

STABLE DEGENERATIONS OF SURFACES ISOGENOUS TO A PRODUCT OF CURVES

MICHAEL VAN OPSTALL

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ABSTRACT. We show that an example of Catanese yields nonrigid surfaces that are diffeomorphic, yet lie on different connected components of the moduli space of *stable* surfaces.

1. INTRODUCTION

In [Cat00] and its sequel, [Cat03], Catanese studies surfaces which admit an unramified cover by a product of curves of genus greater than one. The main interesting result from the point of view of moduli theory is that the moduli space of such surfaces (that is, of the class of canonically polarized surfaces whose smooth members are homeomorphic to a given surface) is either irreducible or has two connected components, swapped by complex conjugation. Of course these articles contain many more important and very interesting results.

In [KSB88], Kollár and Shepherd-Barron introduced a compactification of the moduli space of canonically polarized surfaces. The details of the construction are subtle and due to several authors. In particular, they give a *stable reduction* procedure, by which any one-parameter family of surfaces over a punctured disk can be completed to a family of so-called *stable surfaces* over a finite cover of the disk. The stable reduction is obtained by taking the relative canonical model of a semistable resolution of the original family.

This article is concerned with determining the stable surfaces which occur at the boundary of this moduli space. The moduli space of stable curves provides us with candidates for the boundary surfaces, and we simply verify that these candidates are already stable, so the full force of the minimal model program is not required.

Pairing our results with Catanese's results, we obtain that the two components of the moduli space of canonically polarized surfaces do not meet in the stable compactification. This is the first nonrigid example known to the author of disconnectedness of a moduli space of *stable* surfaces after fixing the diffeomorphism class of a smooth member. The first rigid examples are due to Kharlamov and Kulikov in [KK02], where they construct rigid surfaces not isomorphic to their complex conjugates. One of their goals was to show that not every deformation class of complex manifolds contains a manifold with a real structure. Other rigid examples can be found in the paper [BC04] of Bauer and Catanese. The examples of Catanese used

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here give components of the moduli space exchanged by complex conjugation. Since they do not intersect, the moduli space has no real points. The disconnectedness of the moduli space of canonically polarized surfaces with fixed differentiable structure was first proved by Manetti [Man01]. However, to establish that his surfaces which lie on different components of the moduli space are diffeomorphic, he degenerates them all to a single stable surface and uses a powerful result, proved in the same article, that smooth surfaces on different irreducible components of the moduli space are diffeomorphic if the surfaces parameterized by the intersection of the two components are sufficiently mild. Catanese's examples, on the other hand, are obviously diffeomorphic surfaces.

2. STABLE SURFACES AND SURFACES ISOGENOUS TO A PRODUCT

First we define the higher-dimensional analogue of the nodes which are allowed on stable curves:

Definition 2.1. A surface S has *semi-log canonical* (slc) singularities if

1. S is Cohen-Macaulay;
2. S has normal crossings singularities in codimension one;
3. S is \mathbf{Q} -Gorenstein, i.e. some reflexive power of the dualizing sheaf of S is a line bundle;
4. for any birational morphism $\pi : X \rightarrow S$ from a smooth variety, if we write (numerically)

$$K_X = \pi^* K_S + \sum a_i E_i,$$

then all the $a_i \geq -1$.

The second condition in the definition implies that the dualizing sheaf of S is an invertible sheaf off a subset of codimension 2, so it can be extended to give a Weil divisor class K_S . The third condition states that some multiple of this class is Cartier, so we can make sense of the formula occurring in the fourth condition. A complete classification of slc surface singularities can be found in [KSB88]. Suppose S is a \mathbf{Q} -Gorenstein surface. Let S^ν be the normalization, and D the inverse image of the codimension 1 singular set under the normalization morphism. Then the condition that S be slc is equivalent to the condition that (S^ν, D) is a log canonical (lc) pair.

