first Arens multiplication on A^{**} throughout this paper. The dual of the space $\overline{\text{span}}(A^*A)$ equipped with the multiplication induced by that on A^{**} is also a Banach algebra. In recent years, the topological centre problem for the algebras A^{**} and $[\overline{\text{span}}(A^*A)]^*$, in particular for A being some Banach algebras associated with a locally compact group, has attracted some attention. Let A be either the group algebra $L^1(G)$ or the Fourier algebra $A(G)$ of a locally compact group G. Then the corresponding algebras $[\overline{\text{span}}(A^*A)]^*$ are $LUC(G)^*$ and $UCB(\hat{G})^*$, respectively, where $LUC(G)$ is the space of bounded left uniformly continuous functions on G and $UCB(\hat{G})$ is the space of uniformly continuous linear functionals on $A(G)$.

Let $Z_t(A^{**})$ (resp. $Z_t([\overline{\text{span}}(A^*A)]^*)$) be the topological centre of A^{**} (resp. $(\overline{\text{span}}(A^*A))^*$). In [5], Grosser-Losert showed that $Z_t(LUC(G)^*) = M(G)$ if G is abelian, where $M(G)$ is the measure algebra of G. Lau [15] extended this result to all locally compact groups. For the group algebra $L^1(G)$, Isik-Pym-Ülger [12] proved that if G is compact, then $Z_t(L^1(G)^{**}) = L^1(G)$. This result has also been extended to all locally compact groups by Lau-Losert [16].

When G is *abelian* with dual group Γ, $L^1(\Gamma) \cong A(G)$, $LUC(\Gamma) \cong UCB(\hat{G})$ and $M(\Gamma) \cong B_\rho(G)$ (the reduced Fourier-Stieltjes algebra of G). Therefore, if G is abelian, then $Z(UCB(\hat{G})^*) = B_\rho(G)$ and $Z(A(G)^{**}) = A(G)$. It is natural to consider when the above equalities hold for a non-abelian locally compact group.

Received by the editors January 11, 2005 and, in revised form, May 5, 2005.
2000 *Mathematics Subject Classification.* Primary 22D25, 43A30.
Key words and phrases. Fourier algebra, reduced Fourier-Stieltjes algebra.
This research was supported by an NSERC grant.

Lau-Losert [17] showed that if G is second countable and $\overline{[G,G]}$ is not open in G, where $[G,G]$ denotes the commutator subgroup of G, then

(i) $Z_t(UCB(\hat{G})^*) = B_\rho(G)$;

(ii) $Z_t(A(G)^{**}) = A(G)$ if G is assumed to be amenable.

Clearly, (i) holds for all discrete groups (cf. [14, Proposition 4.5] and [17, Theorem 5.8]). Lau-Losert [17, Theorem 6.5(i)] proved that $Z_t(A(G)^{**}) = A(G)$ is also true when G is discrete and amenable. Moreover, Lau-Losert [18] proved that (i) and (ii) hold if G is a countably infinite product of second countable locally compact groups $\{G_i\}_{i=0}^{\infty}$ with G_i $(i \geq 1)$ compact and non-trivial. However, a special consequence of Losert [19, Theorem 3] says that $A(G) \neq Z_t(A(G)^{**})$ $(\neq A(G)^{**})$ if G is a discrete group containing the free group F_r on r generators $(2 \leq r < \infty)$. Very recently, Losert further showed that (ii) fails for $G = SU(3)$.

In this paper, we are concerned with locally compact groups with a large compact covering number. We study how the topological centre problem (i) for $UCB(\hat{G})^*$ is related to the same problem for the algebras $UCB(\hat{H})^*$ of open subgroups H of G. We prove that $Z_t(UCB(\hat{G})^*) = B_\rho(G)$ if and only if $Z_t(UCB(\hat{H})^*) = B_\rho(H)$ for all σ-compact open subgroups H of G (Theorem 3.4). We further investigate whether the parallel result holds for $A(G)^{**}$ (cf. Theorem 3.9). As an application, we extend some results of Lau-Losert on $Z_t(UCB(\hat{G})^*)$ and $Z_t(A(G)^{**})$ to metrizable locally compact groups (cf. Theorem 3.14 and Theorem 3.16).

2. Preliminaries

Let A be a Banach algebra. Then A^* is a Banach A-bimodule under the actions

$$\langle x \cdot \phi, \psi \rangle = \langle x, \phi\psi \rangle \quad \text{and} \quad \langle \phi \cdot x, \psi \rangle = \langle x, \psi\phi \rangle \;\; (x \in A^* \text{ and } \phi, \psi \in A).$$

Each of these two module actions naturally induces a Banach algebra multiplication on A^{**} which extends that on A (cf. Arens [1]). Let $\cdot$ and $\triangle$ denote the first and the second Arens multiplications on A^{**}, respectively. Evidently, for any fixed $m \in A^{**}$, the maps $n \mapsto n \cdot m$ and $n \mapsto m \triangle n$ are weak*-weak* continuous on A^{**}. The first and the second topological centres of A^{**} are defined as follows:

$$Z_t^1(A^{**}) = \{m \in A^{**} : \text{ the map } n \mapsto m \cdot n \text{ is } w^*\text{-}w^* \text{ continuous on } A^{**}\},$$
$$Z_t^2(A^{**}) = \{m \in A^{**} : \text{ the map } n \mapsto n \triangle m \text{ is } w^*\text{-}w^* \text{ continuous on } A^{**}\}.$$

It is readily seen that $A \subseteq Z_t^1(A^{**}) \cap Z_t^2(A^{**})$. A is said to be *Arens regular* if $Z_t^1(A^{**}) = Z_t^2(A^{**}) = A^{**}$.

