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A REMARK ON THE MONOMIAL CONJECTURE AND COHEN-MACAULAY CANONICAL MODULES

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Abstract. In this paper, some sufficient conditions for rings and modules to satisfy the monomial conjecture are given. A characterization of Cohen-Macaulay canonical modules is presented.

1. Introduction

Throughout this paper, $(R, \mathfrak{m})$ is a Noetherian local ring with $\dim R = d$ and M a finitely generated $R-$module with $\dim M = d'$. The monomial conjecture of Hochster [H1] asserts that for any given system of parameters (s.o.p. for short) $\underline{x} = (x_1, \ldots, x_d)$ of R,

$$x_1^t \ldots x_d^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})R \text{ for all } t > 0.$$

The Monomial Conjecture (MC) has been proved for the equicharacteristic case; cf. [C, Proposition 3]. In the mixed and the positive characteristic cases, (MC) and the Direct Summand Conjecture are equivalent to the Canonical Element Conjecture [H2]. Therefore (MC) makes a central role in the study of these important homological conjectures. (MC) has also been proved when $\dim R \leqslant 2$ (cf. [H1]), and when R is a Buchsbaum ring; cf. [Go, Corollary 4.8]. By using the theory of modules of generalized fractions, Sharp-Zakeri [SZ2] proved some results related to (MC) for rings of dimension d under the assumption that (MC) is valid for rings of dimension $d-1$. There is also a reduction of (MC) given by Strooker-Stückrad [SS]: (MC) is valid for all local rings if and only if for any complete intersection ring A and any ideal $\mathfrak{a}$ of A consisting of zero divisors, $\operatorname{Ann}_A \mathfrak{a}$ is not contained in any parameter ideal of A. Recently, Dutta [D] gave some nice sufficient conditions for (MC).

(MC) has also been raised for modules: An s.o.p. $\underline{x} = (x_1, \ldots, x_{d'})$ of M is said to satisfy (MC) if $x_1^t \ldots x_{d'}^t M \nsubseteq (x_1^{t+1}, \ldots, x_{d'}^{t+1})M$ for all $t > 0$. Hochster [H1] showed that in general (MC) does not hold for every s.o.p. of M, but it holds for high powers of s.o.p., i.e. there exists for each s.o.p. $\underline{x} = (x_1, \ldots, x_{d'})$ of M an integer $n_{\underline{x}} > 0$ such that $(x_1^{n_1}, \ldots, x_{d'}^{n_{d'}})$ satisfies (MC) for all $n_1, \ldots, n_{d'} \geq n_{\underline{x}}$. Then Cuong-Hoa-Loan [CHL, Th. 3.3] proved that there always exists an integer

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$c > 0$ such that $(x_1^{n_1}, \ldots, x_{d'}^{n_{d'}})$ satisfies (MC) for all s.o.p. $(x_1, \ldots, x_{d'})$ of M and all integers $n_1, \ldots, n_{d'} \geq c$ (see also [St, Corollary 11.2.4], [D, Corollary 2] for the case $M = R$). In this case, c is called a *uniform bound for (MC)* of M.

The notion of Cohen-Macaulay canonical modules was introduced by Schenzel [Sc]: M is called *Cohen-Macaulay canonical* if its canonical module exists and is Cohen-Macaulay. Note that if R has a Cohen-Macaulay canonical module of dimension $\dim R$, then R has a maximal Cohen-Macaulay module and hence (MC) is valid for R. Therefore it is worth finding conditions for a module to be Cohen-Macaulay canonical.

The purpose of this paper is to use strict f-sequences introduced in [CMN] as a useful tool to study the Monomial Conjecture and characterize Cohen-Macaulay canonical modules.

