

## OPEN SUBGROUPS AND THE CENTRE PROBLEM FOR THE FOURIER ALGEBRA

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(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. Let  $A(G)$  be the Fourier algebra of a locally compact group and  $UCB(\hat{G})$  the  $C^*$ -algebra of uniformly continuous linear functionals on  $A(G)$ . We study how the centre problem for the algebra  $UCB(\hat{G})^*$  (resp.  $A(G)^{**}$ ) is related to the centre problem for the algebras  $UCB(\hat{H})^*$  (resp.  $A(H)^{**}$ ) of  $\sigma$ -compact open subgroups  $H$  of  $G$ . We extend some results of Lau-Losert on the centres of  $UCB(\hat{G})^*$  and  $A(G)^{**}$ .

### 1. INTRODUCTION

Let  $A$  be a Banach algebra. As is well known, there exist two Banach algebra multiplications on the second dual  $A^{**}$  of  $A$  such that each of them extends the multiplication on  $A$  (cf. Arens [1]). We will always consider the first Arens multiplication on  $A^{**}$  throughout this paper. The dual of the space  $\overline{\text{span}}(A^*A)$  equipped with the multiplication induced by that on  $A^{**}$  is also a Banach algebra. In recent years, the topological centre problem for the algebras  $A^{**}$  and  $[\overline{\text{span}}(A^*A)]^*$ , in particular for  $A$  being some Banach algebras associated with a locally compact group, has attracted some attention. Let  $A$  be either the group algebra  $L^1(G)$  or the Fourier algebra  $A(G)$  of a locally compact group  $G$ . Then the corresponding algebras  $[\overline{\text{span}}(A^*A)]^*$  are  $LUC(G)^*$  and  $UCB(\hat{G})^*$ , respectively, where  $LUC(G)$  is the space of bounded left uniformly continuous functions on  $G$  and  $UCB(\hat{G})$  is the space of uniformly continuous linear functionals on  $A(G)$ .

Let  $Z_t(A^{**})$  (resp.  $Z_t([\overline{\text{span}}(A^*A)]^*)$ ) be the topological centre of  $A^{**}$  (resp.  $(\overline{\text{span}}(A^*A))^*$ ). In [5], Grosser-Losert showed that  $Z_t(LUC(G)^*) = M(G)$  if  $G$  is abelian, where  $M(G)$  is the measure algebra of  $G$ . Lau [15] extended this result to all locally compact groups. For the group algebra  $L^1(G)$ , Isik-Pym-Ülger [12] proved that if  $G$  is compact, then  $Z_t(L^1(G)^{**}) = L^1(G)$ . This result has also been extended to all locally compact groups by Lau-Losert [16].

When  $G$  is *abelian* with dual group  $\Gamma$ ,  $L^1(\Gamma) \cong A(G)$ ,  $LUC(\Gamma) \cong UCB(\hat{G})$  and  $M(\Gamma) \cong B_\rho(G)$  (the reduced Fourier-Stieltjes algebra of  $G$ ). Therefore, if  $G$  is abelian, then  $Z(UCB(\hat{G})^*) = B_\rho(G)$  and  $Z(A(G)^{**}) = A(G)$ . It is natural to consider when the above equalities hold for a non-abelian locally compact group.

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Received by the editors January 11, 2005 and, in revised form, May 5, 2005.

2000 *Mathematics Subject Classification*. Primary 22D25, 43A30.

*Key words and phrases*. Fourier algebra, reduced Fourier-Stieltjes algebra.

This research was supported by an NSERC grant.

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Lau-Losert [17] showed that if  $G$  is second countable and  $\overline{[G, G]}$  is not open in  $G$ , where  $[G, G]$  denotes the commutator subgroup of  $G$ , then

- (i)  $Z_t(UCB(\hat{G})^*) = B_\rho(G)$ ;
- (ii)  $Z_t(A(G)^{**}) = A(G)$  if  $G$  is assumed to be amenable.

Clearly, (i) holds for all discrete groups (cf. [14, Proposition 4.5] and [17, Theorem 5.8]). Lau-Losert [17, Theorem 6.5(i)] proved that  $Z_t(A(G)^{**}) = A(G)$  is also true when  $G$  is discrete and amenable. Moreover, Lau-Losert [18] proved that (i) and (ii) hold if  $G$  is a countably infinite product of second countable locally compact groups  $\{G_i\}_{i=0}^\infty$  with  $G_i$  ( $i \geq 1$ ) compact and non-trivial. However, a special consequence of Losert [19, Theorem 3] says that  $A(G) \neq Z_t(A(G)^{**})$  ( $\neq A(G)^{**}$ ) if  $G$  is a discrete group containing the free group  $F_r$  on  $r$  generators ( $2 \leq r < \infty$ ). Very recently, Losert further showed that (ii) fails for  $G = SU(3)$ .

In this paper, we are concerned with locally compact groups with a large compact covering number. We study how the topological centre problem (i) for  $UCB(\hat{G})^*$  is related to the same problem for the algebras  $UCB(\hat{H})^*$  of open subgroups  $H$  of  $G$ . We prove that  $Z_t(UCB(\hat{G})^*) = B_\rho(G)$  if and only if  $Z_t(UCB(\hat{H})^*) = B_\rho(H)$  for all  $\sigma$ -compact open subgroups  $H$  of  $G$  (Theorem 3.4). We further investigate whether the parallel result holds for  $A(G)^{**}$  (cf. Theorem 3.9). As an application, we extend some results of Lau-Losert on  $Z_t(UCB(\hat{G})^*)$  and  $Z_t(A(G)^{**})$  to metrizable locally compact groups (cf. Theorem 3.14 and Theorem 3.16).

