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ON THE IRREDUCIBILITY OF THE HILBERT SCHEME OF SPACE CURVES

HRISTO ILIEV

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Abstract. Denote by $H_{d,g,r}$ the Hilbert scheme parametrizing smooth irreducible complex curves of degree d and genus g embedded in $\mathbb{P}^r$. In 1921 Severi claimed that $H_{d,g,r}$ is irreducible if $d \geq g+r$. As it has turned out in recent years, the conjecture is true for $r=3$ and 4, while for $r \geq 6$ it is incorrect. We prove that $H_{g,g,3}$, $H_{g+3,g,4}$ and $H_{g+2,g,4}$ are irreducible, provided that $g \geq 13$, $g \geq 5$ and $g \geq 11$, correspondingly. This augments the results obtained previously by Ein (1986), (1987) and by Keem and Kim (1992).

1. Introduction

In 1921 Severi asserted that the Hilbert scheme $H_{d,g,r}$ parametrizing smooth integral complex curves of degree d and genus g embedded in $\mathbb{P}^r$ is irreducible for $d \geq g+r$ [Sev21] . Harris [Ein87, Prop. 9] and Keem [Kee94] show by examples that the claim is incorrect for $r \geq 6$. Meanwhile Ein proved Severi's claim for $r=3$ and 4. Shortly after, Keem and Kim gave a different proof for the case of $r=3$ in [KK92], where they also proved the irreducibility of $H_{g+2,g,3}$ if $g \geq 5$ and of $H_{g+1,g,3}$ if $g \geq 11$. The goal of this paper is to extend the results in the cases of $r=3$ and $r=4$. We focus on the case $\rho(d,g,r)>0$, where $\rho(d,g,r)=g-(r+1)(g-d+r)$ is the Brill-Noether number. Our first result extends the irreducibility range of $H_{d,g,3}$.

Theorem 3.1. *$H_{g,g,3}$ is irreducible provided that $g \geq 13$.*

The second result extends the irreducibility range about curves in $\mathbb{P}^4$.

Theorem 3.2. (a) *$H_{g+3,g,4}$ is irreducible if $g \geq 5$.*
(b) *$H_{g+2,g,4}$ is irreducible if $g \geq 11$.*

Our approach follows the one established through the works of Arbarello-Cornalba, Ein and Keem-Kim. Let g, r and d be non-negative integers. Consider the moduli space $\mathcal{M}_g$ of smooth curves of genus g. For any given $p \in \mathcal{M}_g$ there exist a neighborhood $U \subset \mathcal{M}_g$ of p and a smooth connected variety $\mathcal{M}$ which is a finite ramified covering $h : \mathcal{M} \to U$, and also varieties $\mathcal{C}$, $\mathcal{W}^r_d$ and $\mathcal{G}^r_d$ proper over $\mathcal{M}$ with the following properties.

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(1) $\xi : \mathcal{C} \to \mathcal{M}$ is a universal curve, i.e. for every $p \in \mathcal{M}$, $\xi^{-1}(p)$ is a smooth curve of genus g isomorphic to $h(p)$,
(2) $\mathcal{W}_d^r$ parametrizes the pairs (p, L), where L is a line bundle of degree d and $h^0(L) \geq r+1$,
(3) $\mathcal{G}_d^r$ parametrizes the couples $(p, \mathcal{D})$, where $\mathcal{D}$ is a linear series of dimension r and degree d on $h(p)$.

The main point in the above construction is the existence of a universal family over $\mathcal{M}$. Intuitively, the varieties $\mathcal{W}_d^r$ and $\mathcal{G}_d^r$ can be viewed as patching of the varieties $W_d^r(C)$ and $G_d^r(C)$ when the curve C moves in a subset of $\mathcal{M}_g$. Also, there exists the so-called *relative Picard scheme* $\mathcal{P}ic\,\mathcal{C}$, which is just the relative analogue of $\mathrm{Pic}(C)$ defined for a fixed curve C. For further details see [AC81], [AC83] and [ACGH].