This paper is divided into four sections. In the next section, we give a sufficient condition for an s.o.p. to satisfy (MC) (Theorem 2.3). As a consequence, we show for each strict f-sequence s.o.p. $\underline{x} = (x_1, \ldots, x_{d'})$ of M a concrete integer $n_{\underline{x}}$ such that $(x_1, \ldots, x_{d'-1}, x_{d'}^n)$ satisfies (MC) when $n > n_{\underline{x}}$ (cf. Corollaries 2.4 and 2.6). Then we prove that Theorem 1.6 of Dutta [D] can be extended from rings to modules (Corollary 2.5). In Section 3, we study a stronger kind of uniform bound which is called a strong uniform bound for (MC): An integer $c > 0$ is called a *strong uniform bound for (MC)* of M if $(x_1, \ldots, x_{d'-1}, x_{d'}^n)$ satisfies (MC) for all s.o.p. $(x_1, \ldots, x_{d'})$ of M and all $n \geq c$. In the last section, we characterize Cohen-Macaulay canonical modules (Theorem 4.2). As a consequence, we give a sufficient condition for a ring to satisfy (MC), and again we get theorems of Schenzel [Sc, Theorem 4.3, Theorem 4.6(iii)] by elementary proofs.

2. A sufficient condition for the Monomial Conjecture

In this section we give a sufficient condition such that (MC) is satisfied. We first need the notion of a strict f-sequence introduced in [CMN]. For each Artinian module A, we denote by $\operatorname{Att} A$ the set of all attached primes of A; see [Mac].

Definition 2.1. A sequence $(x_1, \ldots, x_k)$ of elements in $\mathfrak{m}$ is called a *strict f-sequence* of M if $x_{j+1} \notin \mathfrak{p}$ for all $\mathfrak{p} \in \bigcup_{i=1}^{d'-j} \operatorname{Att}(H_{\mathfrak{m}}^i(M/(x_1, \ldots, x_j)M) \setminus \{\mathfrak{m}\}$ for all $j = 0, \ldots, k-1$. A sequence $\underline{x} = (x_1, \ldots, x_k)$ of elements in $\mathfrak{m}$ is called a *permutable strict f-sequence* of M if it is a strict f-sequence of M in any order. It was proved in [CMN] that for each integer $k > 0$, there always exists a strict f-sequence of M of length k. Moreover, if x is a strict f-element of M, then $0 :_M x$ is of finite length. So, each strict f-sequence of M containing at most d' elements is part of an s.o.p. of M. Note that strict f-sequences have been used (cf. [CMN]) to study the finiteness for attached primes of local cohomology modules, to study the polynomial property of the length of generalized fractions defined in [SH], and to characterize the pseudo-generalized Cohen-Macaulay modules defined in [CN].

Lemma 2.2. *The following statements are equivalent:*

(i) *Every s.o.p. of M satisfies (MC).*

(ii) *Every permutable strict f-sequence s.o.p. of M satisfies (MC).*

Proof. It is enough to prove (ii)$\Rightarrow$(i). Let $\underline{x} = (x_1, \ldots, x_{d'})$ be an s.o.p. of M. Without loss of any generality we can assume that $\operatorname{Ann} M = 0$. We claim by induction on k that there exists a permutable strict f-sequence $y_1, \ldots, y_k$ such that $(x_1, \ldots, x_{d'})R = (y_1, \ldots, y_k, x_{k+1}, \ldots, x_{d'})R$. Let $k = 1$. Set $C_1 = \bigcup_{i=1}^{d'} \operatorname{Att}(H_{\mathfrak{m}}^i(M))$.

There exists by [K, Theorem 124] an element $a_1 \in (x_2, \ldots, x_{d'})R$ such that $x_1 + a_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in C_1 \setminus \{\mathfrak{m}\}$. Let $y_1 = x_1 + a_1$. Then $(y_1, x_2, \ldots, x_{d'})R = (x_1, \ldots, x_{d'})R$ and y_1 is a permutable strict f-element of M. Let $1 < k \leqslant d'$ and assume that the claim is true for $k-1$. Set $C_k = \bigcup_{I \subseteq \{1,\ldots,k-1\}} \bigcup_{i=1}^{d'} \operatorname{Att}(0 :_{H^i_{\mathfrak{m}}(M)} \sum_{j \in I} y_j R)$. Let $a_k \in (y_1, \ldots, y_{k-1}, x_{k+1}, \ldots, x_{d'})R$ such that $x_k + a_k \notin \mathfrak{p}$ for all $\mathfrak{p} \in C_k \setminus \{\mathfrak{m}\}$. Let $y_k = x_k + a_k$. It follows by the proof of [CMN, 4.2] that $(y_1, \ldots, y_k)$ is a permutable strict f-sequence of M. By the induction hypothesis we have

