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UNIQUENESS PROBLEM OF MEROMORPHIC FUNCTIONS SHARING SMALL FUNCTIONS

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ABSTRACT. In this paper, a uniqueness theorem of meromorphic functions which share four small functions is given.

1. INTRODUCTION

It is well known that two nonconstant polynomials f, g over an algebraic closed field of characteristic zero are identical if there exist two distinct values a, b such that $f(x) = a$ if and only if $g(x) = a$ and $f(x) = b$ if and only if $g(x) = b$.

In 1926, R. Nevanlinna [1] extended the above result to meromorphic functions. He showed that, for two distinct nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$, they cannot have the same inverse images for five distinct values, and g is a special type of linear fractional transformation of f if they have the same inverse images counted with multiplicities for four distinct values.

Naturally, one may ask the question: Is it possible to replace five distinct values by five small functions?

Over the last few years, there were several generalizations of Nevanlinna's result to the case of small functions as targets.

To state some of them, we must introduce some notions.

Let $f(z)$ be a nonzero holomorphic function on $\mathbb{C}^n$. For $a \in \mathbb{C}^n$, set $f(z) = \sum_{m=0}^{\infty} P_m(z-a)$, where the term $P_m(z)$ is either identically zero or a homogeneous polynomial of degree m. The number $\nu_f(a) := \min\{m | P_m \not\equiv 0\}$ is said to be the zero-multiplicity of f at a.

For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we set $\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$. For $r > 0$, define $B(r) = \{z \in \mathbb{C}^n | \|z\| < r\}$, $S(r) = \{z \in \mathbb{C}^n | \|z\| = r\}$, $d^c = (4\pi\sqrt{-1})^{-1}(\partial - \overline{\partial})$, $v = (dd^c\|z\|^2)^{n-1}$ and $\sigma = d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{n-1}$.

Let $\phi(z)$ be a nonconstant meromorphic function on $\mathbb{C}^n$ with reduced representation $\phi = \frac{\phi_0}{\phi_1}$, where $\phi_0,\ \phi_1$ are holomorphic functions on $\mathbb{C}^n$ having no common zeros.

The characteristic function of ϕ is defined by

$$T(r, \phi) = \int_{S(r)} \log \|\phi\| \sigma - \int_{S(1)} \log \|\phi\| \sigma \quad (r > 1),$$

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where $\|\phi\| = (|\phi_0|^2 + |\phi_1|^2)^{1/2}$. For $n = 1$, $T(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \log\|\phi(re^{i\theta})\|d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log\|\phi(e^{i\theta})\|d\theta$.

The meromorphic function a is said to be "small" with respect to f if $T(r,a) = o(T(r,f))$ as $r \to +\infty$.

Let $\mathcal{R}(f)$ be the set of meromorphic functions on $\mathbb{C}^n$ which are small with respect to f. It is easy to see that $\mathcal{R}(f)$ is a field.

We define

$$\nu^M_{f-a}(z) = min\{M, \nu_{f-a}(z)\},$$

$$\nu^M_{f-a,\leq k}(z) = \begin{cases} 0 & \text{if } \nu_{f-a}(z) > k, \\ \nu^M_{f-a}(z) & \text{if } \nu_{f-a}(z) \leq k, \end{cases}$$

and

$$\nu^M_{f-a,>k}(z) = \begin{cases} 0 & \text{if } \nu_{f-a}(z) \leq k, \\ \nu^M_{f-a}(z) & \text{if } \nu_{f-a}(z) > k, \end{cases}$$

for positive integers k, M or $M = \infty$.

In 2000, Li and Qiao [2] gave a generalized Nevanlinna theorem that if two nonconstant meromorphic functions f and g on $\mathbb{C}$ and five meromorphic functions $\{a_j\}_{j=1}^5$ in $\mathcal{R}(f) \cap \mathcal{R}(g)$ satisfy $\nu^1_{f-a_j} = \nu^1_{g-a_j}, 1 \leq j \leq 5$, then $f = g$. In 2002, Yi [3] obtained an improvement of the above result and showed that if $k \geq 14$ and $\nu^1_{f-a_j,\leq k} = \nu^1_{g-a_j,\leq k}$, $1 \leq j \leq 5$, then $f = g$.

Recently, motivated by the accomplishment of the second main theorem for small functions given by Yamanoi [4], Thai and Tan [6] proved the following results.