Let X be a *topologically left invariant* subspace of A^* (i.e., $X \cdot A \subseteq X$). For $m \in X^*$ and $x \in X$, one can define $m \cdot x \in A^*$ by

$$\langle m \cdot x, \phi \rangle = \langle m, x \cdot \phi \rangle \;\; (\phi \in A).$$

X is called *topologically left introverted* if $m \cdot x \in X$ for all $m \in X^*$ and $x \in X$. It can be seen that $\overline{\text{span}}(A^*A)$ is topologically left introverted in A^* (cf. the proof of Lau [14, Proposition 5.2]).

Let X be a topologically left introverted subspace of A^*. Then X^* becomes a Banach algebra under the multiplication defined by $\langle m \cdot n, x \rangle = \langle m, n \cdot x \rangle$ $(m, n \in X^*$ and $x \in X)$. It is evident that this multiplication on X^* is induced by the first Arens multiplication on A^{**}. That is, if m, $n \in X^*$ and $\tilde{m}$, $\tilde{n} \in A^{**}$ are extensions of m, n, respectively, then $\tilde{m} \cdot \tilde{n} \in A^{**}$ is an extension of $m \cdot n$. Obviously, for any fixed $m \in X^*$, the map $n \mapsto n \cdot m$ is weak*-weak* continuous on X^*. The (left) topological centre of X^* is defined as

$$Z_t(X^*) = \{m \in X^* : \text{ the map } n \mapsto m \cdot n \text{ is } w^*\text{-}w^* \text{ continuous on } X^*\}.$$

If A is a *commutative* Banach algebra, then $Z_t^1(A^{**}) = Z_t^2(A^{**})$ is just the algebraic centre $Z(A^{**})$ of A^{**} (equipped with either of the Arens multiplications). The following lemma is clearly true.

Lemma 2.1. *Let A be a commutative Banach algebra and let X be a topologically introverted subspace of A^*. Then $Z_t(X^*)$ is the algebraic centre $Z(X^*)$ of X^*.*

For a linear subspace Y of A^* containing A^*A, $y^* \in Y^*$ and $f \in A^*$, let $y^* \cdot f \in A^*$ be defined by $\langle y^* \cdot f, a\rangle = \langle y^*, f \cdot a\rangle$ $(a \in A)$. If X is a subset of A^*, $X^\perp$ will denote the annihilator of X in A^{**}. Here we collect some simple facts on A^*A and $Z(A^{**})$, which will be used in the sequel.

Lemma 2.2. *Let A be a commutative Banach algebra. Then*

(i) $(A^*A)^\perp = \{m \in A^{**} : n \cdot m = 0 \textit{ for all } n \in A^{**}\}$.
(ii) $(A^{**}A^*)^\perp = \{m \in A^{**} : m \cdot n = n \cdot m = 0 \textit{ for all } n \in A^{**}\}$.
(iii) $Z(A^{**}) \cap (A^*A)^\perp = (A^{**}A^*)^\perp$.
(iv) $[\overline{\text{span}}(A^*A)]^* \cdot A^* = A^{**}A^*$ *and* $Z(A^{**}) \cap (A^*A)^\perp = ([\overline{\text{span}}(A^*A)]^* \cdot A^*)^\perp$.

Proof. (i) is obviously true, and (ii) is included in [19, Lemma 1]. By (ii), we have $(A^{**}A^*)^\perp \subseteq Z(A^{**})$. Therefore, (iii) follows from (i) and (ii).

For (iv), let $y^* \in [\overline{\text{span}}(A^*A)]^*$. It is easy to see that if $n \in A^{**}$ is an extension of y^*, then $y^* \cdot f = n \cdot f$ for all $f \in A^*$. Therefore, $[\overline{\text{span}}(A^*A)]^* \cdot A^* = A^{**}A^*$ and hence, by (iii), $Z(A^{**}) \cap (A^*A)^\perp = (A^{**}A^*)^\perp = ([\overline{\text{span}}(A^*A)]^* \cdot A^*)^\perp$. □

Let G be a locally compact group. The Fourier-Stieltjes algebra $B(G)$ is the linear span of positive definite continuous functions on G and can be identified with the dual of the group C^*-algebra $C^*(G)$ of G. With the dual norm and the pointwise multiplication, $B(G)$ is a commutative Banach algebra. The reduced Fourier-Stieltjes algebra $B_\rho(G)$ is the closure of $B(G) \cap C_{00}(G)$ in the w^*-topology of $B(G)$, where $C_{00}(G)$ is the set of continuous functions on G with compact support. $B_\rho(G)$ is a closed ideal in $B(G)$ and is precisely the dual of the reduced group C^*-algebra $C^*_\rho(G)$ of G. As is known, $B_\rho(G) = B(G)$ if and only if G is amenable.

The Fourier algebra $A(G)$ is the closed ideal in $B(G)$ generated by $B(G) \cap C_{00}(G)$. $A(G)$ can be identified with the predual of the group von Neumann algebra $VN(G)$ of G. Naturally, $VN(G)$ is a Banach $B(G)$-module under the action defined by $\langle u \cdot T, v\rangle = \langle T, uv\rangle$ $(u \in B(G)$, $v \in A(G)$ and $T \in VN(G))$. See Eymard [2] for more information on $B(G)$, $B_\rho(G)$, $A(G)$, and $VN(G)$.

The support of an operator T in $VN(G)$ is defined by saying that $x \in \operatorname{supp} T$ if and only if $u \cdot T = 0$ implies $u(x) = 0$ for all $u \in A(G)$ (cf. Eymard [2] and Herz [6]). The space $UCB(\hat{G})$ of uniformly continuous linear functionals on $A(G)$ is the norm closure of $A(G) \cdot VN(G)$ in $VN(G)$. It is known that $UCB(\hat{G})$ is a C^*-subalgebra of $VN(G)$ and also a closed $B(G)$-submodule of $VN(G)$ which coincides with the norm closure of $\{T \in VN(G) : \operatorname{supp} T \text{ is compact}\}$ in $VN(G)$ (cf. Granirer [3]–[4]).