$$(x_1, \ldots, x_{d'})R = (y_1, \ldots, y_{k-1}, x_k, \ldots, x_{d'})R = (y_1, \ldots, y_k, x_{k+1}, \ldots, x_{d'})R,$$

and the claim is proved. By the claim, there exists a permutable strict f-sequence $\underline{y} = (y_1, \ldots, y_{d'})$ of M such that $(x_1, \ldots, x_{d'})R = (y_1, \ldots, y_{d'})R$. For each $i = 1, \ldots, d'$ we write $x_i = \sum_{j=1}^{d'} b_{ij} y_j$. Set $B = (b_{ij})_{1 \leqslant i,j \leqslant d'}$. Put $\delta = \det B$. Set $Q_M(\underline{y}) = \cup_{t>0}\big((y_1^{t+1}, ..., y_{d'}^{t+1})M :_M y_1^t \ldots y_{d'}^t\big)$. For each $m + Q_M(\underline{y}) \in M/Q_M(\underline{y})$ we set $\delta(m + Q_M(\underline{y})) = \delta.m + Q_M(\underline{x})$, where $Q_M(\underline{x})$ is defined similarly. Then δ is a homomorphism from $M/Q_M(\underline{y})$ to $M/Q_M(\underline{x})$. This homomorphism is independent of the choice of the matrix B, and it is called the *determinantal map* (cf. [St, 5.1.14]). By [CHL, 3.1], δ is injective. So, $\ell(M/Q_M(\underline{y})) \leqslant \ell(M/Q_M(\underline{x}))$. Since $\underline{y}$ satisfies (MC), i.e. $\ell(M/Q_M(\underline{y})) > 0$, it follows that $\ell(M/Q_M(\underline{x})) > 0$. Therefore $\underline{x}$ satisfies (MC). □

Now we need to recall some knowledge of the theory of modules of generalized fractions [SZ1]. In this theory, for a given positive integer k, the so-called *triangular subsets* of R^k play a similar role as multiplicatively closed subsets of R do in the usual theory of localization of modules. Given a triangular subset U of R^k, Sharp and Zakeri constructed an $R-$module $U^{-k}M$, and they called it *the module of generalized fractions* of M with respect to U. In particular, the set

$$U(M)_{d'+1} = \{(y_1, \ldots, y_{d'}, 1) \in R^{d'+1} : \exists j, 0 \leqslant j \leqslant d', \text{ such that } (y_1, \ldots, y_j) \text{ forms a subset of an s.o.p. of } M \text{ and } y_{j+1} = \ldots = y_{d'} = 1\}$$

is a triangular subset of $R^{d'+1}$. Let $\underline{x} = (x_1, \ldots, x_{d'})$ be an s.o.p. of M. Denote by $M(1/(x_1, \ldots, x_{d'}, 1))$ the submodule $\{m/(x_1, \ldots, x_{d'}, 1) : m \in M\}$ of $U(M)_{d'+1}^{-d'-1}M$. Then the length of the submodule $M(1/(x_1, \ldots, x_{d'}, 1))$ is always finite. Set

$$q(x_1, \ldots, x_{d'}; M) = \ell(M(1/(x_1, \ldots, x_{d'}, 1))).$$

Following Sharp-Hamieh [SH], $q(x_1, \ldots, x_{d'}; M)$ is called *the length of the generalized fraction* $1/(x_1, \ldots, x_{d'}, 1)$ with respect to M. Note that the s.o.p. $(x_1, \ldots, x_{d'})$ of M satisfies (MC) if and only if $q(x_1, \ldots, x_{d'}; M) > 0$; cf. [SZ2].

From now on, we use the following notations from [SH]: Let A be an Artinian $R-$module. The *stability index* of A, denoted by $s(A)$, is the least positive integer s such that $\mathfrak{m}^s A = \mathfrak{m}^n A$ for all $n \geq s$. Denote by $\operatorname{Rl}(A)$ the length of $A/\mathfrak{m}^{s(A)}A$. Then $\operatorname{Rl}(A)$ is finite, and is called the *residual length* of A. It is clear that $\operatorname{Rl}(A) = 0$ if and only if $\mathfrak{m} \notin \operatorname{Att} A$. Moreover, if $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att} A \setminus \{\mathfrak{m}\}$, then $\ell(A/xA) \leqslant \operatorname{Rl}(A)$, and in this case, $\ell(A/x^n A) = \operatorname{Rl}(A)$ for all $n \geq s(A)$.

