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ON L^1-NORMS OF MEROMORPHIC FUNCTIONS WITH FIXED POLES

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Abstract. We study boundedness of the differentiation and embedding operators in the shift-coinvariant subspaces K^1_B generated by Blaschke products with sparse zeros, that is, in the spaces of meromorphic functions with fixed poles in the lower half-plane endowed with L^1-norm. We answer negatively the question of K.M. Dyakonov about the necessity of the condition $B' \in L^\infty(\mathbb{R})$ for the boundedness of the differentiation on K^1_B. Our main tool is a construction of an unconditional basis of rational fractions in K^1_B.

1. Introduction

Let $Z = \{z_n\}_{n\in\mathbb{N}}$ be a sequence of points in the upper half-plane $\mathbb{C}^+$ and let $m_n \in \mathbb{N}$. We consider the closed subspace of $L^p(\mathbb{R})$ spanned by the rational fractions with poles at the points $\overline{z}_n$ and multiplicities at most m_n. Namely, for $p > 1$ let

$$\mathcal{R}^p_Z = \overline{Span}_{L^p(\mathbb{R})} \left\{ \frac{1}{(t-\overline{z}_n)^l} : n \in \mathbb{N}, 1 \le l \le m_n \right\}.$$

For $p = 1$ we should take only rational fractions which belong to $L^1(\mathbb{R})$, that is, with the zero residue at infinity.

Assume that the sequences $\{z_n\}$ and $\{m_n\}$ satisfy the Blaschke condition

$$\sum_n \frac{\operatorname{Im} z_n}{|z_n|^2+1} m_n < \infty. \tag{1.1}$$

In this case $\mathcal{R}^p_Z = H^p \cap B\overline{H^p}$, where H^p denotes the Hardy class in the upper half-plane and

$$B(z) = \prod_n \left(\alpha_n \frac{z-z_n}{z-\overline{z}_n} \right)^{m_n}$$

is the Blaschke product with zeros z_n of multiplicities m_n and $\alpha_n = |z_n^2+1|/(z_n^2+1)$ (by definition, $\alpha_n = 1$ for $z_n = i$). If (1.1) does not hold, then $\mathcal{R}^p_Z$ coincides with the whole Hardy class H^p.

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Thus, the spaces $\mathcal{R}_Z^p$ are a particular case of the so-called shift-coinvariant or model subspaces

$$K_\Theta^p = H^p \cap \Theta\overline{H^p}$$

in the upper half-plane. Here Θ is an inner function in the upper half-plane, that is, a bounded analytic function such that $|\Theta(x)| = 1$ for almost every $x \in \mathbb{R}$ in the sense of non-tangential boundary values. It is well known that for $p > 1$ the subspaces K_Θ^p are the only subspaces of H^p coinvariant with respect to the semigroup of shifts $(U_t)_{t\ge 0}$, $U_t f(x) = e^{itx} f(x)$. These subspaces (especially for $p = 2$) play an outstanding role both in function and operator theory and, in particular, in the Nagy-Foias model for contractions in a Hilbert space. Another important feature of the shift-coinvariant subspaces is that the subspace corresponding to $\Theta(z) = \exp(iaz)$, $a > 0$, essentially coincides with the Paley-Wiener space PW_a^p of entire functions of exponential type at most a whose restrictions on $\mathbb{R}$ belong to $L^p(\mathbb{R})$. Namely, $K_\Theta^p = H^p \cap PW_a^p$. Shift-coinvariant subspaces are discussed in detail in monographs [4, 13].