The notation $\mathcal{F}^{[N]}$ for a coherent sheaf \mathcal{F} denotes the N th reflexive power of \mathcal{F} , that is, the double dual of the N th tensor power of \mathcal{F} .

Definition 2.2. A *stable surface* is a projective, reduced surface S with slc singularities such that some reflexive power $\omega_S^{[N]}$ is an ample line bundle. The smallest such N such that $\omega_S^{[N]}$ is a line bundle is called the *index* of S . A family of stable surfaces is a flat morphism $X \rightarrow B$ whose fibers are stable surfaces and whose relative dualizing sheaf $\omega_{X/B}$ is \mathbf{Q} -Cartier.

After fixing a Hilbert polynomial P , there is a bound for the index of a stable surface with Hilbert polynomial P [Ale94]. This is one of the ingredients in the construction of the moduli space of stable surfaces with fixed Hilbert polynomial P , which compactifies the moduli space of canonically polarized surfaces (after possibly throwing away some components parameterizing only singular surfaces). This moduli space is proper and separated. The moduli space would not be separated

without the requirement that families of stable surfaces have a \mathbf{Q} -Cartier relative dualizing sheaf. It is worth noting that in a later article ([Kol90]) Kollár strengthened the conditions that a family of stable surfaces should satisfy. It is unknown whether the \mathbf{Q} -Gorenstein assumption alone implies these stronger conditions. For our purposes, the weaker condition is sufficient, since it is known to imply the stronger conditions in the case of one-parameter families whose general member is smooth.

We now recall some of the definitions from [Cat00].

Definition 2.3. A surface S is called *isogenous to a product* if it admits an unramified cover by a product of curves of genus two or higher.

Remark 2.4. Catanese calls such a surface isogenous to a *higher* product, but we will not be interested in surfaces covered by, say, a product of elliptic curves. It is clear that any such surface is canonically polarized, since the cover by a product of curves contains no rational curves, so S contains none.

The following is a summary of 3.10-3.13 *loc. cit.*

Proposition 2.5 (Catanese). *A surface S isogenous to a product can be written uniquely as $(C_1 \times C_2)/G$ for some group G which embeds into $\text{Aut } C_1$ and $\text{Aut } C_2$, as long as C_1 and C_2 are not isomorphic. If C_1 and C_2 are isomorphic, the subgroup of G consisting of automorphisms not switching the factors embeds into the automorphism group of each factor to obtain this minimal realization.*

This proposition allows us to describe small deformations of certain surfaces isogenous to a product.

Definition 2.6. The functor of deformations of a stable curve C with the action of a finite group G assigns to an artin ring A :

1. a flat morphism $X \rightarrow \text{Spec } A$,
2. an embedding of G in the group $\text{Aut}_{\text{Spec } A} X$ of automorphisms of the family over the base,
3. and an equivariant isomorphism of the special fiber of the family X with the stable curve C .

Proposition 2.7 (Tuffery). *The functor of deformations of a stable curve C together with a subgroup G of the automorphism group is “well-behaved” (i.e. satisfies the Schlessinger conditions) and unobstructed, with tangent space $\text{Ext}^1(\Omega_C, \mathcal{O}_C)^G$. Such pairs have a proper moduli space, finite over a closed subvariety of the moduli space of stable curves.*

The proof of this may be found in [Tuf93]. Note that the statements given here are stronger than those given in that article, since we work over a field of characteristic zero, which greatly simplifies equivariant cohomology with respect to a finite group. In what follows, *Kuranishi space* will be convenient shorthand for “pointed analytic isomorphism class of the base of a miniversal deformation”.