Theorem A. *Let f, g be meromorphic functions on $\mathbb{C}$ and let $\{a_j\}_{j=1}^5$ be five distinct meromorphic functions in $\mathcal{R}(f) \cap \mathcal{R}(g)$. Assume that*

$$\nu^1_{f-a_j,\leq k} = \nu^1_{g-a_j,\leq k}, \quad 1 \leq j \leq 5.$$

Then $f = g$ for $k \geq 3$.

Theorem B. *Let f^1, f^2, f^3 be three meromorphic functions on $\mathbb{C}$ and let $\{a_j\}_{j=1}^4$ be four distinct meromorphic functions in $\mathcal{R}(f^1) \cap \mathcal{R}(f^2) \cap \mathcal{R}(f^3)$. Assume that*

$$\nu^2_{f^1-a_j,\leq k} = \nu^2_{f^2-a_j,\leq k} = \nu^2_{f^3-a_j,\leq k}, \quad 1 \leq j \leq 4.$$

Then $f^1 = f^2$, $f^2 = f^3$ or $f^3 = f^1$ for $k \geq 23$.

In this paper, we will give an improvement of Theorem B. Our main result is stated as follows.

Theorem 1.1. *Let f^1, f^2, f^3 be three meromorphic functions on $\mathbb{C}^n$, let $\{a_j\}_{j=1}^4$ be four distinct meromorphic functions in $\mathcal{R}(f^1) \cap \mathcal{R}(f^2) \cap \mathcal{R}(f^3)$, and let k_j $(1 \leq j \leq 4)$ be positive integers or ∞ satisfying*

$$k_1 \geq k_2 \geq k_3 \geq k_4.$$

Assume that

$$\nu^2_{f^1-a_j,\leq k_j} = \nu^2_{f^2-a_j,\leq k_j} = \nu^2_{f^3-a_j,\leq k_j}, \ 1 \leq j \leq 4.$$

If k_j $(1 \leq j \leq 4)$ satisfy one of the following conditions:

a) *$k_4 \geq 15$,*
b) *$k_4 = 14$, $k_1 = k_2 = k_3 = 16$,*
c) *$k_4 = 13$, $k_1 = k_2 = k_3 = 17$,*

then $f^1 = f^2$, $f^2 = f^3$ or $f^3 = f^1$.

2. Preliminaries and some lemmas

We first introduce some preliminaries in Nevanlinna theory.

We now define counting function. Let

$$n(t) = \begin{cases} \int_{|\nu_{f-a}|\bigcap B(t)} \nu_{f-a}(z)v & \text{if } n \geq 2, \\ \sum_{|z|\leq t} \nu_{f-a}(z) & \text{if } n = 1. \end{cases}$$

Similarly, we define $n^M(t)$, $n^M_{\leq k}(t)$ and $n^M_{>k}(t)$. We define

$$N_{f-a}(r,0) = \int_1^r \frac{n(t)}{t^{2n-1}}dt \quad (1 < r < +\infty).$$

Similarly, we define $N^M_{f-a}(r,0), N^M_{f-a,\leq k}(r,0)$ and $N^M_{f-a,>k}(r,0)$.

We define the proximity function of meromorphic function f by

$$m(r,f) = \int_{S(r)} \log^+ |f|\sigma.$$

Now we give some useful theorems and lemmas.

Theorem 2.1. *Let f be a nonconstant meromorphic function on $\mathbb{C}^n$. Let $a_1, \ldots, a_q$ be distinct meromorphic functions on $\mathbb{C}^n$. Assume that a_j are small functions with respect to f for all $1 \leq j \leq q$. Then, for each $\varepsilon > 0$,*

$$\|(q-2-\varepsilon)T(r,f) \leq \sum_{j=1}^q N^1_{f-a_j}(r,0) + o(T(r,f)),$$

where the notation "$\|$" means the inequality holds for all large r outside a set of finite Lebesgue measure. (We will use this notation throughout the paper.)

Proof. In [4], Yamanoi proved this second main theorem for $n = 1$. By the standard process of averaging over the complex lines in the complex space $\mathbb{C}^n$, one can easily extend his result to meromorphic functions on $\mathbb{C}^n$ for $n > 1$.