In [6] K.M. Dyakonov posed the problem to describe the inner functions Θ having the following properties analogous to the properties of the Paley-Wiener spaces:

1. $K_\Theta^p \subset K_\Theta^q$ for $q > p$;
2. the differentiation operator $\mathcal{D} : f \mapsto f'$ is bounded as an operator from K_Θ^p to $L^p(\mathbb{R})$, that is, there is a Bernstein-type inequality

$$\|f'\|_p \le C\|f\|_p, \qquad f \in K_\Theta^p.$$

For $p > 1$ Dyakonov [6] (see also [5, 10]) has obtained the complete answer:

Theorem 1.1. *Let $1 < p < \infty$. Then the following statements are equivalent:*

(1) $K_\Theta^p \subset K_\Theta^q$ *for some (any) $q > p$;*
(2) $\mathcal{D}$ *is bounded on* K_Θ^p*;*
(3) $\Theta' \in L^\infty(\mathbb{R})$ *in the sense of non-tangential boundary values.*

The compactness of differentiation and embedding operators was studied in [10] and [9], respectively. In particular, differentiation operator in K_Θ^p, $1 < p < \infty$, is compact if and only if $\lim_{|t|\to\infty} |\Theta'(t)| = 0$. Certain weighted Bernstein-type inequalities for shift-coinvariant subspaces were proved in [8, 2, 3].

The condition $\Theta' \in L^\infty$, which is equivalent to the inclusion $\Theta' \in H^\infty(\mathbb{C}^+)$, implies that up to a unimodular constant factor the function Θ is of the form

$$\Theta(z) = \exp(iaz)B(z), \tag{1.2}$$

where $a \ge 0$ and B is a Blaschke product with zeros tending to infinity. Thus, Θ is meromorphic in the whole complex plane. In other words, $\sigma(\Theta) \cap \mathbb{R} = \emptyset$, where $\sigma(\Theta) = \{z \in \overline{\mathbb{C}^+} : \liminf\limits_{\zeta\to z,\, \zeta\in\mathbb{C}^+} |\Theta(\zeta)| = 0\}$ is the so-called spectrum of the function Θ. In this case

$$|\Theta'(x)| = a + |B'(x)| = a + 2\sum_n \frac{\operatorname{Im} z_n}{|x - z_n|^2} m_n, \qquad x \in \mathbb{R}. \tag{1.3}$$

Note that $|\Theta'(\operatorname{Re} z_n)| \ge 2(\operatorname{Im} z_n)^{-1}$. Hence, if $\Theta' \in L^\infty$, then $\inf_n \operatorname{Im} z_n > 0$.

For the case $p = 1$ the following results were obtained in [5, 6]:

Theorem 1.2. *If $\Theta' \in L^\infty$, then*

(1) $K_\Theta^1 \subset K_\Theta^\infty$*;*
(2) $\mathcal{D}$ *is bounded on* K_Θ^1.

Conversely, if the exponent a *in* (1.2) *is positive or*

$$\sup_k \prod_{n\neq k} \left| \frac{z_k - z_n}{z_k - \overline{z}_n} \right| < 1, \tag{1.4}$$

then both (1) *and* (2) *imply* $\Theta' \in L^\infty$.

It was an open problem (posed in [6]) whether condition $\Theta' \in L^\infty$ is necessary for the embedding $K^1_\Theta \subset K^q_\Theta$, $q > 1$, or for the boundedness of the differentiation operator in K^1_Θ in the general case. The aim of the present note is to answer this question negatively:

Theorem 1.3. *There exists a Blaschke product* B *such that* $K^1_B \subset K^\infty_B$ *and the differentiation operator* $\mathcal{D}$ *is bounded on* K^1_B, *but* $B' \notin L^\infty$.

To prove this theorem we construct in Sections 2 and 3 a class of Blaschke products such that statements (1) and (2) of Theorem 1.2 hold, but $\inf_n \operatorname{Im} z_n = 0$ and, thus, $B' \notin L^\infty$. In Section 4 we also obtain some new conditions which are necessary for the boundedness of differentiation in K^1_Θ in the general case.

It should be noted that (1.4) is a non-sparsness condition on the zeros of B. Therefore any possible example of a Blaschke sequence such that B satisfies the conclusions of Theorem 1.3 has to be necessarily very sparse.

The proof of our main result is based on a construction of "well-behaved" bases in K^1_B consisting of rational fractions. We consider the Blaschke products with sufficiently sparse zeros, and the elements of the bases under consideration will have "almost disjoint supports" (close constructions for the model subspaces in the unit disk may be found in [7] and in [1], where the bases were used to obtain examples concerning Carleson-type embeddings of the model subspaces).