Theorem 2.3. *Assume that $\underline{x} = (x_1, \ldots, x_{d'})$ is a strict f-sequence s.o.p. of M. Set $n_{\underline{x}} = \sum_{i=1}^{d'-1} \operatorname{Rl}(H_{\mathfrak{m}}^{d'-i}(M_{i-1}))$, where $M_0 = M$, $M_i = M/(x_1, \ldots, x_i)M$ for $i = 1, \ldots, d'-1$. If $e(\underline{x}; M) > n_{\underline{x}}$, then $\underline{x}$ satisfies (MC).*

Proof. Without loss of generality, we may assume that $\operatorname{depth} M > 0$. By [SH, Proposition 2.2] we have the exact sequence

$$0 \longrightarrow H_{\mathfrak{m}}^{d'-1}(M)/x_1 H_{\mathfrak{m}}^{d'-1}(M) \longrightarrow U(M_1)_{d'}^{-d'} M_1 \xrightarrow{\Psi_{d'+1}} U(M)_{d'+1}^{-d'-1} M,$$

where $\Psi_{d'+1}$ is defined by $\Psi_{d'+1}(\overline{m}/(u_2,\dots,u_{d'},1)) = m/(x_1,u_2,\dots,u_{d'},1)$, for all $\overline{m} \in M_1$ and $(u_2,\dots,u_{d'},1) \in U(M_1)_{d'}$. Consider $H_{\mathfrak{m}}^{d'-1}(M)/x_1 H_{\mathfrak{m}}^{d'-1}(M)$ as a submodule of $U(M_1)_{d'}^{-d'} M_1$. Then the above exact sequence implies the following exact sequence:

$$\begin{aligned} 0 &\longrightarrow H_{\mathfrak{m}}^{d'-1}(M)/x_1 H_{\mathfrak{m}}^{d'-1}(M) \cap M_1(1/(x_2,\dots,x_{d'-1},x_{d'},1)) \\ &\longrightarrow M_1(1/(x_2,\dots,x_{d'-1},x_{d'},1)) \xrightarrow{\Psi_{d'+1}} M(1/(x_1,\dots,x_{d'-1},x_{d'},1)) \longrightarrow 0. \end{aligned}$$

Let $s = s(H_{\mathfrak{m}}^{d'-1}(M))$, the stability index of $H_{\mathfrak{m}}^{d'-1}(M)$. Since x_1 is a strict f-element of M, $\ell\big(H_{\mathfrak{m}}^{d'-1}(M)/x_1 H_{\mathfrak{m}}^{d'-1}(M)\big) < \infty$. Therefore

$$\begin{aligned} \ell\big(H_{\mathfrak{m}}^{d'-1}(M)/x_1 H_{\mathfrak{m}}^{d'-1}(M) &\cap M(1/(x_2,\dots,x_{d'-1},x_{d'},1))\big) \\ &\leqslant \ell(H_{\mathfrak{m}}^{d'-1}(M)/x_1 H_{\mathfrak{m}}^{d'-1}(M)) \\ &\leqslant \ell(H_{\mathfrak{m}}^{d'-1}(M)/\mathfrak{m}^s H_{\mathfrak{m}}^{d'-1}(M)) = \operatorname{Rl}(H_{\mathfrak{m}}^{d'-1}(M)). \end{aligned}$$

Hence

$$q(x_1,\dots,x_{d'-1},x_{d'};M) \geq q(x_2,\dots,x_{d'-1},x_{d'};M_1) - \operatorname{Rl}(H_{\mathfrak{m}}^{d'-1}(M)).$$