## 2. PRELIMINARIES

Let  $A$  be a Banach algebra. Then  $A^*$  is a Banach  $A$ -bimodule under the actions

$$\langle x \cdot \phi, \psi \rangle = \langle x, \phi\psi \rangle \quad \text{and} \quad \langle \phi \cdot x, \psi \rangle = \langle x, \psi\phi \rangle \quad (x \in A^* \text{ and } \phi, \psi \in A).$$

Each of these two module actions naturally induces a Banach algebra multiplication on  $A^{**}$  which extends that on  $A$  (cf. Arens [1]). Let  $\cdot$  and  $\Delta$  denote the first and the second Arens multiplications on  $A^{**}$ , respectively. Evidently, for any fixed  $m \in A^{**}$ , the maps  $n \mapsto n \cdot m$  and  $n \mapsto m\Delta n$  are weak\*-weak\* continuous on  $A^{**}$ . The first and the second topological centres of  $A^{**}$  are defined as follows:

$$Z_t^1(A^{**}) = \{m \in A^{**} : \text{the map } n \mapsto m \cdot n \text{ is } w^*-w^* \text{ continuous on } A^{**}\},$$

$$Z_t^2(A^{**}) = \{m \in A^{**} : \text{the map } n \mapsto n\Delta m \text{ is } w^*-w^* \text{ continuous on } A^{**}\}.$$

It is readily seen that  $A \subseteq Z_t^1(A^{**}) \cap Z_t^2(A^{**})$ .  $A$  is said to be *Arens regular* if  $Z_t^1(A^{**}) = Z_t^2(A^{**}) = A^{**}$ .

Let  $X$  be a *topologically left invariant* subspace of  $A^*$  (i.e.,  $X \cdot A \subseteq X$ ). For  $m \in X^*$  and  $x \in X$ , one can define  $m \cdot x \in A^*$  by

$$\langle m \cdot x, \phi \rangle = \langle m, x \cdot \phi \rangle \quad (\phi \in A).$$

$X$  is called *topologically left introverted* if  $m \cdot x \in X$  for all  $m \in X^*$  and  $x \in X$ . It can be seen that  $\overline{\text{span}}(A^*A)$  is topologically left introverted in  $A^*$  (cf. the proof of Lau [14, Proposition 5.2]).

Let  $X$  be a topologically left introverted subspace of  $A^*$ . Then  $X^*$  becomes a Banach algebra under the multiplication defined by  $\langle m \cdot n, x \rangle = \langle m, n \cdot x \rangle$  ( $m, n \in X^*$  and  $x \in X$ ). It is evident that this multiplication on  $X^*$  is induced by the first Arens multiplication on  $A^{**}$ . That is, if  $m, n \in X^*$  and  $\tilde{m}, \tilde{n} \in A^{**}$  are extensions of  $m, n$ , respectively, then  $\tilde{m} \cdot \tilde{n} \in A^{**}$  is an extension of  $m \cdot n$ . Obviously, for any

fixed  $m \in X^*$ , the map  $n \mapsto n \cdot m$  is weak\*-weak\* continuous on  $X^*$ . The (left) topological centre of  $X^*$  is defined as

$$Z_t(X^*) = \{m \in X^* : \text{the map } n \mapsto m \cdot n \text{ is } w^*-w^* \text{ continuous on } X^*\}.$$

If  $A$  is a commutative Banach algebra, then  $Z_t^1(A^{**}) = Z_t^2(A^{**})$  is just the algebraic centre  $Z(A^{**})$  of  $A^{**}$  (equipped with either of the Arens multiplications). The following lemma is clearly true.

**Lemma 2.1.** *Let  $A$  be a commutative Banach algebra and let  $X$  be a topologically introverted subspace of  $A^*$ . Then  $Z_t(X^*)$  is the algebraic centre  $Z(X^*)$  of  $X^*$ .*

For a linear subspace  $Y$  of  $A^*$  containing  $A^*A$ ,  $y^* \in Y^*$  and  $f \in A^*$ , let  $y^* \cdot f \in A^*$  be defined by  $\langle y^* \cdot f, a \rangle = \langle y^*, f \cdot a \rangle$  ( $a \in A$ ). If  $X$  is a subset of  $A^*$ ,  $X^\perp$  will denote the annihilator of  $X$  in  $A^{**}$ . Here we collect some simple facts on  $A^*A$  and  $Z(A^{**})$ , which will be used in the sequel.

**Lemma 2.2.** *Let  $A$  be a commutative Banach algebra. Then*

- (i)  $(A^*A)^\perp = \{m \in A^{**} : n \cdot m = 0 \text{ for all } n \in A^{**}\}.$
- (ii)  $(A^{**}A^*)^\perp = \{m \in A^{**} : m \cdot n = n \cdot m = 0 \text{ for all } n \in A^{**}\}.$
- (iii)  $Z(A^{**}) \cap (A^*A)^\perp = (A^{**}A^*)^\perp.$
- (iv)  $[\overline{\text{span}}(A^*A)]^* \cdot A^* = A^{**}A^*$  and  $Z(A^{**}) \cap (A^*A)^\perp = ([\overline{\text{span}}(A^*A)]^* \cdot A^*)^\perp.$

*Proof.* (i) is obviously true, and (ii) is included in [19, Lemma 1]. By (ii), we have  $(A^{**}A^*)^\perp \subseteq Z(A^{**})$ . Therefore, (iii) follows from (i) and (ii).

For (iv), let  $y^* \in [\overline{\text{span}}(A^*A)]^*$ . It is easy to see that if  $n \in A^{**}$  is an extension of  $y^*$ , then  $y^* \cdot f = n \cdot f$  for all  $f \in A^*$ . Therefore,  $[\overline{\text{span}}(A^*A)]^* \cdot A^* = A^{**}A^*$  and hence, by (iii),  $Z(A^{**}) \cap (A^*A)^\perp = (A^{**}A^*)^\perp = ([\overline{\text{span}}(A^*A)]^* \cdot A^*)^\perp. \quad \square$

Let  $G$  be a locally compact group. The Fourier-Stieltjes algebra  $B(G)$  is the linear span of positive definite continuous functions on  $G$  and can be identified with the dual of the group  $C^*$ -algebra  $C^*(G)$  of  $G$ . With the dual norm and the pointwise multiplication,  $B(G)$  is a commutative Banach algebra. The reduced Fourier-Stieltjes algebra  $B_\rho(G)$  is the closure of  $B(G) \cap C_{00}(G)$  in the  $w^*$ -topology of  $B(G)$ , where  $C_{00}(G)$  is the set of continuous functions on  $G$  with compact support.  $B_\rho(G)$  is a closed ideal in  $B(G)$  and is precisely the dual of the reduced group  $C^*$ -algebra  $C_\rho^*(G)$  of  $G$ . As is known,  $B_\rho(G) = B(G)$  if and only if  $G$  is amenable.