This construction seems to be of independent interest and may produce other examples and counterexamples. To justify this point of view we consider an application to the description of equivalent norms in K^p_Θ. It is an important problem to describe the class of measures μ (on the line or in the upper half-plane) such that the norm $\|\cdot\|_{L^p(\mu)}$ is equivalent to the natural L^p-norm on K^p_Θ, that is, there exist positive constants C_1 and C_2 such that $C_1\|f\|_p \le \|f\|_{L^p(\mu)} \le C_2\|f\|_p$, $f \in K^p_\Theta$.

A.L. Volberg [16] solved this problem for the measures of the form $\mu = wm$, where m is the Lebesgue measure on the line and $w \in L^\infty(\mathbb{R})$. Denote by $\tilde{w}$ the harmonic extension of w to the upper half-plane:

$$\tilde{w}(z) = \frac{\operatorname{Im} z}{\pi} \int_{\mathbb{R}} \frac{w(t)}{|t-z|^2} dt, \qquad z \in \mathbb{C}^+.$$

Theorem 1.4. *Let* $1 < p < \infty$, $\mu = wm$, $w \in L^\infty$. *Then* μ *defines an equivalent norm on* K^p_Θ *if and only if*

$$\inf_{z\in\mathbb{C}^+} (\tilde{w}(z) + |\Theta(z)|) > 0. \tag{1.5}$$

The situation when $\Theta(z) = \exp(iaz)$ (that is, the case of the Paley-Wiener spaces) and $w = \chi_E$ is the characteristic function of a measurable set $E \subset \mathbb{R}$ is of special interest (see [12], pp. 112-116, for the history of this problem). The description of discrete measures on the line defining an equivalent norm in the Paley-Wiener space PW^2_a was recently obtained in [14].

It is known that (1.5) is not sufficient for the equivalence of norms on K^1_Θ; a counterexample was constructed by V.P. Havin and B. Jöricke [11]. They show, however, that (1.5) is still sufficient if Θ has the following property: for any sufficiently small $\sigma > 0$ the level set $\{z \in \mathbb{C}^+ : |\Theta(z)| < \sigma\}$ is a finite union of connected sets with mutually disjoint closures. On the other hand, it is shown in [6] that (1.5) is necessary for the equivalence of the corresponding norms if Θ is a function of the form (1.2) such that either $a > 0$ or (1.4) is fulfilled.

Making use of our construction of subspaces K^1_B with unconditional bases of rational functions we produce in Theorem 5.1 a shift-coinvariant subspace such that (1.5) is at the same time neither necessary nor sufficient for the equivalence of the norms. This example simplifies considerably the construction of [11] and answers the question about the necessity of (1.5) asked by Dyakonov in [6].

2. Bases of rational functions in K^1_B

In what follows the symbols C, C_1, C_2, etc. denote different positive constants which may change their values in different occurrences. We write $g \asymp h$ if $C_1 g \le h \le C_2 g$, and we write $a_j = o(b_j)$, $j \to \infty$, if $\lim_{j\to\infty} a_j/b_j = 0$.

We say that a system of vectors $\{h_j\}$ in a Banach space H is an unconditional basis if each element h of H may be represented as an unconditionally convergent series $h = \sum_j a_j h_j$ and $\|h\|_H \asymp \sum_j |a_j| \cdot \|h_j\|_H$.

Let B be a Blaschke product with simple zeros $\{z_j\}_{j\ge0}$, $z_j = x_j + iy_j$. Put

$$f_j(z) = \frac{1}{(z-\overline{z}_0)(z-\overline{z}_j)}, \qquad j \ge 1.$$

Clearly, $f_j \in K^1_B$. We mentioned in the Introduction that K^1_B coincides with the closed linear span of rational fractions with the poles at the points $\overline{z}_j$, $j \ge 0$, and with zero residue at infinity (see [15], Corollary 2.3, for the proof of an analogous result for the case of the unit disk; the statement for the upper half-plane follows by means of the conformal mapping). Thus, in particular, the system $\{f_j\}_{j\ge1}$ is complete in K^1_B. We show that for a class of Blaschke products with sufficiently sparse zeros, the functions f_j form an unconditional basis in K^1_B.