Let $\mathcal{G}$ be the union of components of $\mathcal{G}_d^r$ whose general element $(p, \mathcal{D})$ represents a very ample linear series $\mathcal{D}$ on the curve $C = \xi^{-1}(p)$. Since an irreducible component of $H_{d,g,r}$ is a $\mathrm{PGL}(r+1)$-fiber bundle over a component of $\mathcal{G}$, to establish irreducibility of $H_{d,g,r}$ it is sufficient to prove that $\mathcal{G}$ is irreducible. Regarding the existence of $H_{d,g,r}$, or equivalently of $\mathcal{G}$, we remark that for $d \geq g+r$ it follows by the Halphen's theorem [Har77, IV., Proposition 6.1], while for $d < g+r$ an answer, sufficient for our purposes, is given by Sernesi in [Ser84], namely:

Proposition 1.1. *For all non-negative integers g, r, d such that $r \geq 3$, $d \geq r+1$ and*

$$g - d + r \geq \max\{0, 1 - \rho(d, g, r)\},$$

there exists a regular component V of $H_{d,g,r}$ which has the expected number of moduli. A general point of V corresponds to an embedding $C \hookrightarrow \mathbb{P}^r$ by a complete linear system (i.e. $h^0(C, \mathcal{O}_C(1)) = r+1$), the normal bundle N_C satisfies $H^1(C, N_C) = 0$ and the cup-product map

$$\mu_0(C) : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, K_C(-1)) \to H^0(C, K_C)$$

is of maximal rank.

The following facts will be used in what follows. For proofs see [AC81] and [AC83].

Proposition 1.2. (1) *The dimension of any component of $\mathcal{G}_d^r$ is at least $3g - 3 + \rho(d, g, r)$.*
(2) *Assuming that $\rho(d, g, r) \geq 1$ and $g \geq 2$, there exists a unique component $\mathcal{G}_0 \subset \mathcal{G}$ such that $\mathcal{G}_0$ has general moduli and further the linear series of its general element is very ample (this component is called principal component).*
(3) *$\mathcal{G}_d^1$ is a smooth and irreducible variety of dimension $3g - 3 + \rho(d, g, 1)$.*
(4) *The dimension of any component of $\mathcal{G}_d^2$ whose general point contains a birationally very ample linear series is $3g - 3 + \rho(d, g, 2)$.*
(5) *If $\rho(d, g, 2) \geq 1$ then $\mathcal{G}_d^2$ has a unique component such that its general element carries a birationally very ample linear series.*
(6) *If $\mathcal{W} \subset \mathcal{W}_d^r$ is an irreducible component whose general element is a pair of a curve C and a very ample line bundle L such that $h^0(C, L) = r+1$, then for the dimension of the Zariski tangent space to $\mathcal{W}$ at (C, L) we have*

$$\dim T_{(C,L)}\mathcal{W} = 3g - 3 + \rho(d, g, r) + h^1(C, N_{C,\mathbb{P}^r}),$$

where $N_{C,\mathbb{P}^r}$ is the normal sheaf of $C \hookrightarrow \mathbb{P}^r$ of the embedding induced by L.

Next we recall a result of Ein, which is in fact the technical crux of his proof of Severi's claim for $r = 3$ and 4.

Proposition 1.3. *Let C be a smooth curve and L a very ample line bundle on C of degree d and dimension $r \geq 3$. Consider the embedding $C \hookrightarrow \mathbb{P}^r$ induced by L. Denote $\delta := h^1(C, L)$. Then*

(1) *if $\delta \leq 2$, then $h^1(C, N_{C,\mathbb{P}^r}) = 0$;*
(2) *if $\delta \geq 2$, then $h^1(C, N_{C,\mathbb{P}^r}) \leq (r-2)(\delta - 2)$.*

We will also need an estimate of the dimension of a component $\mathcal{W}$ of $\mathcal{W}^r_d$ used in Keem-Kim's proof of the irreducibility of $H_{d,g,3}$.

Proposition 1.4. *If $\mathcal{G}' \subset \mathcal{G}^r_d$, with $r \geq 2$, is a closed subvariety whose general element carries a special birationally very ample linear series, then*

$$\dim \mathcal{G}' \leq 3d + g - 4r - 1.$$

2. Preliminary results

First we give an upper bound for the dimension of an irreducible component $\mathcal{W} \subset \mathcal{W}^r_d$.

Proposition 2.1. *Let d, g and r be positive integers such that $2 \leq r$, $0 < d \leq g + r - 2$ and let $\mathcal{W}$ be an irreducible component of $\mathcal{W}^r_d$. Let b be the degree of the base locus of the linear series $|D|$ on C, for a general element $(C, |D|) \in \mathcal{W}$. Assume also that for general $(C, |D|) \in \mathcal{W}$ the curve C is not hyperelliptic. If the moving part of $|D|$ is*

(a.1) *very ample and $r \geq 3$, then* $\dim \mathcal{W} \leq 3d + g + 1 - 5r - 2b$*;*
(a.2) *birationally very ample, then* $\dim \mathcal{W} \leq 3d + g - 1 - 4r - 2b$*;*
(b) *compounded, then* $\dim \mathcal{W} \leq 2g - 1 + d - 2r$.