We recall that $UCB(\hat{G})$ is a topologically introverted subspace of $VN(G)$. Thus, $UCB(\hat{G})^*$ is a Banach algebra and $Z_t(UCB(\hat{G})^*)$ is just the algebraic centre $Z(UCB(\hat{G})^*)$ of $UCB(\hat{G})^*$ (cf. Lemma 2.1).

Throughout this paper, H will denote an open subgroup of G. Let $r_H : A(G) \to A(H)$ be the restriction map and $t_H : A(H) \to A(G)$ the trivial extension map (i.e., $(t_H u)(x) = 0$ for $x \in G - H$). The adjoint map r_H^* is a $*$-isomorphism of $VN(H)$ onto the sub von Neumann algebra $VN_H(G)$ of $VN(G)$, where

$$VN_H(G) = \{T \in VN(G) : \ \operatorname{supp} T \subseteq H\}$$

(cf. Eymard [2, Proposition 3.21]). Also, r_H^{**} is an algebraic homomorphism of $A(G)^{**}$ onto $A(H)^{**}$ and t_H^{**} is an algebraic isomorphism of $A(H)^{**}$ into $A(G)^{**}$.

It is known that $r_H^*(UCB(\hat{H})) \subseteq UCB(\hat{G})$ and $t_H^*(UCB(\hat{G})) = UCB(\hat{H})$ (cf. Granirer [3]). We let $\Phi = r_H^*|_{UCB(\hat{H})} : UCB(\hat{H}) \to UCB(\hat{G})$ and $\Psi = t_H^*|_{UCB(\hat{G})} : UCB(\hat{G}) \to UCB(\hat{H})$. Then $\Psi \circ \Phi = id$ and Φ is a $*$-isomorphism of $UCB(\hat{H})$ onto $UCB(\hat{G}) \cap VN_H(G)$. Furthermore, Φ^* is an algebraic homomorphism of $UCB(\hat{G})^*$ onto $UCB(\hat{H})^*$, and Ψ^* is an algebraic isomorphism of $UCB(\hat{H})^*$ into $UCB(\hat{G})^*$.

A direct computation shows the following result on the images of the centres under the maps Φ^* and Ψ^*.

Lemma 2.3. *Let G be a locally compact group and let H be an open subgroup of G. Then*

(i) $\Psi^*[Z(UCB(\hat{H})^*)] \subseteq Z(UCB(\hat{G})^*)$.
(ii) $\Phi^*[Z(UCB(\hat{G})^*)] = Z(UCB(\hat{H})^*)$.

3. The centres of $UCB(\hat{G})^*$ and $A(G)^{**}$

Let G be a locally compact group. In [17], Lau-Losert defined an isometric embedding $\pi = \pi_G : B_\rho(G) \hookrightarrow UCB(\hat{G})^*$ satisfying

$$\langle \pi(\varphi), u \cdot T \rangle = \langle T, \varphi u \rangle \quad \text{for } \varphi \in B_\rho(G), u \in A(G) \text{ and } T \in VN(G),$$

i.e., π is the natural extension of the isometric embedding of $A(G)$ into $UCB(\hat{G})^*$ (see Lau [14] for the amenable case). $\pi : B_\rho(G) \to UCB(\hat{G})^*$ is also an algebraic isomorphism, i.e., $\pi(\varphi\psi) = \pi(\varphi) \cdot \pi(\psi)$ $(\varphi, \psi \in B_\rho(G))$. Furthermore,

$$\varphi \cdot T = \pi(\varphi) \cdot T \text{ for all } \varphi \in B_\rho(G) \text{ and } T \in VN(G),$$

where $\varphi \cdot T$ is the $B_\rho(G)$-module product and $\pi(\varphi) \cdot T$ is as defined in Section 2. In the following, $B_\rho(G)$ will be identified with the closed subalgebra $\pi(B_\rho(G))$ of $UCB(\hat{G})^*$. Lau-Losert [17, Proposition 4.5] showed that $B_\rho(G) \subseteq Z(UCB(\hat{G})^*)$.

Note that $A(G) \subseteq B_\rho(G) \subseteq UCB(\hat{G})^*$. Applying Lemma 2.2 to $A(G)$, we obviously have the following

Corollary 3.1. *Let G be a locally compact group. Then*

(i) $[VN(G)^*VN(G)]^\perp = [UCB(\hat{G})^* \cdot VN(G)]^\perp \subseteq [B_\rho(G) \cdot VN(G)]^\perp \subseteq UCB(\hat{G})^\perp$.
(ii) $[VN(G)^*VN(G)]^\perp = Z(A(G)^{**}) \cap [B_\rho(G) \cdot VN(G)]^\perp = Z(A(G)^{**}) \cap UCB(\hat{G})^\perp$.
(iii) $B_\rho(G) = UCB(\hat{G})^*$ *and* $VN(G)^*VN(G) = B_\rho(G) \cdot VN(G)$ *if G is discrete.*

Let $\mathcal{H}_0$ be the collection of all σ-compact open subgroups of G.

Lemma 3.2. *Let G be a locally compact group and let θ be a function on G such that $\theta|_H \in B_\rho(H)$ for all $H \in \mathcal{H}_0$. Then $\theta \in B_\rho(G)$.*

Proof. First, we observe that there exists a constant $M > 0$ such that $\|\theta|_H\|_{B_\rho(H)} \leq M$ for all $H \in \mathcal{H}_0$. Otherwise, for each positive integer n, there exists an $H_n \in \mathcal{H}_0$ such that $\|\theta|_{H_n}\|_{B_\rho(H_n)} \geq n$. Let H be the subgroup of G generated by H_n ($n = 1, 2, \cdots$). Then $H \in \mathcal{H}_0$ and $\|\theta|_H\|_{B_\rho(H)} \geq \|\theta|_H \cdot 1_{H_n}\|_{B_\rho(H)} = \|\theta|_{H_n}\|_{B_\rho(H_n)} \geq n$ for all n, which is a contradiction.