To make our construction more transparent we consider in detail the following particular case. Let $z_0 = i$. Assume that $x_1 > 0$, $x_{j+1} \ge 4x_j$, and $0 < y_j \le 1$. Note that the system $\{f_j\}$ is minimal, that is, $f_k \notin \overline{Span}\{f_j : j \ne k\}$. Therefore, the property to be a basis is preserved when we add or eliminate a finite number of zeros. Without loss of generality we may assume that $x_1 \ge 4^3$ and, consequently, $x_j \ge 4^{j+2}$.

An easy calculation shows that

$$\|f_j\|_1 \asymp \frac{1}{x_j} \ln \frac{x_j}{y_j}.$$

Theorem 2.1. *The system $\{f_j\}_{j\ge1}$ is an unconditional basis in K^1_B.*

Proof. The theorem will be proved as soon as we show that

$$\|h\|_1 \asymp \sum_{j\ge1} |a_j| \cdot \|f_j\|_1$$

for any finite linear combination $h = \sum_j a_j f_j$ of the functions f_j. We show that

$$\|h\|_1 \ge C \sum_{j\ge1} |a_j| \cdot \|f_j\|_1$$

(the converse inequality is trivial). Put $\Delta_k = [x_k, 5x_k/4]$, $k \in \mathbb{N}$. Then, clearly,

$$\|h\|_1 > \sum_{k\geq 1} \int_{\Delta_k} |h|.$$

Since

$$\int_{\Delta_k} |h| \geq |a_k| \int_{\Delta_k} |f_k| - \sum_{j\neq k} |a_j| \int_{\Delta_k} |f_j|,$$

we get

$$\|h\|_1 > \sum_{j\geq 1} |a_j| \left(\int_{\Delta_j} |f_j| - \sum_{k\neq j} \int_{\Delta_k} |f_j| \right). \tag{2.1}$$

Note that $|t - z_0| < 2x_j$, $t \in \Delta_j$. Hence,

$$\int_{\Delta_j} |f_j| > \frac{1}{2x_j} \int_{\Delta_j} \frac{dt}{|t - z_j|} > \frac{1}{2x_j} \ln \frac{x_j}{4y_j}. \tag{2.2}$$

Let us estimate $\int_{\Delta_k} |f_j|$, $k \neq j$. If $k > j$, then $|x_k - z_j| < 3x_k/4$ and

$$\int_{\Delta_k} |f_j| \leq (3x_k)^{-1}.$$

Hence,

$$\sum_{k\neq j} \int_{\Delta_k} |f_j| \leq \frac{j-1}{x_j} + \sum_{k>j} \frac{1}{3x_k} \leq \frac{j}{3x_j} < \frac{1}{3x_j} \ln \frac{x_j}{4y_j}.$$

Thus, it follows from (2.1) and (2.2) that

$$\|h\|_1 \geq \sum_{j\geq 1} \frac{|a_j|}{6x_j} \ln \frac{x_j}{4y_j} \asymp \sum_{j\geq 1} |a_j| \cdot \|f_j\|_1.$$

□

The same arguments lead to the following slightly more general statement.

Theorem 2.2. *Let B be a Blaschke product with the simple zeros $z_j = x_j + iy_j$, $j \geq 0$, such that $x_j > 0$, $\liminf_{j\to\infty} x_{j+1}/x_j > 1$, and $\{y_j\} \in \ell^\infty$. Then the system $\{f_j\}_{j\geq 1}$ is an unconditional basis in K_B^1.*

3. Main results

In this section B is a Blaschke product with zeros satisfying the conditions of Theorem 2.2. In this case a linear operator T is bounded on K_B^1 if and only if it is bounded on the basic elements f_j, that is, $\|Tf_j\| \leq C\|f_j\|_1$.

Theorem 3.1. *Differentiation operator $\mathcal{D}$ is bounded on K_B^1 if and only if*

$$\liminf_{j\to\infty} y_j \ln x_j > 0. \tag{3.1}$$

The operator $\mathcal{D}$ is compact on K_B^1 if and only if $\lim_{j\to\infty} y_j \ln x_j = \infty$.