For any $\xi \in \mathbb{C}^n$ with $\|\xi\| = 1$, $\xi\mathbb{C}$ is a complex line through the origin in $\mathbb{C}^n$. We use f_ξ and $a_{\xi j}$ to denote the meromorphic functions of f and a_j restricted to line $\xi\mathbb{C}$, respectively. We note that f_ξ and $a_{\xi j}$ are meromorphic functions on $\mathbb{C}$. By Corollary 1 in [4], we have

$$\|(q-2-\varepsilon)T(r,f_\xi) \leq \sum_{j=1}^q N^1_{f_\xi - a_{\xi j}}(r,0) + O\left(\sum_{j=1}^q T(r,a_{\xi j})\right) + o(T(r,f_\xi)).$$

Integrating the above inequality over the projective space $\mathbb{P}^{n-1}$ of lines through the origin in $\mathbb{C}^n$ (cf. [5]), we have

$$\begin{aligned} \|(q-2-\varepsilon)T(r,f) &\leq \sum_{j=1}^q N^1_{f-a_j}(r,0) + O\left(\sum_{j=1}^q T(r,a_j)\right) + o(T(r,f)) \\ &\leq \sum_{j=1}^q N^1_{f-a_j}(r,0) + o(T(r,f)). \end{aligned}$$

□

Logarithmic derivative lemma. *Let f be a nonzero meromorphic function on $\mathbb{C}^n$. Then*

$$\| m(r, D^{\alpha}(f)/f) = o(T(r,f)) \quad (\alpha \in \mathbb{Z}_+^n).$$

For f^1, f^2, f^3, set $T(r) := \sum_{k=1}^{3} T(r, f^k)$.
We denote $\mathcal{R} = \mathcal{R}(f^1) \cap \mathcal{R}(f^2) \cap \mathcal{R}(f^3)$ and $\mathcal{S} = \mathcal{R} \setminus \{a_j\}_{j=1}^{q}$.

Lemma 2.2. *For each a_j, $1 \leq j \leq q$, there exists a sequence $\{c_n^j\}$ in $\mathcal{S}$ such that $\lim_{n \to +\infty} c_n^j = a_j$.*

Proof. It is easy to see that $ca_j \in \mathcal{S}$, where c in $\mathbb{C} \setminus \{0\}$ and $c \neq 1$. Let $\{c_n\}$ be a sequence in $\mathbb{C}$ and $c_n \to 1$ $(n \to +\infty)$. Set $c_n^j = c_n a_j$ and we get $\lim_{n \to +\infty} c_n^j = a_j$. □

Lemma 2.3. *For every $c \in \mathcal{S}$, we put $F_c^{jk} = \frac{f^k - a_j}{f^k - c}$. Then*

$$T(r, F_c^{jk}) = T(r, f^k) + o(T(r)).$$

The proof can be found in [6].

Definition 2.4. Let F, G, H be nonzero meromorphic functions on $\mathbb{C}^n$. Take $\alpha := (\alpha^0, \alpha^1)$ whose components α^k are composed of n nonnegative integers, and set $|\alpha| = |\alpha^0| + |\alpha^1|$. We define Cartan's auxiliary function by

$$\Phi^{\alpha} \equiv \Phi^{\alpha}(F, G, H) := F \cdot G \cdot H \cdot \begin{vmatrix} 1 & 1 & 1 \\ D^{\alpha^0}(\frac{1}{F}) & D^{\alpha^0}(\frac{1}{G}) & D^{\alpha^0}(\frac{1}{H}) \\ D^{\alpha^1}(\frac{1}{F}) & D^{\alpha^1}(\frac{1}{G}) & D^{\alpha^1}(\frac{1}{H}) \end{vmatrix}.$$

In [7], Fujimoto gave the following.

Lemma 2.5. *If $\Phi^{\alpha}(F, G, H) \equiv 0$ and $\Phi^{\alpha}(\frac{1}{F}, \frac{1}{G}, \frac{1}{H}) \equiv 0$ for all α with $|\alpha| \leq 1$, then one of the following assertions holds:*
(i) *$F = G$, $G = H$ or $H = F$.*
(ii) *$\frac{F}{G}$, $\frac{G}{H}$ and $\frac{H}{F}$ are all constant.*

Lemma 2.6. *Suppose that $\Phi^{\alpha}(F, G, H) \not\equiv 0$ with $|\alpha| \leq 1$. If*

$$\nu^{[d]} := min(\nu_{F, \leq k}, d) = min(\nu_{G, \leq k}, d) = min(\nu_{H, \leq k}, d)$$

for some $d \geq |\alpha|$, then $\nu_{\Phi^{\alpha}}(z_0) \geq min(\nu^{[d]}(z_0), d - |\alpha|)$ for every $z_0 \in \{z | \nu_{F, \leq k}(z) > 0\} \backslash A$, where A is an analytic subset of codim $A \geq 2$.