Proposition 2.8. *The Kuranishi space of a surface S minimally realized as a free quotient $(C_1 \times C_2)/G$ is isomorphic to the product of Kuranishi spaces of the pairs (C_1, G) and (C_2, G) if G contains no elements that swap the factors. If G contains elements that swap the factors, then set G_0 equal to the subgroup of G of elements not swapping the factors. Then the Kuranishi space of $S = (C \times C)/G$ is isomorphic*

to the Kuranishi space of (C, G_0) . Consequently, the Kuranishi space is smooth, and the moduli space of canonically polarized surfaces is irreducible at S .

Proof. Since all deformation functors in question are “nice” enough, we may resolve the question by checking to first order. In the case that G does not swap the factors:

$$\begin{aligned} H^1(S, \mathcal{T}_S) &= H^1(C_1 \times C_2, \mathcal{T}_{C_1 \times C_2})^G \\ &= H^1(C_1, \mathcal{T}_{C_1})^G \oplus H^1(C_2, \mathcal{T}_{C_2})^G. \end{aligned}$$

The first line is valid only when G acts freely, but the second line works generally for products of canonically polarized varieties, assuming G acts on both factors (see, e.g., [vO05]).

In the case G swaps the factors ([Cat00], Corollary 3.9 and Remark 3.10), G is the semidirect product of the group G_0 by the group \mathbf{Z}_2 which permutes the factors. In this case, the above computation holds with G replaced by G_0 . The \mathbf{Z}_2 action also permutes factors in cohomology, so we obtain $H^1(C \times C, \mathcal{T}_{C \times C}) \cong H^1(C, \mathcal{T}_C)^{G_0}$.

Since deformations of curves with group action are unobstructed, the Kuranishi space of S is smooth. Since S has a finite automorphism group, the moduli space of S locally near S has only finite quotient singularities, so cannot be reducible. \square

3. DEGENERATIONS

Our main goal is the following theorem.

Theorem 3.1. *Suppose $X \rightarrow \Delta'$ is a family of surfaces isogenous to products over a punctured disk. Then possibly after a finite change of base, totally ramified over the origin in the disc, X (or a pullback thereof) can be completed to a family of stable surfaces over the disk whose central fiber is a quotient of a product of stable curves (possibly by a nonfree group action).*

Proof. By Proposition 2.8, we may assume that X is of the form $(Y_1 \times_{\Delta'} Y_2)/G$, where Y_1 and Y_2 are families of smooth curves with G -action, such that the G -action is fiberwise and free on $Y_1 \times_{\Delta'} Y_2$. Since the moduli functor of stable curves with automorphism group G is proper, after a base change (which we will suppress in our notation), we obtain a family \tilde{X} of the desired form.

It remains to see that the central fiber is a stable surface, and that the family \tilde{X} is a family of stable surfaces. \tilde{X} is obtained by taking the quotient of a family \tilde{Y} of stable surfaces by a group action. Since the group acts freely on the general fiber, the quotient morphism π is étale in codimension one. In this case, [KM98], Proposition 5.20 ensures that \tilde{X} is \mathbf{Q} -Gorenstein, so the special fiber is as well. Well-known results ensure that the special fiber is Cohen-Macaulay. A finite quotient of a variety which is normal crossings in codimension one is normal crossings in codimension one by Corollary 1.7 of [AAL81]. Now [KM98], Proposition 5.20 (appropriately modified to take into account nonnormal varieties), states that a finite quotient of an slc variety is slc as soon as it is \mathbf{Q} -Gorenstein and normal crossings in codimension 1.

Finally, we need to check that $\omega_{\tilde{X}}$ is relatively ample. We can use Nakai-Moishezon: if $\omega_{\tilde{X}}$ is not relatively ample, there is a curve D in the special fiber whose intersection with $K_{\tilde{X}}$ is nonpositive. Since π is unramified in codimension 1, we have that $K_{\tilde{Y}} = \pi^* K_{\tilde{X}}$. Since Y is a product of families of stable curves, $K_{\tilde{Y}}$ is ample. However, if D is nonpositive on $K_{\tilde{X}}$, the pullback of D would be nonpositive on $K_{\tilde{Y}}$, a contradiction. \square

4. APPLICATION TO CATANESE'S EXAMPLES

In [Cat03], Catanese gives a family of examples of moduli spaces of smooth surfaces with fixed K^2 and χ fixed which have two components interchanged by complex conjugation. We review his construction here and study the degenerations of his surfaces.