Proof. Note that

$$f_j'(t) = -\frac{1}{(t-\overline{z}_0)^2(t-\overline{z}_j)} - \frac{1}{(t-\overline{z}_0)(t-\overline{z}_j)^2}.$$

Clearly, the norm of the first summand is majorized by $\|f_j\|_1$ (moreover, it is $o(\|f_j\|_1)$), whereas for the second one we have for sufficiently large j

$$\left\| \frac{1}{(t-\overline{z}_0)(t-\overline{z}_j)^2} \right\|_1 \asymp \frac{1}{x_j y_j}.$$

Thus, differentiation is bounded if and only if

$$\frac{1}{x_j y_j} \le C_2 \frac{1}{x_j} \ln \frac{x_j}{y_j},$$

which is equivalent to (3.1).

Let $\lim\limits_{j\to\infty} y_j \ln x_j = \infty$ and, consequently, $\|f_j'\|_1 = o(\|f_j\|_1)$. Then the operator $\mathcal{D}$ may be approximated by finite rank operators T_N, where

$$T_N \left(\sum_{j\ge 1} a_j f_j \right) = \sum_{j=1}^{N} a_j f_j',$$

$n \in \mathbb{N}$. Indeed, if $h = \sum_{j\ge1} a_j f_j$, then

$$\|h' - T_N h\|_1 \le \sum_{j=N+1}^{\infty} |a_j| \cdot \|f_j'\|_1 \le \sup_{j>N} \frac{\|f_j'\|_1}{\|f_j\|_1} \|h\|_1.$$

Now assume that $0 < \liminf\limits_{j\to\infty} y_j \ln x_j < \infty$. Then there is a subsequence f_{j_n} such that $\|f_{j_n}'\|_1 \asymp \|f_{j_n}\|_1$. Integrating over the intervals $[x_{j_n}, x_{j_n} + y_{j_n}]$, it is easy to show that

$$\inf_{n\ne k} \left\| \frac{f_{j_n}'}{\|f_{j_n}\|_1} - \frac{f_{j_k}'}{\|f_{j_k}\|_1} \right\|_1 > 0.$$

Thus, the sequence $f_{j_n}'/\|f_{j_n}\|_1$ does not contain a convergent subsequence, and $\mathcal{D}$ is not compact. □

Next we consider the problem of the embeddings of K_B^1 into the spaces K_B^q for $q > 1$. An interesting feature of the spaces K_B^1 is that in contrast to the case $p > 1$ the embedding criterion may depend essentially on the exponent q. In particular, for $q > 1$ there exists a Blaschke product B such that $K_B^1 \subset K_B^q$, but the embedding $K_B^1 \subset K_B^r$ does not take place for any $r > q$.

Theorem 3.2. *Let $1 < q \le \infty$. Then the embedding $K_B^1 \subset K_B^q$ takes place if and only if*

$$\liminf_{j\to\infty} y_j^{1-\frac{1}{q}} \ln x_j > 0.$$

Proof. By the closed graph theorem, the embedding $K_B^1 \subset K_B^q$ is equivalent to the inequality $\|f\|_q \le C\|f\|_1$, $f \in K_B^1$. Now, the statement follows from the obvious estimate $\|f_j\|_q \asymp x_j^{-1} y_j^{\frac{1}{q}-1}$. □

Theorem 1.3 follows immediately from Theorems 3.1 and 3.2, and, thus, we get the negative answer to the question of K.M. Dyakonov on the necessity of the condition $\Theta' \in L^\infty(\mathbb{R})$ for the boundedness of the differentiation.

Proof of Theorem 1.3. Since $x_j \ge C\gamma^j$, $\gamma > 1$, one can take $y_j \to 0$ such that $1/y_j \le C_1 j$. Then, by Theorem 3.1, the differentiation operator will be bounded, but $|B'(x_j)| > 2/y_j$ and, thus, $B' \notin L^\infty$. If, moreover, $1/y_j = o(j)$, $j \to \infty$, then $\mathcal{D}$ is compact. By Theorem 3.2, we also have the embedding $K^1_B \subset K^\infty_B$. ☐

4. NECESSARY CONDITIONS FOR THE BERNSTEIN-TYPE INEQUALITY IN K^1_Θ

In this section we obtain a few necessary conditions for the boundedness of the differentiation operator in K^1_Θ.