Lemma 2.7. *With the assumptions as in Lemma 2.6, if $F = G = H \not\equiv 0, \infty$ on an analytic subset of A of pure dimension $n-1$, then $\nu_{\Phi^{\alpha}}(z_0) \geq 2$, $\forall z_0 \in A$.*

For the proof of the above two lemmas, refer to [8].

Lemma 2.8. *Suppose that there exists $\Phi^{\alpha} = \Phi^{\alpha}(F_c^{j_0 1}, F_c^{j_0 2}, F_c^{j_0 3}) \not\equiv 0$ for some $c \in \mathcal{S}$ and some j_0, $1 \leq j_0 \leq 4, |\alpha| \leq 1$, $d \geq |\alpha|$. Then, for each $1 \leq i \leq 3$, the following holds:*

$$\| N^{d-|\alpha|}_{f^i - a_{j_0}, \leq k_{j_0}}(r, 0) + 2 \cdot \sum_{j \neq j_0} N^1_{f^i - a_j, \leq k_j}(r, 0) \leq N_{\Phi^{\alpha}}(r, 0) + o(T(r)) \leq T(r)$$

$$+ \sum_{l=1}^{3} N^1_{f^l - a_{j_0}, > k_{j_0}}(r, 0) + o(T(r)).$$

Proof. The proof is similar to that of Lemma 4.1.8 in [8]. We include the proof here for completeness. The first inequality is deduced immediately from Lemmas 2.6 and 2.7. On the other hand, we have

$$N_{\Phi^\alpha}(r,0) \leq T(r,\Phi^\alpha) + O(1) = N_{\Phi^\alpha}(r,+\infty) + m(r,\Phi^\alpha) + O(1).$$

We easily see that a pole of Φ^α is a zero or a pole of some $F_c^{j_0k}$ and Φ^α is holomorphic at all zeros with multiplicities $\leq k_{j_0}$ of $F_c^{j_0k}$ because of Lemma 2.6. We also see that if z_0 is a pole of $\frac{D^{\alpha^i}(1/F_c^{j_0k})}{1/F_c^{j_0k}}$, then it has the multiplicity $\leq |\alpha^i|$. Thus, if z_0 is a pole of Φ^α, then it has the multiplicity $\leq |\alpha| = \sum_{i=0}^{1}|\alpha^i| \leq 1$. This implies that

$$N_{\Phi^\alpha}(r,+\infty) \leq \sum_{l=1}^{3} N^1_{f^l-a_{j_0},>k_{j_0}}(r,0) + \sum_{l=1}^{3} N_{F_c^{j_0l}}(r,+\infty) + o(T(r))$$

and

$$\begin{aligned} m(r,\Phi^\alpha) &\leq \sum_{l=1}^{3} m(r,F_c^{j_0l}) + O\left(\sum m\left(r,\frac{D^{\alpha^i}(1/F_c^{j_0k})}{1/F_c^{j_0k}}\right)\right) + O(1) \\ &\leq \sum_{l=1}^{3} m(r,F_c^{j_0l}) + o(T(r)). \end{aligned}$$

Note that $\sum_{l=1}^{3} N_{F_c^{j_0l}}(r,+\infty) + \sum_{l=1}^{3} m(r,F_c^{j_0l}) = \sum_{l=1}^{3} T(r,F_c^{j_0l}) = T(r)$. Hence

$$N_{\Phi^\alpha}(r,0) \leq T(r) + \sum_{l=1}^{3} N^1_{f^l-a_{j_0},>k_{j_0}}(r,0) + o(T(r)).$$

□

Lemma 2.9. *Let f be a nonconstant meromorphic function on $\mathbb{C}^n$. Let $a_1,\dots,a_q$ be distinct meromorphic functions on $\mathbb{C}^n$. Assume that a_j are small functions with respect to f for all $1 \leq j \leq q$. Then, for each $\varepsilon > 0$,*

$$\| \left(q - 2 - \varepsilon - \sum_{j=1}^{q} \frac{1}{k_j+1}\right) T(r,f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j+1} N^1_{f-a_j,\leq k_j}(r,0) + o(T(r,f)).$$