Next, we show that $\theta \in B_\rho(G)$. Note that $G = \bigcup_{H \in \mathcal{H}_0} H$ and $\|\cdot\|_\infty \leq \|\cdot\|_{B_\rho}$. Thus, θ is a bounded continuous function on G. Let $f \in L^1(G)$. Then there exists an $H \in \mathcal{H}_0$ such that $f = 0$ on $G - H$. So,

$$|\int_G f(x)\theta(x)\,dx| = |\int_H f(x)\theta|_H(x)\,dx| \leq \|\theta|_H\|_{B_\rho(H)}\|f|_H\|_{C^*_\rho(H)} \leq M\|f\|_{C^*_\rho(G)}.$$

Therefore, $\sup\{\,|\int_G f(x)\theta(x)\,dx| \,:\, f \in L^1(G) \text{ and } \|f\|_{C^*_\rho(G)} \leq 1\} \leq M < \infty$. According to [2, Proposition 2.1], $\theta \in B_\rho(G)$. □

For any open subgroup H of G, let λ_H be the left regular representation of H and let m_H denote $\Phi^*(m)$ for $m \in UCB(\hat{G})^*$, where $\Phi : UCB(\hat{H}) \to UCB(\hat{G})$ is the $*$-isomorphism as defined in Section 2. It is easy to see that $\Phi(\lambda_H(h)) = \lambda_G(h)$ for all $h \in H$. Also, for all $x \in G$ and $\varphi \in B_\rho(G)$, we have $\langle \pi(\varphi), \lambda_G(x)\rangle = \varphi(x)$.

Lemma 3.3. *Let G be a locally compact group and let $m \in UCB(\hat{G})^*$ be such that $m_H \in B_\rho(H)$ for all $H \in \mathcal{H}_0$. Then $m \in B_\rho(G)$.*

Proof. By the assumption, for each $H \in \mathcal{H}_0$, there exists a $\theta_H \in B_\rho(H)$ such that $m_H = \pi_H(\theta_H)$, where $\pi_H : B_\rho(H) \hookrightarrow UCB(\hat{H})^*$ is the isometric embedding.

For $H \in \mathcal{H}_0$ and $x \in H$, we have

$$\langle m, \lambda_G(x)\rangle = \langle m, \Phi(\lambda_H(x))\rangle = \langle m_H, \lambda_H(x)\rangle = \langle \pi_H(\theta_H), \lambda_H(x)\rangle = \theta_H(x)\,.$$

Thus, $\theta_H(x) = \langle m, \lambda_G(x)\rangle$ for all $H \in \mathcal{H}_0$ and $x \in H$. Let $\theta(x) = \langle m, \lambda_G(x)\rangle$ ($x \in G$). Then $\theta|_H = \theta_H \in B_\rho(H)$ for all $H \in \mathcal{H}_0$. By Lemma 3.2, $\theta \in B_\rho(G)$.

To show that $m = \pi(\theta)$, let $T \in UCB(\hat{G})$. Note that $\{R \in VN(G) : \operatorname{supp} R$ is compact$\}$ is norm dense in $UCB(\hat{G})$. So, we assume that $\operatorname{supp} T$ is compact. Then $\operatorname{supp} T \subseteq H$ for some $H \in \mathcal{H}_0$, i.e., $T \in VN_H(G)$. Since $UCB(\hat{G}) \cap VN_H(G) = \Phi[UCB(\hat{H})]$, there exists an $S \in UCB(\hat{H})$ such that $T = \Phi(S)$. Thus,

$$\langle m, T\rangle = \langle \Phi^*(m), S\rangle = \langle m_H, S\rangle = \langle \pi_H(\theta_H), S\rangle = \langle \pi_H(\theta|_H), S\rangle.$$

On the other hand, note that $\Phi(v \cdot P) = t_H(v) \cdot r^*_H(P)$ for all $v \in A(H)$ and $P \in VN(H)$. Hence, we have

$$\begin{aligned}\langle \Phi^*(\pi(\theta)), v \cdot P\rangle &= \langle \pi(\theta), t_H(v) \cdot r^*_H(P)\rangle = \langle r^*_H(P), t_H(v)\theta\rangle \\ &= \langle P, \theta|_H\, v\rangle = \langle \pi_H(\theta|_H), v \cdot P\rangle\end{aligned}$$

for all $v \in A(H)$ and $P \in VN(H)$, i.e., $\pi_H(\theta|_H) = \Phi^*(\pi(\theta))$. It follows that

$$\langle m, T\rangle = \langle \pi_H(\theta|_H), S\rangle = \langle \Phi^*(\pi(\theta)), S\rangle = \langle \pi(\theta), \Phi(S)\rangle = \langle \pi(\theta), T\rangle.$$

Therefore, $m = \pi(\theta) \in B_\rho(G)$. □

Immediately, we have the following

Theorem 3.4. *Let G be a locally compact group and let $\mathcal{H}_0$ be the collection of σ-compact open subgroups of G. Then $Z(UCB(\hat{G})^*) = B_\rho(G)$ if and only if $Z(UCB(\hat{H})^*) = B_\rho(H)$ for all $H \in \mathcal{H}_0$.*

Proof. Assume that $Z(UCB(\hat{G})^*) = B_\rho(G)$. Let H be any open subgroup of G and let $\Phi^* : UCB(\hat{G})^* \longrightarrow UCB(\hat{H})^*$ be the algebraic homomorphism as defined in Section 2. It is readily seen that for all $\varphi \in B_\rho(G)$, $\Phi^*(\varphi) = \varphi|_H \in B_\rho(H)$. By Lemma 2.3(ii), we have $Z(UCB(\hat{H})^*) = \Phi^*(Z(UCB(\hat{G})^*)) = \Phi^*(B_\rho(G)) \subseteq B_\rho(H)$, i.e., $Z(UCB(\hat{H})^*) = B_\rho(H)$.