Proof. (a.1) Consider first the projection $\mathcal{W} \times_{\mathcal{M}} \mathcal{C}_b \to \mathcal{W}^{r-b}_{d-b}$ given by

$$((C, |D|), (P_1 + P_2 + ... + P_b)) \mapsto ((C, |D - (P_1 + P_2 + ... + P_b)|).$$

There exists a component $\mathcal{W}_b \subset \mathcal{W}^r_{d-b}$ whose general element carries a basepoint free linear series. For its dimension we have $\dim \mathcal{W}_b = \dim \mathcal{W} - b$, where b is the dimension of the fiber. Now the dimension of $\dim \mathcal{W}_b$ is estimated using Proposition 1.2 and Proposition 1.3. Consider a general element $l_b := (C, |D|) \in \mathcal{W}_b$, so the linear series $|D|$ of degree $d - b$ is very ample on C. Then

$$\dim \mathcal{W}_b \leq \dim T_l \mathcal{W}_b = 3g - 3 + \rho(d - b, g, r) + h^1(C, N_{C,\mathbb{P}^r}).$$

By Proposition 1.3, $h^1(C, N_{C,\mathbb{P}^r}) \leq (r-2)(g - d + b + r - 2)$, which yields the desired inequality.

(a.2) Similar to (a.1), but we apply Proposition 1.4 instead.

(b) Consider a general $l := (C, |D|) \in \mathcal{W}$, and assume that $|D|$ is a compounded linear series on C. This means that the morphism $\Phi_{|D|} : C \to \Gamma = \Phi_{|D|}(C)$ is of degree $n \geq 2$. Let γ be the geometric genus of Γ and consider first the case $\gamma \geq 1$ (in fact, we can assume that Γ is smooth). Denote by $\mathcal{X}_{n,\gamma}$ the set of points in moduli space $\mathcal{M}_g$ representing smooth curves which are $n : 1$ covers of smooth curves of genus γ. By the well-known Riemann's moduli count for $\gamma \geq 1$,

$$\dim \mathcal{X}_{n,\gamma} \leq 2g - 2 + (2n - 3)(1 - \gamma).$$

According to H.Martens's theorem [ACGH, IV.5], the dimension of the fiber $W^r_d(C)$ of $\mathcal{W}$ over a point $C \in \mathcal{X}_{n,\gamma}$ does not exceed $d-2r$; therefore for $\gamma \geq 1$ we find

$$\begin{aligned}\dim \mathcal{W} &\leq \mathcal{X}_{n,\gamma} + \dim W^r_b(C)\\ &\leq 2g-2+(2n-3)(1-\gamma)+d-2r\\ &< 2g-1+d-2r.\end{aligned}$$

If $\gamma = 0$, C is an $n-$sheeted covering of $\mathbb{P}^1$, and the complete linear series g^r_{d-b} is transformed into a complete linear series $g^r_{\frac{d-b}{n}} = rg^1_1$ on $\mathbb{P}^1$. Pulling it back, we get an injective map $\mathcal{W}^r_{d-b} \to \mathcal{W}^1_n$. Since $\dim \mathcal{W}^1_n \leq \dim \mathcal{G}^1_n = 2g+2n-5$ and $n = \frac{d-b}{r}$, we find that

$$\dim \mathcal{W} \leq \dim \mathcal{G}^1_n + b = 2g-5+2\frac{d}{r} - 2\frac{b}{r} + b.$$

By Clifford's inequality $0 \leq b \leq d-2r-1$, and since $-2\frac{b}{r}+b$ is a non-decreasing function in b for $r \geq 2$, we get

$$\begin{aligned}\dim \mathcal{W} &\leq 2g-5+2\frac{d}{r} - 2\frac{d-2r-1}{r} + d-2r-1\\ &= 2g-2+(d-2r)+\frac{2}{r}\\ &\leq 2g-1+(d-2r).\end{aligned}$$

This completes the proof of the proposition. □

The next lemma deals with a specific situation which occurs in the course of the proof.