Proof. By Theorem 2.1, we have

$$\begin{aligned} &\|(q-2-\varepsilon)T(r,f) \leq \sum_{j=1}^{q} N^1_{f-a_j}(r,0) + o(T(r,f)) \\ \leq\ & \sum_{j=1}^{q} N^1_{f-a_j,\leq k_j}(r,0) + \sum_{j=1}^{q} \frac{1}{k_j+1} N_{f-a_j,>k_j}(r,0) + o(T(r,f)) \\ =\ & \sum_{j=1}^{q} N^1_{f-a_j,\leq k_j}(r,0) + \sum_{j=1}^{q} \frac{1}{k_j+1}\left(N_{f-a_j}(r,0) - N_{f-a_j,\leq k_j}(r,0)\right) \\ & + o(T(r,f)) \\ \leq\ & \sum_{j=1}^{q} \frac{k_j}{k_j+1} N^1_{f-a_j,\leq k_j}(r,0) + \left(\sum_{j=1}^{q} \frac{1}{k_j+1}\right) T(r,f) + o(T(r,f)). \end{aligned}$$

Hence,

$$\| \left(q-2-\varepsilon-\sum_{j=1}^{q}\frac{1}{k_j+1}\right) T(r,f) \le \sum_{j=1}^{q}\frac{k_j}{k_j+1}N^1_{f-a_j,\le k_j}(r,0)+o(T(r,f)).$$

□

3. Proof of the main result

Denote by $\mathcal{Q}$ the set of all indices $j_0 \in \{1,2,3,4\}$ satisfying the following: There exist $c \in \mathcal{S}$ and $\alpha = (\alpha^0,\alpha^1)$ with $|\alpha| \le 1$ such that $\Phi^{\alpha}(F_c^{j_0 1}, F_c^{j_0 2}, F_c^{j_0 3}) \not\equiv 0$.

For each $1 \le i \le 3$ and $j_0 \in \mathcal{Q}$, by Lemma 2.8, we have

$$\|N^1_{f^i-a_{j_0},\le k_{j_0}}(r,0)+2\sum_{j\ne j_0}N^1_{f^i-a_j,\le k_j}(r,0) \le T(r)+\sum_{l=1}^{3}N^1_{f^l-a_{j_0},>k_{j_0}}(r,0)+o(T(r)).$$

This implies that

$$\begin{aligned}
&\|\sum_{i=1}^{3}\left(N^1_{f^i-a_{j_0},\le k_{j_0}}(r,0)+2\sum_{j\ne j_0}N^1_{f^i-a_j,\le k_j}(r,0)\right)\\
\le\ & 3T(r)+3\sum_{i=1}^{3}N^1_{f^i-a_{j_0},>k_{j_0}}(r,0)+o(T(r))\\
\le\ & 3T(r)+\left(\frac{3}{k_{j_0}+1}\right)\sum_{i=1}^{3}N_{f^i-a_{j_0},>k_{j_0}}(r,0)+o(T(r))\\
\le\ & 3T(r)+\left(\frac{3}{k_{j_0}+1}\right)\sum_{i=1}^{3}(N_{f^i-a_{j_0}}(r,0)-N_{f^i-a_{j_0},\le k_{j_0}}(r,0))+o(T(r))\\
\le\ & 3\left(\frac{k_{j_0}+2}{k_{j_0}+1}\right)T(r)-\left(\frac{3}{k_{j_0}+1}\right)\sum_{i=1}^{3}N^1_{f^i-a_{j_0},\le k_{j_0}}(r,0)+o(T(r)).
\end{aligned}$$

Hence

$$\begin{aligned}
&\|\sum_{i=1}^{3}\left((k_{j_0}+4)N^1_{f^i-a_{j_0},\le k_{j_0}}(r,0)+2(k_{j_0}+1)\sum_{j\ne j_0}N^1_{f^i,a_j,\le k_j}(r,0)\right)\\
\le\ & 3(k_{j_0}+2)T(r)+o(T(r)).
\end{aligned}$$

This means that

$$\begin{aligned}
\|2(k_{j_0}+1)\sum_{i=1}^{3}\sum_{j=1}^{4}N^1_{f^i-a_j,\le k_j}(r,0) &\le 3(k_{j_0}+2)T(r)\\
&+(k_{j_0}-2)\sum_{i=1}^{3}N^1_{f^i-a_{j_0},\le k_{j_0}}(r,0)+o(T(r)).
\end{aligned}$$