The Fourier algebra  $A(G)$  is the closed ideal in  $B(G)$  generated by  $B(G) \cap C_{00}(G)$ .  $A(G)$  can be identified with the predual of the group von Neumann algebra  $VN(G)$  of  $G$ . Naturally,  $VN(G)$  is a Banach  $B(G)$ -module under the action defined by  $\langle u \cdot T, v \rangle = \langle T, uv \rangle$  ( $u \in B(G)$ ,  $v \in A(G)$  and  $T \in VN(G)$ ). See Eymard [2] for more information on  $B(G)$ ,  $B_\rho(G)$ ,  $A(G)$ , and  $VN(G)$ .

The support of an operator  $T$  in  $VN(G)$  is defined by saying that  $x \in \text{supp } T$  if and only if  $u \cdot T = 0$  implies  $u(x) = 0$  for all  $u \in A(G)$  (cf. Eymard [2] and Herz [6]). The space  $UCB(\hat{G})$  of uniformly continuous linear functionals on  $A(G)$  is the norm closure of  $A(G) \cdot VN(G)$  in  $VN(G)$ . It is known that  $UCB(\hat{G})$  is a  $C^*$ -subalgebra of  $VN(G)$  and also a closed  $B(G)$ -submodule of  $VN(G)$  which coincides with the norm closure of  $\{T \in VN(G) : \text{supp } T \text{ is compact}\}$  in  $VN(G)$  (cf. Granirer [3]–[4]).

We recall that  $UCB(\hat{G})$  is a topologically introverted subspace of  $VN(G)$ . Thus,  $UCB(\hat{G})^*$  is a Banach algebra and  $Z_t(UCB(\hat{G})^*)$  is just the algebraic centre  $Z(UCB(\hat{G})^*)$  of  $UCB(\hat{G})^*$  (cf. Lemma 2.1).

Throughout this paper,  $H$  will denote an open subgroup of  $G$ . Let  $r_H : A(G) \rightarrow A(H)$  be the restriction map and  $t_H : A(H) \rightarrow A(G)$  the trivial extension map (i.e.,  $(t_H u)(x) = 0$  for  $x \in G - H$ ). The adjoint map  $r_H^*$  is a  $*$ -isomorphism of  $VN(H)$  onto the sub von Neumann algebra  $VN_H(G)$  of  $VN(G)$ , where

$$VN_H(G) = \{T \in VN(G) : \text{supp } T \subseteq H\}$$

(cf. Eymard [2, Proposition 3.21]). Also,  $r_H^{**}$  is an algebraic homomorphism of  $A(G)^{**}$  onto  $A(H)^{**}$  and  $t_H^{**}$  is an algebraic isomorphism of  $A(H)^{**}$  into  $A(G)^{**}$ .

It is known that  $r_H^*(UCB(\hat{H})) \subseteq UCB(\hat{G})$  and  $t_H^*(UCB(\hat{G})) = UCB(\hat{H})$  (cf. Granirer [3]). We let  $\Phi = r_H^*|_{UCB(\hat{H})} : UCB(\hat{H}) \rightarrow UCB(\hat{G})$  and  $\Psi = t_H^*|_{UCB(\hat{G})} : UCB(\hat{G}) \rightarrow UCB(\hat{H})$ . Then  $\Psi \circ \Phi = id$  and  $\Phi$  is a  $*$ -isomorphism of  $UCB(\hat{H})$  onto  $UCB(\hat{G}) \cap VN_H(G)$ . Furthermore,  $\Phi^*$  is an algebraic homomorphism of  $UCB(\hat{G})^*$  onto  $UCB(\hat{H})^*$ , and  $\Psi^*$  is an algebraic isomorphism of  $UCB(\hat{H})^*$  into  $UCB(\hat{G})^*$ .

A direct computation shows the following result on the images of the centres under the maps  $\Phi^*$  and  $\Psi^*$ .

**Lemma 2.3.** *Let  $G$  be a locally compact group and let  $H$  be an open subgroup of  $G$ . Then*

- (i)  $\Psi^*[Z(UCB(\hat{H})^*)] \subseteq Z(UCB(\hat{G})^*)$ .
- (ii)  $\Phi^*[Z(UCB(\hat{G})^*)] = Z(UCB(\hat{H})^*)$ .

### 3. THE CENTRES OF $UCB(\hat{G})^*$ AND $A(G)^{**}$

Let  $G$  be a locally compact group. In [17], Lau-Losert defined an isometric embedding  $\pi = \pi_G : B_\rho(G) \hookrightarrow UCB(\hat{G})^*$  satisfying

$$\langle \pi(\varphi), u \cdot T \rangle = \langle T, \varphi u \rangle \quad \text{for } \varphi \in B_\rho(G), u \in A(G) \text{ and } T \in VN(G),$$

i.e.,  $\pi$  is the natural extension of the isometric embedding of  $A(G)$  into  $UCB(\hat{G})^*$  (see Lau [14] for the amenable case).  $\pi : B_\rho(G) \rightarrow UCB(\hat{G})^*$  is also an algebraic isomorphism, i.e.,  $\pi(\varphi\psi) = \pi(\varphi) \cdot \pi(\psi)$  ( $\varphi, \psi \in B_\rho(G)$ ). Furthermore,

$$\varphi \cdot T = \pi(\varphi) \cdot T \quad \text{for all } \varphi \in B_\rho(G) \text{ and } T \in VN(G),$$

where  $\varphi \cdot T$  is the  $B_\rho(G)$ -module product and  $\pi(\varphi) \cdot T$  is as defined in Section 2. In the following,  $B_\rho(G)$  will be identified with the closed subalgebra  $\pi(B_\rho(G))$  of  $UCB(\hat{G})^*$ . Lau-Losert [17, Proposition 4.5] showed that  $B_\rho(G) \subseteq Z(UCB(\hat{G})^*)$ .