Conversely, suppose $Z(UCB(\hat{H})^*) = B_\rho(H)$ for all $H \in \mathcal{H}_0$. To get the non-trivial inclusion $Z(UCB(\hat{G})^*) \subseteq B_\rho(G)$, let $m \in Z(UCB(\hat{G})^*)$. Then, for all $H \in \mathcal{H}_0$, by Lemma 2.3(ii), $m_H = \Phi^*(m) \in Z(UCB(\hat{H})^*) = B_\rho(H)$. It follows from Lemma 3.3 that $m \in B_\rho(G)$. □

Remark 3.5. (I) Theorem 3.4 would be trivial if we had $\overline{\bigcup_{H\in\mathcal{H}_0} Z(UCB(\hat{H})^*)}^{\|\cdot\|} = Z(UCB(\hat{G})^*)$ (cf. Lemma 2.3(i)). However, even though $\bigcup_{H\in\mathcal{H}_0} UCB(\hat{H}) = UCB(\hat{G})$, $\overline{\bigcup_{H\in\mathcal{H}_0} Z(UCB(\hat{H})^*)}^{\|\cdot\|}$ is in general a proper subspace of $Z(UCB(\hat{G})^*)$ (e.g., it is the case when G is abelian but non-σ-compact).

(II) We note that Lemma 3.3 (and hence Theorem 3.4) stays true if $\mathcal{H}_0$ is replaced by the class $\mathcal{H}_c$ of compactly generated open subgroups of G. In fact, that θ is in $B_\rho(G)$ follows from the boundedness of the family $\{\|\theta_H\|_{B_\rho(H)} : H \in \mathcal{H}_c\}$ and the same argument as in the second paragraph of the proof of Lemma 3.2.

Now, we turn our attention to the Fourier algebra $A(G)$. We investigate whether parallel results hold for $A(G)$ and $Z(A(G)^{**})$.

Note that $\|u\|_{B(H)} \leq \|u^\circ\|_{B(G)}$ for $u \in B(H)$, where H is any open subgroup of G and u° is the trivial extension of u to G (cf. [8, Lemma 3.1]). So, one can see that Lemma 3.2 remains valid if $B_\rho(H)$ and $B_\rho(G)$ are replaced by $B(H)$ and $B(G)$, respectively. The assertion also holds if $B_\rho(H)$ and $B_\rho(G)$ are replaced by $A(H)$ and $A(G)$, respectively. However, the proof is different from the proof of Lemma 3.2. For the reader's convenience and our later use, we include the following lemma with a very short proof as suggested by the referee.

Lemma 3.6. *Let G be a locally compact group and let u be a function on G such that $u|_H \in A(H)$ for all $H \in \mathcal{H}_0$. Then $u \in A(G)$.*

Proof. Let H_0 be a fixed σ-compact open subgroup of G. Note that $1_{xH_0} \in B(G)$ for all $x \in G$. So, for any $x \in G$ and $H \in \mathcal{H}_0$ with $xH_0 \subseteq H$, by the assumption, $u \cdot 1_{xH_0} \in t_H(A(H)) \subseteq A(G)$, where $t_H : A(H) \to A(G)$ is the trivial extension map. We also note that, for any countable subset D of G, $DH_0 \subseteq H$ for some $H \in \mathcal{H}_0$. Since $A(G) \cap C_{00}(G)$ is norm dense in $A(G)$ and $\|1_{xH_0}\|_{B(G)} = 1$, it is readily seen that $\|u \cdot 1_{xH_0}\|_{A(G)} > 0$ can hold for countably many cosets xH_0 only. Therefore, there exists an $H \in \mathcal{H}_0$ such that $\operatorname{supp} u \subseteq H$ and hence $u \in t_H(A(H)) \subseteq A(G)$. □

Remark 3.7. Let $B_\rho(G) = A(G) \oplus B_\rho^s(G)$ be the Lebesgue decomposition (see Kaniuth-Lau-Schlichting [13, Corollary 2.5]). Then each $u \in B_\rho(G)$ can be written as $u = u^a + u^s$ with $u^a \in A(G)$ and $u^s \in B_\rho^s(G)$. It follows from Lemma 3.6 that

$$\text{if } u \in B_\rho^s(G) \text{ and } u \neq 0, \text{ then there exists an } H \in \mathcal{H}_0 \text{ such that } (u|_H)^s \neq 0.$$

Our original proof of Lemma 3.6 was derived from the above assertion.

Note that $\overline{\bigcup_{H\in\mathcal{H}_0} VN_H(G)}^{\|\cdot\|} = \bigcup_{H\in\mathcal{H}_0} VN_H(G)$. As is shown in [10, Proposition 7.3], we can have $\bigcup_{H\in\mathcal{H}_0} VN_H(G) \neq VN(G)$ when G is non-σ-compact (and even abelian). Thus, we do *not* have the corresponding result for $A(G)^{**}$ parallel to Lemma 3.3, i.e., for an $m \in A(G)^{**}$, we can have $m \notin A(G)$ even $r_H^{**}(m) \in A(H)$ for all $H \in \mathcal{H}_0$. The following proposition tells us that for such elements $m \in A(G)^{**}$, we should take $UCB(\hat{G})^\perp$ (the annihilator of $UCB(\hat{G})$ in $VN(G)^*$) into account. Note that $UCB(\hat{G}) = \bigcup_{H\in\mathcal{H}_0} r_H^*(UCB(\hat{H}))$. Therefore, for any $m \in A(G)^{**}$,

$$m \in UCB(\hat{G})^\perp \text{ iff } r_H^{**}(m) \in UCB(\hat{H})^\perp \text{ for all } H \in \mathcal{H}_0.$$

Clearly, $A(G) \cap UCB(\hat{G})^\perp = \{0\}$. We show next how $A(G) \oplus UCB(\hat{G})^\perp$ is related to the family $\{A(H) \oplus UCB(\hat{H})^\perp : H \in \mathcal{H}_0\}$.