For $\zeta \in \mathbb{C}^+$ consider the function

$$k(z,\zeta) = \frac{1-\Theta(z)\overline{\Theta(\zeta)}}{z-\overline{\zeta}}, \qquad z \in \mathbb{C}^+,$$

which is the reproducing kernel of the space K^2_Θ corresponding to the point ζ, that is, $(f, k(\cdot,\zeta)) = 2\pi i f(\zeta)$, $f \in K^2_\Theta$.

Proposition 4.1. *If $\mathcal{D}$ is bounded on K^1_Θ, then Θ is of the form* (1.2) *(equivalently, $\sigma(\Theta) \cap \mathbb{R} = \emptyset$).*

Proof. Assume that $\sigma(\Theta) \cap \mathbb{R} \ne \emptyset$. If Θ is not of the form $\Theta(z) = \exp\left(\frac{ib}{t_0 - z}\right)$ for some $t_0 \in \mathbb{R}$ and $b > 0$, then there exist $\alpha, \beta \in \mathbb{R}$ and non-trivial inner functions Θ_1, Θ_2 such that $\Theta = \Theta_1\Theta_2$, $\sigma(\Theta_1) \cap (\alpha,\beta) \ne \emptyset$ and Θ_2 is analytic in a neighborhood of $[\alpha,\beta]$.

Let $k_l(\cdot,\zeta)$ denote the reproducing kernels corresponding to Θ_l, $l = 1, 2$. Then the function $k_1(\cdot,\zeta_1)k_2(\cdot,\zeta_2)$ is in K^1_Θ for any $\zeta_1, \zeta_2 \in \mathbb{C}^+$. Applying the Bernstein-type inequality $\|f'\|_1 \le C\|f\|_1$ to the function $f = k_1(\cdot,i)k_2(\cdot,i)$, we get

$$\int_\alpha^\beta |k_1'(t,i)k_2(t,i)|dt \le \int_\alpha^\beta |k_1(t,i)k_2'(t,i)|dt + C\|k_1(\cdot,i)k_2(\cdot,i)\|_1 < \infty.$$

Therefore $\int_\alpha^\beta |\Theta_1'(t)|dt < \infty$. On the other hand, it follows immediately from the formula for the modulus of the angular derivative that $\int_\alpha^\beta |\Theta'(t)|dt = \infty$ whenever $\sigma(\Theta) \cap (\alpha,\beta) \ne \emptyset$ (see also [3], Lemma 6.1), and we have a contradiction.

Finally, if $\Theta(z) = \exp\left(\frac{ib}{t_0-z}\right)$, then we apply similar arguments to $\Theta_1(z) = \Theta_2(z) = \exp\left(\frac{ib}{2(t_0-z)}\right)$ and $f = k_1^2(\cdot,i)$. ☐

Proposition 4.2. *Let B be a Blaschke product with zeros z_n such that* $\inf_n \operatorname{Im} z_n = \delta > 0$. *Then the differentiation operator is bounded on K^1_B if and only if $B' \in L^\infty$.*

We make use of the following estimate.

Lemma 4.3. *Let B be a Blaschke product with zeros z_n such that* $\inf_n \operatorname{Im} z_n = \delta > 0$. *Then there is $\varepsilon = \varepsilon(\delta)$ such that*

$$|k(t,x)| \asymp |B'(x)|, \qquad t, x \in \mathbb{R},\ |t-x| < \varepsilon(1+|B'(x)|)^{-1}.$$