By $k_1 \ge k_2 \ge k_3 \ge k_4$, we have

$$1 \ge \frac{k_1}{k_1+1} \ge \frac{k_2}{k_2+1} \ge \frac{k_3}{k_3+1} \ge \frac{k_4}{k_4+1} \ge \frac{1}{2}.$$

Using Lemma 2.9, we obtain, for $1 \leq i \leq 3$,

$$\| \left(2-\varepsilon-\sum_{j=1}^{4}\frac{1}{k_j+1}\right)T(r,f^i) \leq \sum_{j=1}^{4}\frac{k_j}{k_j+1}N^1_{f^i-a_j,\leq k_j}(r,0)+o(T(r,f^i))$$
$$\leq \sum_{j=1}^{4}\frac{k_1}{k_1+1}N^1_{f^i-a_j,\leq k_j}(r,0)+o(T(r,f^i)).$$

Hence

$$\|2(k_{j_0}+1)\left(2-\varepsilon-\sum_{j=1}^{4}\frac{1}{k_j+1}\right)\left(\frac{k_1+1}{k_1}\right)T(r)$$
$$\leq 3(k_{j_0}+2)T(r)+(k_{j_0}-2)\sum_{i=1}^{3}N^1_{f^i-a_{j_0},\leq k_{j_0}}(r,0)+o(T(r))$$

and

$$\|\sum_{i=1}^{3}N^1_{f^i-a_{j_0},\leq k_{j_0}}(r,0)$$
$$\geq \left(\frac{2(k_{j_0}+1)}{k_{j_0}-2}\left(2-\varepsilon-\sum_{j=1}^{4}\frac{1}{k_j+1}\right)\left(\frac{k_1+1}{k_1}\right)-\frac{3(k_{j_0}+2)}{k_{j_0}-2}\right)T(r)$$
$$+o(T(r)).$$

Now, we will show that $\sharp\mathcal{Q}\leq 2$. If $\sharp\mathcal{Q}\geq 3$, i.e., $\mathcal{Q}\supset\{j_0,j_1,j_2\}$, we get

$$\|\sum_{s=0}^{2}\sum_{i=1}^{3}N^1_{f^i-a_{j_s},\leq k_{j_s}}(r,0)$$
$$\geq \sum_{s=0}^{2}\left(\frac{2(k_{j_s}+1)}{k_{j_s}-2}\left(2-\varepsilon-\sum_{j=1}^{4}\frac{1}{k_j+1}\right)\left(\frac{k_1+1}{k_1}\right)-\frac{3(k_{j_s}+2)}{k_{j_s}-2}\right)T(r) \tag{3.1}$$
$$+o(T(r)).$$

Since $f^1\not\equiv f^2\not\equiv f^3\not\equiv f^1$, it implies that

$$\sum_{s=0}^{2}N^1_{f^i-a_{j_s},\leq k_{j_s}}(r,0)\leq N_{f^1-f^2}(r,0)+o(T(r))\leq T(r,f^1)+T(r,f^2)+o(T(r)).$$

Similarly,

$$\sum_{s=0}^{2}N^1_{f^i-a_{j_s},\leq k_{j_s}}(r,0)\leq T(r,f^2)+T(r,f^3)+o(T(r)),$$
$$\sum_{s=0}^{2}N^1_{f^i-a_{j_s},\leq k_{j_s}}(r,0)\leq T(r,f^1)+T(r,f^3)+o(T(r)).$$

Hence

$$\sum_{s=0}^{2}N^1_{f^i-a_{j_s},\leq k_{j_s}}(r,0)\leq \frac{2}{3}T(r)+o(T(r)), \quad 1\leq i\leq 3.$$

By (3.1), we have

$$
\begin{aligned}
&\|3\cdot\frac{2}{3}T(r)\\
\geq\ & \sum_{s=0}^{2}\left(\frac{2(k_{j_s}+1)}{k_{j_s}-2}\left(2-\varepsilon-\sum_{j=1}^{4}\frac{1}{k_j+1}\right)\left(\frac{k_1+1}{k_1}\right)-\frac{3(k_{j_s}+2)}{k_{j_s}-2}\right)T(r)\\
&+o(T(r)).
\end{aligned}
$$

Let $r\to+\infty$ and $\varepsilon\to 0$, and we get

$$
2\geq\sum_{s=0}^{2}\left(\frac{2(k_{j_s}+1)}{k_{j_s}-2}\left(2-\sum_{j=1}^{4}\frac{1}{k_j+1}\right)\left(\frac{k_1+1}{k_1}\right)-\frac{3(k_{j_s}+2)}{k_{j_s}-2}\right).
$$

i) $k_4\geq 15$.