Note that  $A(G) \subseteq B_\rho(G) \subseteq UCB(\hat{G})^*$ . Applying Lemma 2.2 to  $A(G)$ , we obviously have the following

**Corollary 3.1.** *Let  $G$  be a locally compact group. Then*

- (i)  $[VN(G)^*VN(G)]^\perp = [UCB(\hat{G})^* \cdot VN(G)]^\perp \subseteq [B_\rho(G) \cdot VN(G)]^\perp \subseteq UCB(\hat{G})^\perp$ .
- (ii)  $[VN(G)^*VN(G)]^\perp = Z(A(G)^{**}) \cap [B_\rho(G) \cdot VN(G)]^\perp = Z(A(G)^{**}) \cap UCB(\hat{G})^\perp$ .
- (iii)  $B_\rho(G) = UCB(\hat{G})^*$  and  $VN(G)^*VN(G) = B_\rho(G) \cdot VN(G)$  if  $G$  is discrete.

Let  $\mathcal{H}_0$  be the collection of all  $\sigma$ -compact open subgroups of  $G$ .

**Lemma 3.2.** *Let  $G$  be a locally compact group and let  $\theta$  be a function on  $G$  such that  $\theta|_H \in B_\rho(H)$  for all  $H \in \mathcal{H}_0$ . Then  $\theta \in B_\rho(G)$ .*

*Proof.* First, we observe that there exists a constant  $M > 0$  such that  $\|\theta|_H\|_{B_\rho(H)} \leq M$  for all  $H \in \mathcal{H}_0$ . Otherwise, for each positive integer  $n$ , there exists an  $H_n \in \mathcal{H}_0$  such that  $\|\theta|_{H_n}\|_{B_\rho(H_n)} \geq n$ . Let  $H$  be the subgroup of  $G$  generated by  $H_n$  ( $n = 1, 2, \dots$ ). Then  $H \in \mathcal{H}_0$  and  $\|\theta|_H\|_{B_\rho(H)} \geq \|\theta|_H \cdot 1_{H_n}\|_{B_\rho(H)} = \|\theta|_{H_n}\|_{B_\rho(H_n)} \geq n$  for all  $n$ , which is a contradiction.

Next, we show that  $\theta \in B_\rho(G)$ . Note that  $G = \bigcup_{H \in \mathcal{H}_0} H$  and  $\|\cdot\|_\infty \leq \|\cdot\|_{B_\rho}$ . Thus,  $\theta$  is a bounded continuous function on  $G$ . Let  $f \in L^1(G)$ . Then there exists an  $H \in \mathcal{H}_0$  such that  $f = 0$  on  $G - H$ . So,

$$\left| \int_G f(x)\theta(x) dx \right| = \left| \int_H f(x)\theta|_H(x) dx \right| \leq \|\theta|_H\|_{B_\rho(H)} \|f|_H\|_{C_\rho^*(H)} \leq M \|f\|_{C_\rho^*(G)}.$$

Therefore,  $\sup \{ \left| \int_G f(x)\theta(x) dx \right| : f \in L^1(G) \text{ and } \|f\|_{C_\rho^*(G)} \leq 1 \} \leq M < \infty$ . According to [2, Proposition 2.1],  $\theta \in B_\rho(G)$ . □

For any open subgroup  $H$  of  $G$ , let  $\lambda_H$  be the left regular representation of  $H$  and let  $m_H$  denote  $\Phi^*(m)$  for  $m \in UCB(\hat{G})^*$ , where  $\Phi : UCB(\hat{H}) \rightarrow UCB(\hat{G})$  is the  $*$ -isomorphism as defined in Section 2. It is easy to see that  $\Phi(\lambda_H(h)) = \lambda_G(h)$  for all  $h \in H$ . Also, for all  $x \in G$  and  $\varphi \in B_\rho(G)$ , we have  $\langle \pi(\varphi), \lambda_G(x) \rangle = \varphi(x)$ .

**Lemma 3.3.** *Let  $G$  be a locally compact group and let  $m \in UCB(\hat{G})^*$  be such that  $m_H \in B_\rho(H)$  for all  $H \in \mathcal{H}_0$ . Then  $m \in B_\rho(G)$ .*

*Proof.* By the assumption, for each  $H \in \mathcal{H}_0$ , there exists a  $\theta_H \in B_\rho(H)$  such that  $m_H = \pi_H(\theta_H)$ , where  $\pi_H : B_\rho(H) \hookrightarrow UCB(\hat{H})^*$  is the isometric embedding.

For  $H \in \mathcal{H}_0$  and  $x \in H$ , we have

$$\langle m, \lambda_G(x) \rangle = \langle m, \Phi(\lambda_H(x)) \rangle = \langle m_H, \lambda_H(x) \rangle = \langle \pi_H(\theta_H), \lambda_H(x) \rangle = \theta_H(x).$$

Thus,  $\theta_H(x) = \langle m, \lambda_G(x) \rangle$  for all  $H \in \mathcal{H}_0$  and  $x \in H$ . Let  $\theta(x) = \langle m, \lambda_G(x) \rangle$  ( $x \in G$ ). Then  $\theta|_H = \theta_H \in B_\rho(H)$  for all  $H \in \mathcal{H}_0$ . By Lemma 3.2,  $\theta \in B_\rho(G)$ .