Proof. Note that B is meromorphic in the complex plane $\mathbb{C}$. Therefore, $k(\cdot,x) \in K^2_B$, $x \in \mathbb{R}$, and, $\|k(\cdot,x)\|_2^2 = 2\pi|B'(x)|$. In particular, $k(t,x)$ is well defined for $t \in \mathbb{R}$. Since $\inf_n \operatorname{Im} z_n \ge \delta$, we have $|t - z_n| \asymp |x - z_n|$, $|t-x| \le 1$ (with the constants depending on δ) and, by (1.3), $|B'(t)| \asymp |B'(x)|$, $|t-x| \le 1$. Hence, there is $\varepsilon > 0$ such that

$$\int\limits_{|t-x|<\varepsilon(1+|B'(x)|)^{-1}} |B'(t)|dt \le \frac{\pi}{2}.$$

Since B is a meromorphic Blaschke product, there is a well-defined branch of the argument of B on the real axis. Namely, there exists an increasing differentiable function φ such that $B(t) = e^{2i\varphi(t)}$, $t \in \mathbb{R}$. Note that

$$k(t,x) = -2ie^{i(\varphi(t)-\varphi(x))}\frac{\sin(\varphi(t)-\varphi(x))}{t-x}.$$

Therefore, if $|t-x| < \varepsilon(1+|B'(x)|)^{-1}$, then $|\varphi(t)-\varphi(x)| \le \pi/4$, and

$$|k(t,x)| \asymp \frac{\varphi(t)-\varphi(x)}{t-x} \asymp \varphi'(x) = |B'(x)|/2.$$

□

Proof of Proposition 4.2. Let us fix $x \in \mathbb{R}$. Note that each of the summands in the formula (1.3), which are of the form $2|x-z_n|^{-2}\operatorname{Im} z_n$, does not exceed $2/\delta$. Thus, if $|B'(x)| \ge 4/\delta$, then there exist two subproducts B_1 and B_2 such that $B = B_1B_2$ and

$$|B_1'(x)| \asymp |B_2'(x)| \asymp |B'(x)|.$$

Denote by $k_l(\cdot,x)$ and φ_l, $l = 1,2$, the corresponding reproducing kernels and arguments, and put $f = k_1(\cdot,x)k_2(\cdot,x)$. Clearly, $f \in K_B^1$, and, by the Cauchy inequality (recall that $\|k_l(\cdot,x)\|_2^2 = 2\pi|B_l'(x)|$),

$$\|f\|_1 \le C_1|B'(x)|. \tag{4.1}$$

Making use of the formula

$$f(t) = 4e^{i(\varphi(t)-\varphi(x))}\frac{\sin(\varphi_1(t)-\varphi_1(x))}{t-x}\cdot\frac{\sin(\varphi_2(t)-\varphi_2(x))}{t-x}, \quad t \in \mathbb{R},$$

it is easy to see that

$$|f'(t)| \ge 4\varphi'(t)\left|\frac{\sin(\varphi_1(t)-\varphi_1(x))}{t-x}\right|\cdot\left|\frac{\sin(\varphi_2(t)-\varphi_2(x))}{t-x}\right|.$$

Now, applying Lemma 4.1 to the kernels $k_l(\cdot,x)$, we get

$$\|f'\|_1 > \int\limits_{|t-x|<\varepsilon(1+|B'(x)|)^{-1}} |f'| \ge C_2\frac{|B'(x)|^3}{1+|B'(x)|}.$$

If the differentiation operator is bounded, then combining the last inequality with (4.1) we get a uniform estimate for $|B'(x)|$, $x \in \mathbb{R}$. □

Next we show that in the case when the differentiation operator is bounded, the zeros which are close to the real axis should be sufficiently sparse.

Proposition 4.4. *If $\mathcal{D}$ is bounded on K_B^1, then*

$$\frac{1}{\operatorname{Im} z_k} \le C\max(1, \ln|z_j-\overline{z}_k|) \tag{4.2}$$

for any distinct zeros z_k and z_j.