Lemma 2.2. *Let $\mathcal{W} \subset \mathcal{W}^s_e$, $0 < e < 2g-2$, be an irreducible component whose general element $l \in \mathcal{W}$ represents a birationally very ample but not very ample line bundle L on a curve C, and assume further that the moving part of its dual, i.e. $K_C \otimes L^{-1}$, is birationally very ample. Consider the open subset $\mathcal{V} \subset \mathcal{W}$ consisting of $(C, L) \in \mathcal{W}$ for which L is basepoint free, birationally very ample on C and $h^0(C, L) = s+1$. Since for $(C, L) \in \mathcal{V}$ the line bundle L cannot be very ample, there exist points $p, p' \in C$ such that $(C, L(-p-p')) \in \mathcal{W}^{s-1}_{e-2} = \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_m$, where $\mathcal{W}_1, \ldots, \mathcal{W}_m$ are the irreducible components of $\mathcal{W}^{s-1}_{e-2}$. Let $X \subset \mathcal{V} \times_{\mathcal{M}} \mathcal{C}_2$ be the subset defined as $X := \{(L, p+q) \in \mathcal{V} \times_{\mathcal{M}} \mathcal{C}_2 | h^0(C, L(-p-q)) \geq s\}$ and consider the projection*

$$\begin{aligned}X &\to \mathcal{W}^{s-1}_{e-2},\\ (C, L; p+p') &\mapsto (C, L(-p-p')).\end{aligned}$$

Let $\mathcal{W}_1$ be the component of $\mathcal{W}^{s-1}_{e-2}$, where the elements of X are generically projected. Then

$$\dim \mathcal{W} = \dim \mathcal{V} \leq \dim \mathcal{W}_1.$$

Proof. For the proof it is enough to show that for any $(C, L) \in \mathcal{V}$ and $p, p' \in C$ such that $(C, L(-p-p')) \in \mathcal{W}_1$, there are at most finitely many $(C, M) \in \mathcal{V}$ and $q, q' \in C$ such that

$$L(-p-p') \cong M(-q-q').$$

Assume that $(C, L; p+p')$, $(C, M; q+q') \in X$ have the above property. A simple calculation gives

$$\begin{aligned} h^0(C, K_C \otimes L^{-1}(p+p'-q-q')) &= h^0(C, K_C \otimes M^{-1}) \\ &= h^0(C, K_C \otimes L^{-1}) \\ &= h^0(C, K_C \otimes L^{-1}(p+p')) - 1. \end{aligned}$$

By assumption, the moving part of $K_C \otimes L^{-1}$ is birationally very ample, hence the moving part of $K_C \otimes L^{-1}(p+p')$ will also be birationally very ample. Therefore, for the given C there exist at most finitely many pairs (q, q') such that the above equality holds. But giving such a pair (q, q') determines the line bundle M up to isomorphism by

$$L(-p-p'+q+q') \cong M,$$

and it follows that there are at most finitely many line bundles M with this property. This implies $\dim \mathcal{W} = \dim \mathcal{V} = \dim X \leq \dim \mathcal{W}_1$. □

We will also use the next lemma.

Lemma 2.3. *Let g^r_d be a birationally very ample linear series of degree $d \geq g$ on a smooth curve of genus g. Then*

$$r \leq \frac{1}{3}(2d - g + 1).$$

Proof. The lemma is obtained using Castelnuovo's inequality

$$g \leq \binom{m}{2}(r-1) + m\varepsilon,$$

where $m := [\frac{d-1}{r-1}]$ and $\varepsilon = d - 1 - m(r-1)$. □

3. Irreducibility of $H_{g,g,3}$, $H_{g+3,g,4}$ and $H_{g+2,g,4}$

Theorem 3.1. *$H_{g,g,3}$ is irreducible if $g \geq 13$.*

Proof. As was pointed out in the beginning, it is sufficient to prove that there is a unique irreducible component $\mathcal{G}' \subset \mathcal{G}^3_g$ whose general element carries a very ample linear series. The existence of such a component follows by Proposition 1.1. Further, by Proposition 1.2 we have $\dim \mathcal{G}' \geq 3g - 3 + \rho(g, g, 3)$. Let r be the dimension $r = h^0(C, |\mathcal{D}|) - 1$, where $(C, \mathcal{D}) \in \mathcal{G}'$ is a general element, and let $\mathcal{W}' \subset \mathcal{W}^r_g$ be the component whose general elements contain the completions $|\mathcal{D}|$. Thus, $\dim \mathcal{G}' = \dim \mathcal{W}' + \dim \mathrm{Gr}(4, r+1)$, and by Proposition 2.1 we find

$$4g - 15 = 3g - 3 + \rho(g, g, 3) \leq \dim \mathcal{G}' \leq 4g - r - 11,$$

which implies

$$r \leq 4. \tag{1}$$

Let $\mathcal{W}_0 \subset \mathcal{W}^{r-1}_{g-2}$ be the dual of $\mathcal{W}'$ in the Picard scheme $\mathcal{P}ic\,\mathcal{C}$.