To show that  $m = \pi(\theta)$ , let  $T \in UCB(\hat{G})$ . Note that  $\{R \in VN(G) : \text{supp } R \text{ is compact}\}$  is norm dense in  $UCB(\hat{G})$ . So, we assume that  $\text{supp } T$  is compact. Then  $\text{supp } T \subseteq H$  for some  $H \in \mathcal{H}_0$ , i.e.,  $T \in VN_H(G)$ . Since  $UCB(\hat{G}) \cap VN_H(G) = \Phi[UCB(\hat{H})]$ , there exists an  $S \in UCB(\hat{H})$  such that  $T = \Phi(S)$ . Thus,

$$\langle m, T \rangle = \langle \Phi^*(m), S \rangle = \langle m_H, S \rangle = \langle \pi_H(\theta_H), S \rangle = \langle \pi_H(\theta|_H), S \rangle.$$

On the other hand, note that  $\Phi(v \cdot P) = t_H(v) \cdot r_H^*(P)$  for all  $v \in A(H)$  and  $P \in VN(H)$ . Hence, we have

$$\begin{aligned} \langle \Phi^*(\pi(\theta)), v \cdot P \rangle &= \langle \pi(\theta), t_H(v) \cdot r_H^*(P) \rangle = \langle r_H^*(P), t_H(v)\theta \rangle \\ &= \langle P, \theta|_H v \rangle = \langle \pi_H(\theta|_H), v \cdot P \rangle \end{aligned}$$

for all  $v \in A(H)$  and  $P \in VN(H)$ , i.e.,  $\pi_H(\theta|_H) = \Phi^*(\pi(\theta))$ . It follows that

$$\langle m, T \rangle = \langle \pi_H(\theta|_H), S \rangle = \langle \Phi^*(\pi(\theta)), S \rangle = \langle \pi(\theta), \Phi(S) \rangle = \langle \pi(\theta), T \rangle.$$

Therefore,  $m = \pi(\theta) \in B_\rho(G)$ . □

Immediately, we have the following

**Theorem 3.4.** *Let  $G$  be a locally compact group and let  $\mathcal{H}_0$  be the collection of  $\sigma$ -compact open subgroups of  $G$ . Then  $Z(UCB(\hat{G})^*) = B_\rho(G)$  if and only if  $Z(UCB(\hat{H})^*) = B_\rho(H)$  for all  $H \in \mathcal{H}_0$ .*

*Proof.* Assume that  $Z(UCB(\hat{G})^*) = B_\rho(G)$ . Let  $H$  be any open subgroup of  $G$  and let  $\Phi^* : UCB(\hat{G})^* \rightarrow UCB(\hat{H})^*$  be the algebraic homomorphism as defined in Section 2. It is readily seen that for all  $\varphi \in B_\rho(G)$ ,  $\Phi^*(\varphi) = \varphi|_H \in B_\rho(H)$ . By Lemma 2.3(ii), we have  $Z(UCB(\hat{H})^*) = \Phi^*(Z(UCB(\hat{G})^*)) = \Phi^*(B_\rho(G)) \subseteq B_\rho(H)$ , i.e.,  $Z(UCB(\hat{H})^*) = B_\rho(H)$ .

Conversely, suppose  $Z(UCB(\hat{H})^*) = B_\rho(H)$  for all  $H \in \mathcal{H}_0$ . To get the non-trivial inclusion  $Z(UCB(\hat{G})^*) \subseteq B_\rho(G)$ , let  $m \in Z(UCB(\hat{G})^*)$ . Then, for all  $H \in \mathcal{H}_0$ , by Lemma 2.3(ii),  $m_H = \Phi^*(m) \in Z(UCB(\hat{H})^*) = B_\rho(H)$ . It follows from Lemma 3.3 that  $m \in B_\rho(G)$ .  $\square$

*Remark 3.5.* (I) Theorem 3.4 would be trivial if we had  $\overline{\bigcup_{H \in \mathcal{H}_0} Z(UCB(\hat{H})^*)}^{\|\cdot\|} = Z(UCB(\hat{G})^*)$  (cf. Lemma 2.3(i)). However, even though  $\bigcup_{H \in \mathcal{H}_0} UCB(\hat{H}) = UCB(\hat{G})$ ,  $\overline{\bigcup_{H \in \mathcal{H}_0} Z(UCB(\hat{H})^*)}^{\|\cdot\|}$  is in general a proper subspace of  $Z(UCB(\hat{G})^*)$  (e.g., it is the case when  $G$  is abelian but non- $\sigma$ -compact).

(II) We note that Lemma 3.3 (and hence Theorem 3.4) stays true if  $\mathcal{H}_0$  is replaced by the class  $\mathcal{H}_c$  of compactly generated open subgroups of  $G$ . In fact, that  $\theta$  is in  $B_\rho(G)$  follows from the boundedness of the family  $\{\|\theta_H\|_{B_\rho(H)} : H \in \mathcal{H}_c\}$  and the same argument as in the second paragraph of the proof of Lemma 3.2.

Now, we turn our attention to the Fourier algebra  $A(G)$ . We investigate whether parallel results hold for  $A(G)$  and  $Z(A(G)^{**})$ .

Note that  $\|u\|_{B(H)} \leq \|u^\circ\|_{B(G)}$  for  $u \in B(H)$ , where  $H$  is any open subgroup of  $G$  and  $u^\circ$  is the trivial extension of  $u$  to  $G$  (cf. [8, Lemma 3.1]). So, one can see that Lemma 3.2 remains valid if  $B_\rho(H)$  and  $B_\rho(G)$  are replaced by  $B(H)$  and  $B(G)$ , respectively. The assertion also holds if  $B_\rho(H)$  and  $B_\rho(G)$  are replaced by  $A(H)$  and  $A(G)$ , respectively. However, the proof is different from the proof of Lemma 3.2. For the reader's convenience and our later use, we include the following lemma with a very short proof as suggested by the referee.