Proposition 3.8. *Let G be a locally compact group and let $m \in A(G)^{**}$. Then*

$$m \in A(G) \oplus UCB(\hat{G})^\perp \textit{ iff } r_H^{**}(m) \in A(H) \oplus UCB(\hat{H})^\perp \textit{ for all } H \in \mathcal{H}_0.$$

Proof. Obviously, if $m \in A(G) \oplus UCB(\hat{G})^\perp$, then $r_H^{**}(m) \in A(H) \oplus UCB(\hat{H})^\perp$ for all open subgroups H of G. Conversely, suppose $r_H^{**}(m) \in A(H) \oplus UCB(\hat{H})^\perp$ for all $H \in \mathcal{H}_0$. Then, for each $H \in \mathcal{H}_0$, there exists $u_H \in A(H)$ such that $r_H^{**}(m) - u_H \in UCB(\hat{H})^\perp$. Let $u(x) = \langle m, \lambda_G(x)\rangle$ $(x \in G)$. It is evident that $u|_H = u_H \in A(H)$ for all $H \in \mathcal{H}_0$. By Lemma 3.6, $u \in A(G)$. Since

$$r_H^{**}(m-u) = r_H^{**}(m) - r_H(u) = r_H^{**}(m) - u_H \in UCB(\hat{H})^\perp$$

for all $H \in \mathcal{H}_0$, $m-u \in UCB(\hat{G})^\perp$. So, $m = u + (m-u) \in A(G) \oplus UCB(\hat{G})^\perp$. □

Corollary 3.1 together with the equality $A(G) \cap UCB(\hat{G})^\perp = \{0\}$ implies that

$$\text{if } Z(A(G)^{**}) = A(G), \text{ then span}[VN(G)^*VN(G)] \text{ is norm dense in } VN(G).$$

The counterexample $G = SU(3)$ by Losert shows that the converse of the above assertion is not true. In general, for non-σ-compact and non-metrizable locally compact groups (cf. Theorem 3.12), we have the following

Theorem 3.9. *Let G be a locally compact group and let $\mathcal{H}_0$ be the collection of σ-compact open subgroups of G. Then the following statements are equivalent:*

(i) $Z(A(G)^{**}) = A(G)$.

(ii) $\overline{\text{span}[VN(G)^*VN(G)]}^{\|\cdot\|} = VN(G)$ *and* $Z(A(H)^{**}) = A(H)$ *for all* $H \in \mathcal{H}_0$.

Proof. It is easy to see that if $Z(A(G)^{**}) = A(G)$, then $Z(A(H)^{**}) = A(H)$ for all open subgroups H of G (cf. the proof of [11, Proposition 8.2]). So, we only need to prove that (ii) $\Longrightarrow$ (i). Assume that span$[VN(G)^*VN(G)]$ is norm dense in $VN(G)$ and $Z(A(H)^{**}) = A(H)$ for all $H \in \mathcal{H}_0$. Let $m \in Z(A(G)^{**})$. Then $r_H^{**}(m) \in Z(A(H)^{**}) = A(H)$ for all $H \in \mathcal{H}_0$. By Proposition 3.8, there exists $u \in A(G)$ such that $m-u \in UCB(\hat{G})^\perp$. Note that we also have $m-u \in Z(A(G)^{**})$. It follows from Corollary 3.1(ii) that $m-u \in Z(A(G)^{**}) \cap UCB(\hat{G})^\perp = [VN(G)^*VN(G)]^\perp = \{0\}$, i.e., $m = u \in A(G)$. □

If G is amenable, then $B_\rho(G) \cdot VN(G) = VN(G)$ and hence, by Corollary 3.1(i), $[VN(G)^*VN(G)]^\perp \subseteq [B_\rho(G) \cdot VN(G)]^\perp = \{0\}$, i.e., span$[VN(G)^*VN(G)]$ is norm dense in $VN(G)$. Immediately, we have the following corollary.

Corollary 3.10. *Let G be an amenable locally compact group. Then $Z(A(G)^{**}) = A(G)$ if and only if $Z(A(H)^{**}) = A(H)$ for all $H \in \mathcal{H}_0$.*

We should point out that Corollary 3.10 can also be derived from Lau-Losert [17, Lemma 6.3]. For a discrete group G containing the free group F_r on r generators ($2 \leq r < \infty$), more than obtaining $Z(A(G)^{**}) \neq A(G)$, Losert [19] actually showed that $\mathrm{span}[VN(G)^*VN(G)]$ ($= \mathrm{span}[B_\rho(G) \cdot VN(G)]$) is *not* norm dense in $VN(G)$ (cf. [19, Corollary 1]). We do not know if this is true for *all* non-amenable locally compact groups. Opposite to the case $\overline{\mathrm{span}[VN(G)^*VN(G)]}^{\|\cdot\|} = VN(G)$, Ülger [20, Theorem 3.3] implies that if G is discrete, then $\overline{\mathrm{span}[VN(G)^*VN(G)]}^{\|\cdot\|} = UCB(\hat{G})$ if and only if $A(G)$ is Arens regular.

As is shown in [11], the assertion in Corollary 3.10 remains valid for *all* locally compact groups G if $\mathcal{H}_0$ is replaced by the following family $\mathcal{H}$:

$$\mathcal{H} = \{H : H \text{ is an open subgroup of } G \text{ and } \kappa(H) \leq \chi(G) \cdot \aleph_0\},$$

where $\kappa(H)$ is the compact covering number of H and $\chi(G)$ is the character of G (i.e., the least cardinality of an open basis at the identity of G). In fact, we have the following analogue of Lemma 3.3 with $\mathcal{H}_0$ replaced by $\mathcal{H}$. The proof included here is different from that given in [11], where a higher level Mazur property of $A(G)$ is used.