The curve C in the general element $(C, K_C(-\mathcal{D}))$ of $\mathcal{G}'$ cannot be hyperelliptic since a hyperelliptic curve cannot have a very ample special linear series. Then it is easy to see that the linear series of the general element of $\mathcal{W}_0$ is birationally very ample, after the clearing of a possible base part. If we assume the opposite we find by Proposition 2.1 that

$$\begin{aligned} 4g - 15 &\leq \dim \mathcal{W}' + 4(r-3) \\ &= \dim \mathcal{W}_0 + 4(r-3) \\ &\leq 2g - 1 + ((g-2) - 2(r-1)) + 4(r-3) \\ &= 3g + 2r - 13, \end{aligned}$$

and this implies $g \leq 2r + 2$. The last is impossible due to (1) and the assumption $g \geq 13$. This means that the moving part of the linear series in a general element $(C, |D|) \in \mathcal{W}_0$ is birationally very ample. Let b be the degree of its base locus. Again applying Proposition 2.1, we obtain

$$4g - 15 \leq \dim \mathcal{W}_0 + 4(r-3) \leq 4g - 15 - 2b,$$

which implies $b = 0$, i.e. $|D|$ is basepoint free and also

$$\dim \mathcal{W}' = \dim \mathcal{W}_0 = 4g - 4r - 3.$$

Now I claim that we can only have $r = 3$. Indeed, by inequality (1) it is enough to check that $r \neq 4$ since $r \geq 3$ by default. So, assume that $r = 4$. Then

$$\dim \mathcal{W}_0 = 4g - 4r - 3 = 4g - 19.$$

For a general $(C, L) \in \mathcal{W}_0$ the line bundle L cannot be very ample, because if it were very ample, then applying Proposition 2.1 we would get

$$\dim \mathcal{W}_0 \leq 3(g-2) + g + 1 - 5(4-1) = 4g - 20,$$

which is impossible. Thus, for a general $(C, L) \in \mathcal{W}_0$ the line bundle L is basepoint free and birationally very ample but not very ample on C. Let $\mathcal{W}_1 \subset \mathcal{W}^2_{g-4}$ be a component chosen like the component $\mathcal{W}_1$ in Lemma 2.2. Therefore,

$$4g - 19 = \dim \mathcal{W}_0 \leq \dim \mathcal{W}_1.$$

Now we estimate the dimension of $\mathcal{W}_1$. If its general element has a compounded linear series, applying Proposition 2.1 we get

$$4g - 19 \leq \dim \mathcal{W}_1 \leq 2g - 1 + g - 4 - 4 = 3g - 9,$$

which is impossible due to the assumption $g \geq 13$. So, it remains for its general element to carry a birationally very ample linear series, but then by the same proposition we must have

$$4g - 19 \leq \dim \mathcal{W}_1 \leq 3(g-4) + g - 8 - 1 = 4g - 21,$$

which is again a contradiction. This proves that $r = 3$ is the only possibility.

Now we can complete the proof of the theorem. For any component $\mathcal{G}' \subset \mathcal{G}^3_g$ whose general element carries a very ample linear series, we have established that:

- its general element has linear series that is necessarily complete;
- the general element of its dual $\mathcal{W}_0 \subset \mathcal{W}^2_{g-2}$ carries a basepoint free and birationally very ample linear series.

Let $\mathcal{G}$ be the union of components of $\mathcal{G}^2_{g-2}$ whose general element has a birationally very ample linear series. By Proposition 1.2, $\dim \mathcal{G} = 4g - 15$, and since $\dim \mathcal{W}_0 = 4g - 15$, it turns out that $\mathcal{W}_0$ is an irreducible component of $\mathcal{G}$. But since $\rho(g-2, g, 2) = \rho(g, g, 3) \geq 1$, as we assume $g \geq 13$, it follows again by Proposition 1.2 that $\mathcal{G}$ is irreducible. On its turn this means that the union of irreducible components of $\mathcal{G}^3_g$ whose general element has a very ample linear series must be irreducible. Therefore $H_{g,g,3}$ is irreducible. □

Theorem 3.2. (a) $H_{g+3,g,4}$ *is irreducible if* $g \geq 5$.
(b) $H_{g+2,g,4}$ *is irreducible if* $g \geq 11$.