**Lemma 3.6.** *Let  $G$  be a locally compact group and let  $u$  be a function on  $G$  such that  $u|_H \in A(H)$  for all  $H \in \mathcal{H}_0$ . Then  $u \in A(G)$ .*

*Proof.* Let  $H_0$  be a fixed  $\sigma$ -compact open subgroup of  $G$ . Note that  $1_{xH_0} \in B(G)$  for all  $x \in G$ . So, for any  $x \in G$  and  $H \in \mathcal{H}_0$  with  $xH_0 \subseteq H$ , by the assumption,  $u \cdot 1_{xH_0} \in t_H(A(H)) \subseteq A(G)$ , where  $t_H : A(H) \rightarrow A(G)$  is the trivial extension map. We also note that, for any countable subset  $D$  of  $G$ ,  $DH_0 \subseteq H$  for some  $H \in \mathcal{H}_0$ . Since  $A(G) \cap C_{00}(G)$  is norm dense in  $A(G)$  and  $\|1_{xH_0}\|_{B(G)} = 1$ , it is readily seen that  $\|u \cdot 1_{xH_0}\|_{A(G)} > 0$  can hold for countably many cosets  $xH_0$  only. Therefore, there exists an  $H \in \mathcal{H}_0$  such that  $\text{supp } u \subseteq H$  and hence  $u \in t_H(A(H)) \subseteq A(G)$ .  $\square$

*Remark 3.7.* Let  $B_\rho(G) = A(G) \oplus B_\rho^s(G)$  be the Lebesgue decomposition (see Kaniuth-Lau-Schlichting [13, Corollary 2.5]). Then each  $u \in B_\rho(G)$  can be written as  $u = u^a + u^s$  with  $u^a \in A(G)$  and  $u^s \in B_\rho^s(G)$ . It follows from Lemma 3.6 that

if  $u \in B_\rho^s(G)$  and  $u \neq 0$ , then there exists an  $H \in \mathcal{H}_0$  such that  $(u|_H)^s \neq 0$ .

Our original proof of Lemma 3.6 was derived from the above assertion.

Note that  $\overline{\bigcup_{H \in \mathcal{H}_0} VN_H(G)}^{\|\cdot\|} = \bigcup_{H \in \mathcal{H}_0} VN_H(G)$ . As is shown in [10, Proposition 7.3], we can have  $\bigcup_{H \in \mathcal{H}_0} VN_H(G) \neq VN(G)$  when  $G$  is non- $\sigma$ -compact (and even abelian). Thus, we do *not* have the corresponding result for  $A(G)^{**}$  parallel to Lemma 3.3, i.e., for an  $m \in A(G)^{**}$ , we can have  $m \notin A(G)$  even  $r_H^{**}(m) \in A(H)$  for all  $H \in \mathcal{H}_0$ . The following proposition tells us that for such elements  $m \in A(G)^{**}$ , we should take  $UCB(\hat{G})^\perp$  (the annihilator of  $UCB(\hat{G})$  in  $VN(G)^*$ ) into account. Note that  $UCB(\hat{G}) = \bigcup_{H \in \mathcal{H}_0} r_H^*(UCB(\hat{H}))$ . Therefore, for any  $m \in A(G)^{**}$ ,

$$m \in UCB(\hat{G})^\perp \text{ iff } r_H^{**}(m) \in UCB(\hat{H})^\perp \text{ for all } H \in \mathcal{H}_0.$$

Clearly,  $A(G) \cap UCB(\hat{G})^\perp = \{0\}$ . We show next how  $A(G) \oplus UCB(\hat{G})^\perp$  is related to the family  $\{A(H) \oplus UCB(\hat{H})^\perp : H \in \mathcal{H}_0\}$ .

**Proposition 3.8.** *Let  $G$  be a locally compact group and let  $m \in A(G)^{**}$ . Then*

$$m \in A(G) \oplus UCB(\hat{G})^\perp \text{ iff } r_H^{**}(m) \in A(H) \oplus UCB(\hat{H})^\perp \text{ for all } H \in \mathcal{H}_0.$$

*Proof.* Obviously, if  $m \in A(G) \oplus UCB(\hat{G})^\perp$ , then  $r_H^{**}(m) \in A(H) \oplus UCB(\hat{H})^\perp$  for all open subgroups  $H$  of  $G$ . Conversely, suppose  $r_H^{**}(m) \in A(H) \oplus UCB(\hat{H})^\perp$  for all  $H \in \mathcal{H}_0$ . Then, for each  $H \in \mathcal{H}_0$ , there exists  $u_H \in A(H)$  such that  $r_H^{**}(m) - u_H \in UCB(\hat{H})^\perp$ . Let  $u(x) = \langle m, \lambda_G(x) \rangle$  ( $x \in G$ ). It is evident that  $u|_H = u_H \in A(H)$  for all  $H \in \mathcal{H}_0$ . By Lemma 3.6,  $u \in A(G)$ . Since

$$r_H^{**}(m - u) = r_H^{**}(m) - r_H^{**}(u) = r_H^{**}(m) - u_H \in UCB(\hat{H})^\perp$$

for all  $H \in \mathcal{H}_0$ ,  $m - u \in UCB(\hat{G})^\perp$ . So,  $m = u + (m - u) \in A(G) \oplus UCB(\hat{G})^\perp$ .  $\square$

Corollary 3.1 together with the equality  $A(G) \cap UCB(\hat{G})^\perp = \{0\}$  implies that

if  $Z(A(G)^{**}) = A(G)$ , then  $\text{span}[VN(G)^*VN(G)]$  is norm dense in  $VN(G)$ .

The counterexample  $G = SU(3)$  by Losert shows that the converse of the above assertion is not true. In general, for non- $\sigma$ -compact and non-metrizable locally compact groups (cf. Theorem 3.12), we have the following

**Theorem 3.9.** *Let  $G$  be a locally compact group and let  $\mathcal{H}_0$  be the collection of  $\sigma$ -compact open subgroups of  $G$ . Then the following statements are equivalent:*

- (i)  $Z(A(G)^{**}) = A(G)$ .
- (ii)  $\overline{\text{span}[VN(G)^*VN(G)]}^{\|\cdot\|} = VN(G)$  and  $Z(A(H)^{**}) = A(H)$  for all  $H \in \mathcal{H}_0$ .