Lemma 3.11 ([11, Proposition 8.2])**.** *Let G be a locally compact group and let $m \in A(G)^{**}$ be such that $r_H^{**}(m) \in A(H)$ for all $H \in \mathcal{H}$. Then $m \in A(G)$.*

Proof. Following the proof of Proposition 3.8 and noting that $\mathcal{H}_0 \subseteq \mathcal{H}$, we see that there exists $u \in A(G)$ such that $r_H^{**}(m) = r_H(u)$ for all $H \in \mathcal{H}$. We only need to show that $\langle m, T\rangle = \langle u, T\rangle$ for all $T \in VN(G)$. Let $T \in VN(G)$. Then there exists an $H \in \mathcal{H}$ such that $\mathrm{supp}\, T \subseteq H$ (cf. [9, Proposition 4.1]), i.e., $T \in VN_H(G)$. Since $VN_H(G) = r_H^*(VN(H))$, $T = r_H^*(T_1)$ for some $T_1 \in VN(G)$. It follows that

$$\langle m, T\rangle = \langle r_H^{**}(m), T_1\rangle = \langle r_H(u), T_1\rangle = \langle u, r_H^*(T_1)\rangle = \langle u, T\rangle\,.$$

Therefore, $m = u \in A(G)$. □

Consequently, we arrive at the following

Theorem 3.12 ([11, Theorem 8.3])**.** *Let G be a locally compact group and let $\mathcal{H}$ be the collection of open subgroups of G as defined above. Then $Z(A(G)^{**}) = A(G)$ if and only if $Z(A(H)^{**}) = A(H)$ for all $H \in \mathcal{H}$.*

In particular, if G is a metrizable locally compact group, then $Z(A(G)^{**}) = A(G)$ if and only if $Z(A(H)^{**}) = A(H)$ for all $H \in \mathcal{H}_0$.

Remark 3.13. (I) By [9, Proposition 4.1], we have $\overline{\mathrm{span}[VN(G)^*VN(G)]}^{\|\cdot\|} = VN(G)$ if and only if $\overline{\mathrm{span}[VN(H)^*VN(H)]}^{\|\cdot\|} = VN(H)$ for all $H \in \mathcal{H}$. Thus, Theorem 3.12 also follows from Theorem 3.9.

(II) Note that $[B_\rho(G) \cdot VN(G)]^\perp \subseteq UCB(\hat{G})^\perp$. Comparing with Proposition 3.8, it is natural to ask whether for all $m \in A(G)^{**}$, we have

(1) $m \in A(G) \oplus [B_\rho(G) \cdot VN(G)]^\perp$ iff $r_H^{**}(m) \in A(H) \oplus [B_\rho(H) \cdot VN(H)]^\perp$ for all $H \in \mathcal{H}_0$.

By Proposition 3.8 and Corollary 3.1(ii), it is readily seen that (1) holds for all $m \in Z(A(G)^{**})$. However, (1) does not hold in general for all $m \in A(G)^{**}$ (see the paragraph following Remark 3.7). On the other hand, owing to [9, Proposition 4.1], we have $B_\rho(G) \cdot VN(G) = \bigcup_{H \in \mathcal{H}} B_\rho(H) \cdot VN(H)$. Therefore, (1) remains valid if $\mathcal{H}_0$ is replaced by $\mathcal{H}$, i.e., for all $m \in A(G)^{**}$, we have

$$\text{(2)} \quad m \in A(G) \oplus [B_\rho(G) \cdot VN(G)]^\perp \text{ iff } r_H^{**}(m) \in A(H) \oplus [B_\rho(H) \cdot VN(H)]^\perp \text{ for all } H \in \mathcal{H}.$$

The above arguments are also valid when $B_\rho(G)$ (resp. $B_\rho(H)$) is replaced by $VN(G)^*$ (resp. $VN(H)^*$).

Lau-Losert [17, Theorem 5.8] showed that if G is second countable and $\overline{[G,G]}$ is not open in G, then $Z(UCB(\hat{G})^*) = B_\rho(G)$. They proved in the same paper that if G is amenable, and $Z(UCB(\hat{G})^*) = B_\rho(G)$, then $Z(A(G)^{**}) = A(G)$ (cf. [17, Theorem 6.4]). Therefore, $Z(A(G)^{**}) = A(G)$ if G is second countable and amenable, and $\overline{[G,G]}$ is not open in G ([17, Theorem 6.5(iii)]). Applying Theorem 3.4, we have the following minor extension of [17, Theorem 5.8 and Theorem 6.5(iii)].

Theorem 3.14. *Let G be a metrizable locally compact group such that $\overline{[G,G]}$ is not open in G. Then*

(i) $Z(UCB(\hat{G})^*) = B_\rho(G)$.
(ii) $Z(A(G)^{**}) = A(G)$ *if G is amenable.*

Proof. (i) Let H be any σ-compact open subgroup of G. Then H is a second countable locally compact group. Note that $\overline{[H,H]}$ is not open in H since $\overline{[H,H]} \subseteq \overline{[G,G]}$, and $\overline{[G,G]}$ is not open in G. By [17, Theorem 5.8], $Z(UCB(\hat{H})^*) = B_\rho(H)$. So, by Theorem 3.4, $Z(UCB(\hat{G})^*) = B_\rho(G)$.

(ii) It follows from (i) and [17, Theorem 6.4]. □

Corollary 3.15. *Let $G = \prod_n G_n$ be a finite or countable product of metrizable locally compact groups such that G_n is compact for all but finitely many n. Assume that either $\overline{[G_1,G_1]}$ is not open in G_1 or G_1 is abelian and non-discrete. Then*

(i) $Z(UCB(\hat{G})^*) = B_\rho(G)$.
(ii) $Z(A(G)^{**}) = A(G)$ *if all G_n are amenable.*

Proof. Clearly, G is a metrizable locally compact group. According to Theorem 3.14, we only need to show that $\overline{[G,G]}$ is not open in G.