The construction of the example begins with the construction of a *triangle curve* (i.e. a Galois cover of \mathbf{P}^1 branched at three points) which is not antiholomorphic to itself. Let C denote this curve and G denote the Galois group of the cover $C \rightarrow \mathbf{P}^1$. G is therefore a quotient of the fundamental group of \mathbf{P}^1 minus three points, and is consequently generated by two elements. Choose $h \geq 2$ and a curve C'_1 of genus h . Then the fundamental group of C'_1 surjects onto G , so there exists an étale cover $C_1 \rightarrow C'_1$ with Galois group G . Then the surface $S = (C_1 \times C)/G$ is isogenous to a product of curves of general type (the triangle curve constructed is not the elliptic curve with j -invariant 1728, which is the only triangle curve not of general type).

The critical result for finding multiple components of the moduli space is Catanese's Proposition 3.2: the existence of an antiholomorphic isomorphism of two surfaces minimally realized as surfaces isogenous to products of curves of general type implies antiholomorphic isomorphisms of the factors (up to reordering the factors). In what follows, denote by \overline{X} the complex conjugate of the manifold X .

Choosing any C'_2 of genus h and a cover C_2 of C'_2 with Galois group G as above, suppose $(C_1 \times C)/G \cong \overline{(C_2 \times C)/G}$. Then there is an antiholomorphic isomorphism of $(C_1 \times C)/G$ with $(C_2 \times C)/G$, and hence, an antiholomorphic automorphism of C , which is impossible by the construction of C . The various choices of C_2 fill out a component of the moduli space. But $\overline{(C_2 \times C)/G}$ is diffeomorphic to $(C_2 \times C)/G$, and hence also has a point in the moduli space, which cannot be on this component. Therefore the moduli space has at least two components.

Now let us consider the stable degenerations of these surfaces, and address the question of whether the two components are joined together by deformations through stable surfaces. The results in this chapter show that the (small) deformations of S are just the G -equivariant deformations of C_1 , or equivalently, the deformations of C_1/G . Let \overline{M} denote the moduli space of smoothable stable surfaces occurring as degenerations of $(C_1 \times C)/G$ or its conjugate. \overline{M} has two irreducible components; is it connected?

Suppose \overline{M} were connected: then there would exist a surface $(C' \times C)/G$ on the boundary of the moduli space which lies on the closure of both components. Since both components come from curves with G -action, the map induced from the Kuranishi space of $(C' \times C, G)$ must surject onto a neighborhood of the corresponding boundary point. Since the G action on $C' \times C$ is not necessarily free, we cannot claim directly that the Kuranishi space of $(C' \times C)/G$ is irreducible. However, the Kuranishi space of $(C' \times C, G)$ is irreducible (by Proposition 2.7 and the fact that the Kuranishi space is a product when G acts on both factors and both factors are stable curves), so it cannot map onto two components, but it does map onto the Kuranishi space of $(C' \times C)/G$. So the disconnection of various moduli spaces considered in [Cat03] continues in the stable compactification.

Note that this argument is not strong enough in general to claim that a moduli space of surfaces isogenous to a product of curves is always irreducible at the boundary: it just rules out deformations to other surfaces isogenous to a product

with the same Galois group. By the results of [Cat03], there are at most two components of the moduli space (after fixing topology) parameterizing smooth varieties, but there may be a component parameterizing only singular surfaces meeting both other components along the boundary.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, 155 SOUTH 1400 EAST, ROOM 233,
SALT LAKE CITY, UTAH 84112-0090

E-mail address: `opstall@math.utah.edu`