*Proof.* It is easy to see that if  $Z(A(G)^{**}) = A(G)$ , then  $Z(A(H)^{**}) = A(H)$  for all open subgroups  $H$  of  $G$  (cf. the proof of [11, Proposition 8.2]). So, we only need to prove that (ii)  $\implies$  (i). Assume that  $\text{span}[VN(G)^*VN(G)]$  is norm dense in  $VN(G)$  and  $Z(A(H)^{**}) = A(H)$  for all  $H \in \mathcal{H}_0$ . Let  $m \in Z(A(G)^{**})$ . Then  $r_H^{**}(m) \in Z(A(H)^{**}) = A(H)$  for all  $H \in \mathcal{H}_0$ . By Proposition 3.8, there exists  $u \in A(G)$  such that  $m - u \in UCB(\hat{G})^\perp$ . Note that we also have  $m - u \in Z(A(G)^{**})$ . It follows from Corollary 3.1(ii) that  $m - u \in Z(A(G)^{**}) \cap UCB(\hat{G})^\perp = [VN(G)^*VN(G)]^\perp = \{0\}$ , i.e.,  $m = u \in A(G)$ .  $\square$

If  $G$  is amenable, then  $B_\rho(G) \cdot VN(G) = VN(G)$  and hence, by Corollary 3.1(i),  $[VN(G)^*VN(G)]^\perp \subseteq [B_\rho(G) \cdot VN(G)]^\perp = \{0\}$ , i.e.,  $\text{span}[VN(G)^*VN(G)]$  is norm dense in  $VN(G)$ . Immediately, we have the following corollary.

**Corollary 3.10.** *Let  $G$  be an amenable locally compact group. Then  $Z(A(G)^{**}) = A(G)$  if and only if  $Z(A(H)^{**}) = A(H)$  for all  $H \in \mathcal{H}_0$ .*

We should point out that Corollary 3.10 can also be derived from Lau-Losert [17, Lemma 6.3]. For a discrete group  $G$  containing the free group  $F_r$  on  $r$  generators ( $2 \leq r < \infty$ ), more than obtaining  $Z(A(G)^{**}) \neq A(G)$ , Losert [19] actually showed that  $\text{span}[VN(G)^*VN(G)] (= \text{span}[B_\rho(G) \cdot VN(G)])$  is *not* norm dense in  $VN(G)$  (cf. [19, Corollary 1]). We do not know if this is true for *all* non-amenable locally compact groups. Opposite to the case  $\overline{\text{span}[VN(G)^*VN(G)]}^{\|\cdot\|} = VN(G)$ , Ülger [20, Theorem 3.3] implies that if  $G$  is discrete, then  $\overline{\text{span}[VN(G)^*VN(G)]}^{\|\cdot\|} = UCB(\hat{G})$  if and only if  $A(G)$  is Arens regular.

As is shown in [11], the assertion in Corollary 3.10 remains valid for *all* locally compact groups  $G$  if  $\mathcal{H}_0$  is replaced by the following family  $\mathcal{H}$ :

$$\mathcal{H} = \{H : H \text{ is an open subgroup of } G \text{ and } \kappa(H) \leq \chi(G) \cdot \aleph_0\},$$

where  $\kappa(H)$  is the compact covering number of  $H$  and  $\chi(G)$  is the character of  $G$  (i.e., the least cardinality of an open basis at the identity of  $G$ ). In fact, we have the following analogue of Lemma 3.3 with  $\mathcal{H}_0$  replaced by  $\mathcal{H}$ . The proof included here is different from that given in [11], where a higher level Mazur property of  $A(G)$  is used.

**Lemma 3.11** ([11, Proposition 8.2]). *Let  $G$  be a locally compact group and let  $m \in A(G)^{**}$  be such that  $r_H^{**}(m) \in A(H)$  for all  $H \in \mathcal{H}$ . Then  $m \in A(G)$ .*

*Proof.* Following the proof of Proposition 3.8 and noting that  $\mathcal{H}_0 \subseteq \mathcal{H}$ , we see that there exists  $u \in A(G)$  such that  $r_H^{**}(m) = r_H(u)$  for all  $H \in \mathcal{H}$ . We only need to show that  $\langle m, T \rangle = \langle u, T \rangle$  for all  $T \in VN(G)$ . Let  $T \in VN(G)$ . Then there exists an  $H \in \mathcal{H}$  such that  $\text{supp } T \subseteq H$  (cf. [9, Proposition 4.1]), i.e.,  $T \in VN_H(G)$ . Since  $VN_H(G) = r_H^*(VN(H))$ ,  $T = r_H^*(T_1)$  for some  $T_1 \in VN(G)$ . It follows that

$$\langle m, T \rangle = \langle r_H^{**}(m), T_1 \rangle = \langle r_H(u), T_1 \rangle = \langle u, r_H^*(T_1) \rangle = \langle u, T \rangle.$$

Therefore,  $m = u \in A(G)$ . □

Consequently, we arrive at the following

**Theorem 3.12** ([11, Theorem 8.3]). *Let  $G$  be a locally compact group and let  $\mathcal{H}$  be the collection of open subgroups of  $G$  as defined above. Then  $Z(A(G)^{**}) = A(G)$  if and only if  $Z(A(H)^{**}) = A(H)$  for all  $H \in \mathcal{H}$ .*

*In particular, if  $G$  is a metrizable locally compact group, then  $Z(A(G)^{**}) = A(G)$  if and only if  $Z(A(H)^{**}) = A(H)$  for all  $H \in \mathcal{H}_0$ .*

*Remark 3.13.* (I) By [9, Proposition 4.1], we have  $\overline{\text{span}[VN(G)^*VN(G)]}^{\|\cdot\|} = VN(G)$  if and only if  $\overline{\text{span}[VN(H)^*VN(H)]}^{\|\cdot\|} = VN(H)$  for all  $H \in \mathcal{H}$ . Thus, Theorem 3.12 also follows from Theorem 3.9.