Proof. (a) For the first part it is sufficient to check that $\mathcal{G}^4_{g+3}$ has a unique component whose general element represents a curve and a very ample linear series on it. Let $\mathcal{G}' \subset \mathcal{G}^4_{g+3}$ be a component with this property; the existence of such components follows from Proposition 1.1. For its dimension we have

$$\dim \mathcal{G}' \geq 3g - 3 + \rho(g+3, g, 4) = 4g - 8.$$

I claim now that the linear series in a general element of $\mathcal{G}'$ is complete. To see this denote by $\mathcal{W}' \subset \mathcal{W}^r_{g+3}$ the component containing the completed linear series $|\mathcal{D}|$ for $(C, \mathcal{D}) \in \mathcal{G}'$, i.e. $\mathcal{G}'$ is generically a Grassmannian fiber bundle over $\mathcal{W}'$ with fiber $\mathrm{Gr}(5, r+1)$, where $r = h^0(C, |\mathcal{D}|) - 1$. We need to check that $r = 4$. Assume that $r \geq 5$. Then for a general $l := (C, |\mathcal{D}|) \in \mathcal{W}'$ the linear series $|\mathcal{D}|$ is very ample on C, and using Proposition 2.1 we obtain

$$4g - 8 \leq \dim \mathcal{G}' = \dim \mathcal{W}' + 5(r-4) \leq 4g - 10,$$

which is impossible.

Assume that $\mathcal{G}' \subset \mathcal{G}^4_{g+3}$ is a component different from the principal component $\mathcal{G}_0$ dominating the moduli space. This implies for the image $\pi(\mathcal{G}') \subset \mathcal{M}_g$ of the forgetful map $\pi : (C, \mathcal{D}) \mapsto C$ that $\dim \pi(\mathcal{G}') \leq 3g - 4$. From

$$3g - 3 + \rho(g+3, g, 4) \leq \dim \mathcal{G}'$$

we obtain for the dimension of the fiber $\pi^{-1}(C)$ over general $C \in \pi(\mathcal{G}')$ that $\dim \pi^{-1}(C) = \dim G^4_{g+3}(C) \geq \rho(g+3, g, 4) + 1$. By [ACGH, Ch. IV., Theorem 4.1]

$$\dim G^4_{g+3}(C) \leq \dim T_{\mathcal{D}} G^4_{g+3}(C) = \rho(g+3, g, 4) + \dim \ker \mu_0(C),$$

where $\mu_0(C)$ is the cup-product map

$$\mu_0(C) : H^0(C, \mathcal{D}) \otimes H^0(C, K_C(-\mathcal{D})) \to H^0(C, K_C).$$

Since the general $\mathcal{D} \in G^4_{g+3}(C)$ is complete, we find $h^0(C, K(-\mathcal{D})) = 1$, and therefore the map $\mu_0(C)$ is injective, i.e. $\dim \ker \mu_0(C) = 0$. This implies

$$\rho(g+3, g, 4) + 1 \leq \dim G^4_{g+3}(C) \leq \rho(g+3, g, 4) + \dim \ker \mu_0(C) = \rho(g+3, g, 4),$$

which is a contradiction. This means that $\mathcal{G}'$ must coincide with $\mathcal{G}_0$. This completes the proof of (a).

(b) Just like in the proofs of the preceding irreducibility statements, $\mathcal{G}^4_{g+2}$ contains a principal component $\mathcal{G}_0$ whose general element represents a curve and a complete very ample linear series on it. Assume that $\mathcal{G}'$ is another component of $\mathcal{G}^4_{g+2}$ for whose general element $(C, \mathcal{D}) \in \mathcal{G}'$ the linear series $\mathcal{D}$ is very ample on C. Let r be the dimension of the completion $\mathcal{D}$, and let $\mathcal{W}' \subset \mathcal{W}^r_{g+2}$ be the component whose general elements are the completions $(C, |\mathcal{D}|)$. Thus, $\dim \mathcal{G}' = \dim \mathcal{W}' + \dim \mathrm{Gr}(5, r+1)$. Using Proposition 2.1 we find

$$4g - 13 \leq \dim \mathcal{G}' = \dim \mathcal{W}' + \dim \mathrm{Gr}(5, r+1) \leq 4g - 13,$$

which implies

$$\dim \mathcal{W}' = 4g - 5r + 7. \tag{2}$$

As we want to dismiss the existence of $\mathcal{G}'$, we will show first that $r \geq 5$, and then we will see that the latter also leads to a contradiction, proving in this way that $\mathcal{G}_0 \subset \mathcal{G}^4_{g+2}$ is the only component whose general element carries a very ample series. Denote by $\mathcal{W}_0 \subset \mathcal{W}^{r-3}_{g-4}$ the dual of the $\mathcal{W}'$ variety in the Picard scheme.