In fact, we only need to verify the case of $k_1=k_2=k_3=k_4=15$. We have

$$
\begin{aligned}
\frac{2(k_{j_s}+1)}{k_{j_s}-2}\left(2-\sum_{j=1}^{4}\frac{1}{k_j+1}\right)\left(\frac{k_1+1}{k_1}\right)-\frac{3(k_{j_s}+2)}{k_{j_s}-2} &= \frac{2(16)}{13}\cdot\frac{7}{4}\cdot\frac{16}{15}-\frac{3(17)}{13}\\
&\geq 0.67179.
\end{aligned}
$$

This gives a contradiction:

$$
2\geq 3\times 0.67179=2.01537.
$$

Hence $\sharp\mathcal{Q}\leq 2$.

ii) $k_4=14$, $k_3=k_2=k_1=16$.

For $k_4=14$, we get

$$
\frac{2(k_4+1)}{k_4-2}\left(2-\sum_{j=1}^{4}\frac{1}{k_j+1}\right)\left(\frac{k_1+1}{k_1}\right)-\frac{3(k_4+2)}{k_4-2}\geq 0.6665.
$$

For $k_1=k_2=k_3=16$, we get

$$
\frac{2(k_i+1)}{k_i-2}\left(2-\sum_{j=1}^{4}\frac{1}{k_j+1}\right)\left(\frac{k_1+1}{k_1}\right)-\frac{3(k_i+2)}{k_i-2}\geq 0.67602,\quad 1\leq i\leq 3.
$$

If $\mathcal{Q}\supset\{1,2,3\}$, this gives a contradiction

$$
2\geq 3\times 0.67602=2.02806.
$$

If $\mathcal{Q}\supset\{1,2,4\}$ ($\{1,3,4\}$ or $\{2,3,4\}$), this gives a contradiction

$$
2\geq 0.6665+2\times 0.67602=2.01854.
$$

Hence $\sharp\mathcal{Q}\leq 2$.

iii) $k_4=13$, $k_3=k_2=k_1=17$. For $k_4=13$, we get

$$
\frac{2(k_4+1)}{k_4-2}\left(2-\sum_{j=1}^{4}\frac{1}{k_j+1}\right)\left(\frac{k_1+1}{k_1}\right)-\frac{3(k_4+2)}{k_4-2}\geq 0.6577411.
$$

For $k_1=k_2=k_3=17$, we get

$$
\frac{2(k_i+1)}{k_i-2}\left(2-\sum_{j=1}^{4}\frac{1}{k_j+1}\right)\left(\frac{k_1+1}{k_1}\right)-\frac{3(k_i+2)}{k_i-2}\geq 0.6772988,\quad 1\leq i\leq 3.
$$

If $\mathcal{Q} \supset \{1,2,3\}$, this gives a contradiction

$$2 \geq 3 \times 0.6772988 = 2.0318964.$$

If $\mathcal{Q} \supset \{1,2,4\}$ ($\{1,3,4\}$ or $\{2,3,4\}$), this gives a contradiction

$$2 \geq 0.6577411 + 2 \times 0.6772988 = 2.0123387.$$

Hence $\sharp\mathcal{Q} \leq 2$.

It is easy to see that $\{1,2,3,4\} \setminus \mathcal{Q}$ contains at least 2 indices. Without loss of generality, we may assume that $\{1,2\} \subset \{1,2,3,4\} \setminus \mathcal{Q}$. By Lemma 2.2, it implies that $\Phi^\alpha(F_2^{11}, F_2^{12}, F_2^{13}) \equiv 0$ and $\Phi^\alpha(F_1^{21}, F_1^{22}, F_1^{23}) \equiv 0$ for all α with $|\alpha| \leq 1$.

Applying Lemma 2.5 for F_2^{11}, F_2^{12} and F_2^{13}, there are the following two cases:

i) There exist $1 \leq l_1 < l_2 \leq 3$ such that $F_2^{1l_1} = F_2^{1l_2}$. Then $f^{l_1} \equiv f^{l_2}$.

ii) There are two distinct constants $\alpha, \beta \in \mathbb{C} \setminus \{0,1\}$ such that $F_2^{11} = \alpha F_2^{12} = \beta F_2^{13}$.