(II) Note that  $[B_\rho(G) \cdot VN(G)]^\perp \subseteq UCB(\hat{G})^\perp$ . Comparing with Proposition 3.8, it is natural to ask whether for all  $m \in A(G)^{**}$ , we have

$$(1) \quad m \in A(G) \oplus [B_\rho(G) \cdot VN(G)]^\perp \text{ iff } r_H^{**}(m) \in A(H) \oplus [B_\rho(H) \cdot VN(H)]^\perp \text{ for all } H \in \mathcal{H}_0.$$

By Proposition 3.8 and Corollary 3.1(ii), it is readily seen that (1) holds for all  $m \in Z(A(G)^{**})$ . However, (1) does not hold in general for all  $m \in A(G)^{**}$  (see

the paragraph following Remark 3.7). On the other hand, owing to [9, Proposition 4.1], we have  $B_\rho(G) \cdot VN(G) = \bigcup_{H \in \mathcal{H}} B_\rho(H) \cdot VN(H)$ . Therefore, (1) remains valid if  $\mathcal{H}_0$  is replaced by  $\mathcal{H}$ , i.e., for all  $m \in A(G)^{**}$ , we have

$$(2) \quad m \in A(G) \oplus [B_\rho(G) \cdot VN(G)]^\perp \text{ iff } r_H^{**}(m) \in A(H) \oplus [B_\rho(H) \cdot VN(H)]^\perp$$

for all  $H \in \mathcal{H}$ .

The above arguments are also valid when  $B_\rho(G)$  (resp.  $B_\rho(H)$ ) is replaced by  $VN(G)^*$  (resp.  $VN(H)^*$ ).

Lau-Losert [17, Theorem 5.8] showed that if  $G$  is second countable and  $\overline{[G, G]}$  is not open in  $G$ , then  $Z(UCB(\hat{G})^*) = B_\rho(G)$ . They proved in the same paper that if  $G$  is amenable, and  $Z(UCB(\hat{G})^*) = B_\rho(G)$ , then  $Z(A(G)^{**}) = A(G)$  (cf. [17, Theorem 6.4]). Therefore,  $Z(A(G)^{**}) = A(G)$  if  $G$  is second countable and amenable, and  $\overline{[G, G]}$  is not open in  $G$  ([17, Theorem 6.5(iii)]). Applying Theorem 3.4, we have the following minor extension of [17, Theorem 5.8 and Theorem 6.5(iii)].

**Theorem 3.14.** *Let  $G$  be a metrizable locally compact group such that  $\overline{[G, G]}$  is not open in  $G$ . Then*

- (i)  $Z(UCB(\hat{G})^*) = B_\rho(G)$ .
- (ii)  $Z(A(G)^{**}) = A(G)$  if  $G$  is amenable.

*Proof.* (i) Let  $H$  be any  $\sigma$ -compact open subgroup of  $G$ . Then  $H$  is a second countable locally compact group. Note that  $\overline{[H, H]}$  is not open in  $H$  since  $\overline{[H, H]} \subseteq \overline{[G, G]}$ , and  $\overline{[G, G]}$  is not open in  $G$ . By [17, Theorem 5.8],  $Z(UCB(\hat{H})^*) = B_\rho(H)$ . So, by Theorem 3.4,  $Z(UCB(\hat{G})^*) = B_\rho(G)$ .

(ii) It follows from (i) and [17, Theorem 6.4]. □

**Corollary 3.15.** *Let  $G = \prod_n G_n$  be a finite or countable product of metrizable locally compact groups such that  $G_n$  is compact for all but finitely many  $n$ . Assume that either  $\overline{[G_1, G_1]}$  is not open in  $G_1$  or  $G_1$  is abelian and non-discrete. Then*

- (i)  $Z(UCB(\hat{G})^*) = B_\rho(G)$ .
- (ii)  $Z(A(G)^{**}) = A(G)$  if all  $G_n$  are amenable.

*Proof.* Clearly,  $G$  is a metrizable locally compact group. According to Theorem 3.14, we only need to show that  $\overline{[G, G]}$  is not open in  $G$ .

Let  $q : G \rightarrow G_1$  be the canonical projection. Then  $q([G, G]) \subseteq [G_1, G_1]$  and hence  $q(\overline{[G, G]}) \subseteq \overline{q([G, G])} \subseteq [G_1, G_1]$ . Since  $q : G \rightarrow G_1$  is an open map,  $\overline{[G, G]}$  is not open in  $G$  if  $\overline{[G_1, G_1]}$  is not open in  $G_1$ .

Assume now  $G_1$  is abelian and non-discrete. Let  $G' = \prod_{n \neq 1} G_n$ . Then  $\overline{[G, G]} \subseteq G'$ .

Since  $G_1 \cong G/G'$  is non-discrete,  $G'$  is not open in  $G$  and hence  $\overline{[G, G]}$  is not open in  $G$ . □

A close inspection of the proof of Lemma 3.2, Lemma 3.3 and Theorem 3.4 shows that Lau-Losert [18, Theorem 4.2] is still true if the group  $G_0$  is assumed to be metrizable (but may not be second countable). More precisely, for the group  $G = G_0 \times \prod_{i=1}^\infty G_i$  as in [18, Theorem 4.2] with  $G_0$  only metrizable, our Lemma 3.2,

Lemma 3.3 and hence Theorem 3.4 remain valid if  $\mathcal{H}_0$  is replaced by the family

$$\{H \times \prod_{i=1}^{\infty} G_i \mid H \text{ is a } \sigma\text{-compact open subgroup of } G_0\}.$$

Therefore, slightly extending Lau-Losert [18, Theorem 4.2], we have the following

**Theorem 3.16.** *Let  $G = G_0 \times \prod_{i=1}^{\infty} G_i$ , where each  $G_i$  ( $i \geq 0$ ) is a metrizable locally compact group and  $G_i$  is compact and non-trivial for  $i \geq 1$ . Then*

- (i)  $Z(UCB(\hat{G})^*) = B_\rho(G)$ .
- (ii)  $Z(A(G)^{**}) = A(G)$  if  $G_0$  is amenable.

#### ACKNOWLEDGEMENT

The author is indebted to the referee for the valuable comments, in particular, for suggesting Remark 3.5(II) and the current short proof of Lemma 3.6.

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