I claim that $r \geq 5$. To see this assume the opposite, i.e. $r = 4$. Then $\mathcal{G}' \equiv \mathcal{W}' \subset \mathcal{W}^4_{g+2}$, and the dimension of the dual $\mathcal{W}_0 \subset \mathcal{W}^1_{g-4} \subset \mathcal{G}^1_{g-4}$ is $\dim \mathcal{W}_0 = \dim \mathcal{W}' = 4g - 13 = \dim \mathcal{G}^1_{g-4}$. As has been pointed out, the linear series in a general element of the principal component $\mathcal{G}_0$ is complete, and therefore for the dual $\mathcal{X}_0$ of $\mathcal{G}_0$ we have $\mathcal{X}_0 \subset \mathcal{G}^1_{g-4}$ and $\dim \mathcal{X}_0 = \dim \mathcal{G}_0 = 4g - 13 = \dim \mathcal{G}^1_{g-4}$. But according to Theorem 1.2, the variety $\mathcal{G}^1_{g-4}$ is smooth and irreducible of dimension $4g - 13$, which means that $\mathcal{W}_0$ and $\mathcal{X}_0$ must coincide. Hence, their dual components $\mathcal{G}'$ and $\mathcal{G}_0$ also coincide. Therefore $r \geq 5$.

I claim now that the general element of $\mathcal{W}_0 \subset \mathcal{W}^{r-3}_{g-4}$ cannot be a compounded linear series. To see this, assume the opposite. By (2) and Proposition 2.1, it follows that

$$4g - 5r + 7 = \dim \mathcal{W}' = \dim \mathcal{W}_0 \leq 3g - 2r + 1,$$

which gives $\frac{g+6}{3} \leq r$. Then applying Lemma 2.3 for a general element $(C, g^r_{g+2}) \in \mathcal{W}' \subset \mathcal{W}^r_{g+2}$, we get

$$r \leq \frac{g+3}{3},$$

which is a contradiction.

Thus, the general element of $\mathcal{W}_0$ carries a birationally very ample linear series, after clearing a possible base part, and we can apply Proposition 2.1 to obtain

$$4g - 5r + 7 = \dim \mathcal{W}_0 \leq 3(g-4) + g - 4(r-3) - 1 = 4g - 4r - 1.$$

This gives $r \geq 8$.

It is easy to see that the moving part of the linear series in a general element of $\mathcal{W}_0$ cannot be very ample, since if it were very ample, applying Proposition 2.1 for $\mathcal{W}_0 \subset \mathcal{W}^{r-3}_{g-4}$ we would get

$$4g - 5r + 7 = \dim \mathcal{W}_0 \leq 3(g-4) + g + 1 - 5(r-3) = 4g - 5r + 4,$$

which is absurd.

Further, set $r_0 = r - 3$ and $d_0 = g - 4$ and assume that the degree of the base locus of the general element of $\mathcal{W}_0$ is b_0. Consider the map

$$\mathcal{W}_0 \times_{\mathcal{M}} \mathcal{C}_{b_0} \to \mathcal{W}^{r_0-b_0}_{d_0-b_0},$$
$$((C, L), (P_1 + \cdots + P_{b_0})) \mapsto (C, L(-P_1 - \cdots - P_{b_0})).$$

Let $\mathcal{V}'_0 \subset \mathcal{W}^{r_0-b_0}_{d_0-b_0}$ be the closed subset defined as

$$\mathcal{V}'_0 := \{(C, L) \in \mathcal{W}^{r_0-b_0}_{d_0-b_0} | h^0(C, L) \geq r_0 + 1\},$$

and let $\mathcal{V}_0 \subset \mathcal{V}'_0$ be the component whose general elements are the general elements of $\mathcal{W}_0$ but without base loci. For the dimensions of $\mathcal{V}_0$ and $\mathcal{W}_0$ we have

$$\dim \mathcal{W}_0 \leq \dim \mathcal{V}_0 + b_0.$$

Also, the general element of $\mathcal{V}_0$ is necessarily base-point free and birationally very ample but not very ample. Consider the mapping

$$\mathcal{V}_0 \times_{\mathcal{M}} \mathcal{C}_2 \to \mathcal{W}^{r_0-2}_{d_0-b_0-2},$$
$$((C, L), (P + Q)) \mapsto (C, L(-P - Q)).$$