Let $q : G \longrightarrow G_1$ be the canonical projection. Then $q([G,G]) \subseteq [G_1,G_1]$ and hence $q(\overline{[G,G]}) \subseteq \overline{q([G,G])} \subseteq \overline{[G_1,G_1]}$. Since $q : G \longrightarrow G_1$ is an open map, $\overline{[G,G]}$ is not open in G if $\overline{[G_1,G_1]}$ is not open in G_1.

Assume now G_1 is abelian and non-discrete. Let $G' = \prod_{n \neq 1} G_n$. Then $\overline{[G,G]} \subseteq G'$. Since $G_1 \cong G/G'$ is non-discrete, G' is not open in G and hence $\overline{[G,G]}$ is not open in G. □

A close inspection of the proof of Lemma 3.2, Lemma 3.3 and Theorem 3.4 shows that Lau-Losert [18, Theorem 4.2] is still true if the group G_0 is assumed to be metrizable (but may not be second countable). More precisely, for the group $G = G_0 \times \prod_{i=1}^{\infty} G_i$ as in [18, Theorem 4.2] with G_0 only metrizable, our Lemma 3.2, Lemma 3.3 and hence Theorem 3.4 remain valid if $\mathcal{H}_0$ is replaced by the family

$$\{H \times \prod_{i=1}^{\infty} G_i \,|\, H \text{ is a } \sigma\text{-compact open subgroup of } G_0\}.$$

Therefore, slightly extending Lau-Losert [18, Theorem 4.2], we have the following

Theorem 3.16. *Let $G = G_0 \times \prod_{i=1}^{\infty} G_i$, where each G_i $(i \geq 0)$ is a metrizable locally compact group and G_i is compact and non-trivial for $i \geq 1$. Then*

(i) $Z(UCB(\hat{G})^*) = B_\rho(G)$.

(ii) $Z(A(G)^{**}) = A(G)$ *if G_0 is amenable.*

Acknowledgement

The author is indebted to the referee for the valuable comments, in particular, for suggesting Remark 3.5(II) and the current short proof of Lemma 3.6.

References

1. R. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2** (1951), 839-848. MR0045941 (13:659f)
2. P. Eymard, *L'algèbra de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181-236. MR0228628 (37:4208)
3. E. E. Granirer, *Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group*, Trans. Amer. Math. Soc. **189** (1974), 371-382. MR0336241 (49:1017)
4. E. E. Granirer, *Density theorems for some linear subspaces and some C^*-algebras of $VN(G)$*, Istituto Nazionale di Alta Mathematica, Symposia Mathematica **22** (1977), 61-70. MR0487287 (58:6935)
5. M. Grosser and V. Losert, *The norm-strict bidual of a Banach algebra and the dual of $C_u(G)$*, Manuscript Math. **45** (1984), 127-146. MR0724731 (86b:46073)
6. C. Herz, *Harmonic synthesis for subgroups*, Annales de l'Institut Fourier (Grenoble) **23** (1973), 91-123. MR0355482 (50:7956)
7. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, New York, 1979. MR0551496 (81k:43001)
8. Z. Hu, *Open subgroups of G and almost periodic functionals on $A(G)$*, Proc. Amer. Math. Soc. **128** (2000), 2473-2478. MR1662249 (2000k:22009)
9. Z. Hu, *Inductive extreme non-Arens regularity of the Fourier algebra $A(G)$*, Studia Math. **151** (2002), 247-264. MR1917836 (2003f:46069)
10. Z. Hu, *Maximally decomposable von Neumann algebras on locally compact groups and duality*, Houston J. Math. **31** (2005), 857-881. MR2148807
11. Z. Hu and M. Neufang, *Decomposability of von Neumann algebras and Mazur property of higher level*, Canad. J. Math, to appear.
12. N. Isik, J. S. Pym and A. Ülger, *The second dual of the group algebra of a compact group*, J. London Math. Soc. (2) **35** (1987), 135-148. MR0871771 (88f:43012)
13. E. Kaniuth, A. T. Lau and G. Schlichting, *Lebesgue type decomposition of subspaces of Fourier-Stieltjes algebras*, Trans. Amer. Math. Soc. **355** (2003), 1467-1490. MR1946400 (2004c:43004)
14. A. T. Lau, *Uniformly continuous functionals of the Fourier algebra of any locally compact group*, Trans. Amer. Math. Soc. **251** (1979), 39-59. MR0531968 (80m:43009)
15. A. T. Lau, *Continuity of Arens multiplication on the dual space of bounded uniformly continuous functions on locally compact groups and topological semigroups*, Math. Proc. Cambridge Philos. Soc. **99** (1986), 273-283. MR0817669 (87i:43001)
16. A. T. Lau and V. Losert, *On the second conjugate algebra of a locally compact group*, J. London Math. Soc. **37** (1988), 464-470. MR0939122 (89e:43007)
17. A. T. Lau and V. Losert, *The C^*-algebra generated by operators with compact support on a locally compact group*, J. Funct. Anal. **112** (1993), 1-30. MR1207935 (94d:22005)

18. A. T. Lau and V. Losert, *The centre of the second conjugate algebra of the Fourier algebra for infinite product of groups*, Math. Proc. Cambridge Philos. Soc., **138** (2005), 27-39. MR2127225 (2006c:43003)
19. V. Losert, *The centre of the bidual of Fourier algebras (discrete groups)*, preprint.
20. A. Ülger, *Central elements of A^{**} for certain Banach algebras A without bounded approximate identities*, Glasgow Math. J. **41**(1999), 369-377. MR1720442 (2001b:46082)

Department of Mathematics and Statistics, University of Windsor, Windsor, Ontario, Canada N9B 3P4
E-mail address: zhiguohu@uwindsor.ca