Proof. We consider the action of $\mathcal{D}$ on the test functions

$$f_{jk}(z) = \frac{1}{(z-\overline{z}_j)(z-\overline{z}_k)}.$$

Without loss of generality assume that $x_k \ge x_j$ and $y_k \le y_j$. Then, by a direct calculation,

$$\|f_{jk}\|_1 \le \frac{C_1}{x_k - x_j + y_j - y_k} \ln \frac{x_k - x_j + y_j}{y_k}.$$

On the other hand,

$$\|f'_{jk}\|_1 \ge C_2 \int\limits_{x_k}^{x_k+y_k} \frac{dt}{|t - z_k|^2 |t - z_j|} \ge \frac{C_3}{y_k |z_j - \overline{z}_k|}.$$

Combining these two inequalities, we arrive at the necessary condition (4.2). ☐

Condition (4.2) implies that if a zero z_k is close to the real axis, then there are no other zeros in the disk of the radius $e^{C/\operatorname{Im} z_k}$. For example, if $z_k = x_k + ik^{-1}$, $k \in \mathbb{N}$, then x_k should grow as a progression and we are in the situation considered in Sections 2 and 3: there is a basis of rational fractions with two poles, and the boundedness of differentiation in K^1_Θ reduces to the boundedness on basic elements, that is, to (3.1). Thus, the gap between necessary and sufficient conditions for the Bernstein-type inequality in K^1_B seems to be rather small.

5. A COUNTEREXAMPLE TO VOLBERG'S THEOREM

In this section we use the construction of Section 2 to produce new counterexamples to the L^1-version of Volberg's theorem. In particular, we give a negative answer to the question of K.M. Dyakonov on necessity of (1.5).

Theorem 5.1. *Let B be a Blaschke product satisfying the conditions of Theorem* 2.2. *Assume also that $y_j \to 0$ and $\ln \frac{1}{y_j} = o(\ln x_j)$. Then the condition* (1.5) *(with $\Theta = B$) is neither necessary nor sufficient for the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_{L^1(wm)}$ on K^1_B.*

Proof. Let

$$E_1 = \bigcup_{j\ge1} [x_j - 1, x_j + 1], \qquad E_2 = \bigcup_{j\ge1} [x_j + 1, x_{j+1} - 1],$$

and let $w_l = \chi_{E_l}$, $l = 1, 2$. First, we show that the weight w_2 defines an equivalent norm on K^1_B, but (1.5) is not fullfilled. Indeed,

$$\tilde{w}_2(z_j) = \frac{y_j}{\pi} \int\limits_{E_2} \frac{dt}{|t - z_j|^2} \le \frac{y_j}{\pi} \int\limits_{|t - x_j|>1} \frac{dt}{(t - x_j)^2} \le C y_j.$$

Thus, $\tilde{w}(z_j) + |B(z_j)| \to 0$, $j \to \infty$. On the other hand,

$$\int\limits_{x_j+1}^{x_{j+1}-1} |f_j| \asymp \frac{1}{x_j} \ln x_j \asymp \frac{1}{x_j} \ln \frac{x_j}{y_j} \asymp \|f_j\|_1,$$

and, by the arguments analogous to the proof of Theorem 2.1, it is easily shown that $\int_{E_2} |f| \ge C_1 \|f\|_1$, $f \in K^1_B$. Hence, the weight w_2 defines an equivalent norm on K^1_B.

Now consider the weight w_1. Note that

$$\int_{E_1} |f_j| \asymp \int_{x_j-1}^{x_j+1} |f_j| \asymp \frac{1}{x_j} \ln \frac{1}{y_j} = o(\|f_j\|_1),$$

and, thus, the norm $\|\cdot\|_{L^1(w_1 m)}$ is not equivalent to the natural L^1-norm. Let us show that the weight w_1 satisfies (1.5). Put $F = \bigcup_{j\ge 0}\{z \in \mathbb{C}^+ : |z - x_j| < 1\}$ and $G = \mathbb{C}^+ \setminus F$. Then for $z = x + iy \in F$

$$\tilde{w}_1(z) > \frac{y}{\pi} \int_{x_j-1}^{x_j+1} \frac{dt}{(t-x)^2 + y^2} = \frac{1}{\pi}\left(\operatorname{arctg} \frac{x_j + 1 - x}{y} + \operatorname{arctg} \frac{x + 1 - x_j}{y}\right) > \frac{1}{4}.$$

Finally, making use of the formula

$$2 \ln |B(z)| = \sum_{j\ge 0} \ln \left(1 - \frac{4yy_j}{|z - \overline{z}_j|^2}\right),$$

it is easy to see that $\inf_{z\in G} |B(z)| > 0$. ☐

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