Let $\mathcal{W}_1 \subset \mathcal{W}^{r_0-1}_{d_0-2-b_0}$ be a component of maximal dimension (if there are more than one) whose elements contain the projections of the general elements of $\mathcal{V}_0$. Set $r_1 = r_0 - 1$ and $d_1 = d_0 - 2 - b_0$, i.e.

$$\mathcal{W}_1 \subset \mathcal{W}^{r_1}_{d_1}.$$

Remark that its dual $\mathcal{W}'_1 \subset \mathcal{W}^{r+b_0+1}_{g+2+b_0+2}$ arises by "adding" some effective divisors of degree $b_0 + 2$ to the linear series of $\mathcal{W}' \subset \mathcal{W}^r_{g+2}$, and $\dim \mathcal{W}'_1 = \dim \mathcal{W}_1$. Also, by Lemma 2.2 we have

$$\dim \mathcal{W}_0 \leq \dim \mathcal{W}_1.$$

Now we proceed with $\mathcal{W}_1$ just as we did with $\mathcal{W}_0$. If its general element has a very ample or compounded linear series (after clearing base locus), we reach a numerical contradiction using Proposition 2.1. If the moving part is birationally very ample but not very ample, we apply the same procedure to get

$$\mathcal{W}_2 \subset \mathcal{W}^{r_2}_{d_2},$$

where $r_2 = r_1 - 1$, $d_2 = d_1 - 2 - b_1$, with b_1 the degree of the base locus. We would like to have just as before

$$\dim \mathcal{W}_1 \leq \dim \mathcal{W}_2.$$

For this we must guarantee that the general element of $\mathcal{W}'_1$ has a birationally very ample moving part, so we could apply Lemma 2.2. But this is almost immediate: if we suppose the opposite, it follows by Proposition 2.1 that

$$\begin{aligned}\dim \mathcal{W}_1 &= \dim \mathcal{W}'_1\\ &\leq 2g - 1 + (d_1 - 2r_1),\end{aligned}$$

and this leads to a numerical contradiction due to $\dim \mathcal{W}_1 \geq \dim \mathcal{W}_0 = 4g - 5r + 7$.

In this way we can construct $\mathcal{W}_2, \mathcal{W}_3, \ldots$, unless the process terminates due to entering a case where the general element of $\mathcal{W}_i$ is very ample or compounded. Precisely, if the general element of $\mathcal{W}_i$ is compounded, we obtain

$$\begin{aligned}4g - 5r + 7 = \dim \mathcal{W}_0 &\leq \dim \mathcal{W}_i + B_i\\ &\leq 2g - 1 + (d_i - 2r_i) + B_i\\ &= 2g - 1 + (d_0 - 2i - B_i - 2(r_0 - i)) + B_i\\ &= 2g - 1 + (d_0 - 2r_0),\end{aligned}$$

where $B_i := b_0 + \cdots + b_{i-1}$. This yields a numerical contradiction just as before. The case in which the general element has a linear series with a very ample moving part is dealt with in a similar way. Remark that, as soon as we construct $\mathcal{W}_i$, we get that the moving part of the linear series in a general element of $\mathcal{W}'_i$ is birationally very ample, which allows us to apply Lemma 2.2 and construct $\mathcal{W}_{i+1}$. We continue the inductive process until i becomes larger than $\frac{r}{2} - 4$. Then for $i = [\frac{r}{2} - 3]$ we can apply Proposition 2.1 for birationally very ample linear series to get

$$\begin{aligned}4g - 5r + 7 &\leq \dim \mathcal{W}_i + B_i\\ &\leq 3(d_i - b_i) + g - 4r_i - 1 + B_i\\ &= 3(d_0 - 2i - B_{i+1}) + g - 4(r_0 - i) - 1 + B_i\\ &\leq 3d_0 + g - 4r_0 - 2i - 1\\ &\leq 3(g - 4) + g - 4(r - 3) - 2(\frac{r-1}{2} - 3) - 1\\ &= 4g - 5r + 6,\end{aligned}$$

which is a contradiction. This precludes the existence of a component $\mathcal{G}' \subset \mathcal{G}^4_{g+2}$ different from the principal component $\mathcal{G}_0$, and completes the proof.

□

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Department of Mathematics, Seoul National University, Seoul 151-747, Korea
E-mail address: itso@math.snu.ac.kr