Now, we claim that this is impossible.

First, we show that $\nu_{f^i - a_3}(z) \geq k_3 + 1$ $(1 \leq i \leq 3)$ for $z \in A$, where

$$A := \left(\bigcup_{i=1}^{3} \{z | (f^i - a_3)(z) = 0\}\right) \setminus \{z | (a_3 - a_1)(z) = 0 \text{ or } (a_3 - a_2)(z) = 0\}.$$

In fact, if there exist $1 \leq j \leq 3$ and $z_0 \in A$ such that $0 < \nu_{f^j - a_3}(z_0) \leq k_3$, then

$$(f^1 - a_3)(z_0) = (f^2 - a_3)(z_0) = (f^3 - a_3)(z_0) = 0.$$

Hence $F_2^{11}(z_0) = F_2^{12}(z_0) = \frac{(a_3 - a_1)}{(a_3 - a_2)}(z_0) \neq 0, \infty$, so that $\alpha = 1$, which is a contradiction to $\alpha \in \mathbb{C} \setminus \{0,1\}$.

Let

$$b_1 = \frac{1}{\alpha}\frac{a_3 - a_1}{a_3 - a_2},\ b_2 = \frac{\beta}{\alpha}\frac{a_3 - a_1}{a_3 - a_2},\ b_3 = \frac{a_3 - a_1}{a_3 - a_2}.$$

Obviously, $b_1, b_2, b_3 \in \mathcal{R}$, and we have

$$\begin{aligned} \nu_{F_2^{12} - b_3} &= \nu_{\frac{(f^2 - a_3)}{(f^2 - a_2)}\frac{(a_1 - a_2)}{(a_3 - a_2)}}, \\ \nu_{F_2^{12} - b_1} &= \nu_{F_2^{11} - \alpha b_1} = \nu_{\frac{(f^1 - a_3)}{(f^1 - a_2)}\frac{(a_1 - a_2)}{(a_3 - a_2)}}, \\ \nu_{F_2^{12} - b_2} &= \nu_{F_2^{13} - \frac{\alpha}{\beta} b_1} = \nu_{\frac{(f^3 - a_3)}{(f^3 - a_2)}\frac{(a_1 - a_2)}{(a_3 - a_2)}}. \end{aligned} \tag{3.2}$$

It is easy to see that $a_i - a_j = 0$ on $\{z | (f^k - a_i)(z) = 0 \text{ and } (f^k - a_j)(z) = 0\}$, $1 \leq i,\ j \leq 4$, $1 \leq k \leq 3$.

By $\nu_{f^i - a_3}(z) \geq k_3 + 1$ $(1 \leq i \leq 3)$ and (3.2), we have $\nu_{F_2^{12} - b_j} \geq k_3 + 1$ on

$$\{z | (F_2^{12} - b_j)(z) = 0\} \setminus \{z | (a_1 - a_2)(z) \cdot (a_1 - a_3)(z) \cdot (a_2 - a_3)(z) = 0\}$$

for $1 \leq j \leq 3$. Hence, for $1 \leq j \leq 3$,

$$\begin{aligned} N^1_{F_2^{12} - b_j}(r,0) &\leq \frac{1}{k_3 + 1} N_{F_2^{12} - b_j}(r,0) + N_{a_1 - a_2}(r,0) + N_{a_1 - a_3}(r,0) \\ &\quad + N_{a_2 - a_3}(r,0) \\ &\leq \frac{1}{k_3 + 1} N_{F_2^{12} - b_j}(r,0) + o(T(r)) \\ &= \frac{1}{k_3 + 1} N_{F_2^{12} - b_j}(r,0) + o(T(r, F_2^{12})). \end{aligned}$$

Using Theorem 2.1, we have

$$\begin{aligned}
\|(3-2-\varepsilon)T(r,F_2^{12}) &\leq \sum_{j=1}^{3} N^1_{F_2^{12}-b_j}(r,0)+o(T(r,F_2^{12})) \\
&\leq \frac{1}{k_3+1}\sum_{j=1}^{3} N_{F_2^{12}-b_j}(r,0)+o(T(r,F_2^{12})) \\
&\leq \frac{3}{k_3+1}T(r,F_2^{12})+o(T(r,F_2^{12})).
\end{aligned}$$

This is a contradiction for $k_3 \geq 3$.

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