Continuing this process we get

$$\begin{aligned} q(x_1,\dots,x_{d'-1},x_{d'};M) &\geq q(x_2,\dots,x_{d'-1},x_{d'};M_1) - \operatorname{Rl}(H_{\mathfrak{m}}^{d'-1}(M)) \\ &\geq q(x_3,\dots,x_{d'-1},x_{d'};M_2) - \sum_{i=1}^{2} \operatorname{Rl}(H_{\mathfrak{m}}^{d'-i}(M_{i-1})) \\ &\geq \dots \geq q(x_{d'};M_{d'-1}) - \sum_{i=1}^{d'-1} \operatorname{Rl}(H_{\mathfrak{m}}^{d'-i}(M_{i-1})) \\ &= q(x_{d'};M_{d'-1}) - n_{\underline{x}}. \end{aligned}$$

Note that $\dim M_{d'-1} = 1$. Therefore $q(x_{d'};M_{d'-1}) = e(x_{d'};M_{d'-1})$. Since $\underline{x}$ is an f-sequence of M, we get by the hypothesis that

$$q(x_{d'};M_{d'-1}) = e(x_{d'};M_{d'-1}) = e(\underline{x};M) > n_{\underline{x}}.$$

Hence $q(x_1,\dots,x_{d'};M) > 0$, i.e. $(x_1,\dots,x_{d'})$ satisfies (MC). $\square$

Note that the number $n_{\underline{x}}$ in Theorem 2.3 does not depend on $x_{d'-1}, x_{d'}$.

Corollary 2.4. *Assume that $\underline{x} = (x_1,\dots,x_{d'})$ is a strict f-sequence s.o.p. of M and $n_{\underline{x}}$ is the integer defined as in Theorem 2.3. Then $(x_1,\dots,x_{d'-2},x_{d'-1}^n,x_{d'}^m)$ satisfies (MC) when $nm > n_{\underline{x}}$. In particular, $(x_1,\dots,x_{d'-1},x_{d'}^n)$ satisfies (MC) when $n > n_{\underline{x}}$.*

Proof. By [CMN, Cor. 3.5], $(x_1,\dots,x_{d'-2},x_{d'-1}^n,x_{d'}^m)$ is a strict f-sequence of M for all integers $n,m > 0$. We have

$$e(x_1,\dots,x_{d'-2},x_{d'-1}^n,x_{d'}^m;M) = nm\, e(x_1,\dots,x_{d'};M) \geq nm.$$

So, by Theorem 2.3, $(x_1,\dots,x_{d'-2},x_{d'-1}^n,x_{d'}^m)$ satisfies (MC) for all $nm > n_{\underline{x}}$. $\square$

Theorem 1.6 of Dutta [D] can be extended from rings to modules as follows.

Corollary 2.5. *Given an s.o.p.* $\underline{x} = (x_1, \dots, x_{d'})$ *of* M*, we can construct a strict f-sequence* $(y_1, \dots, y_{d'-1})$*, for each* $y_i \in (x_1, \dots, x_{d'})R$*,* $i = 1, \dots, d'-1$*, such that* $(x_1, \dots, x_{d'})R = (y_1, \dots, y_{d'-1}, x_{d'})R$ *and* $(y_1, \dots, y_{d'-1}, x_{d'}^n)$ *satisfies (MC) for* $n \gg 0$.

Proof. Without loss of generality we can assume that $\operatorname{Ann} M = 0$. Let $(x_1, \dots, x_{d'})$ be an s.o.p. of M. By using [K, Theorem 124], we can choose $(y_1, \dots, y_{d'-1})$ such that

$$\begin{aligned}
&y_1 = x_1 + a_1, \text{ for some } a_1 \in (x_2, \dots, x_{d'})R,\\
&y_2 = x_2 + a_2, \text{ for some } a_2 \in (y_1, x_3, \dots, x_{d'})R,\\
&\dots\\
&y_{d'-1} = x_{d'-1} + a_{d'-1}, \text{ for some } a_{d'-1} \in (y_1, \dots, y_{d'-2}, x_{d'})R,
\end{aligned}$$

where $y_k \notin \mathfrak{p}$ for all $\mathfrak{p} \in \cup_{i=1}^{d'-k+1} \operatorname{Att}(H^i_{\mathfrak{m}}(M/(y_1, \dots, y_{k-1})M)) \setminus \{\mathfrak{m}\}$, and all $k \leqslant d'-1$. Then $(y_1, \dots, y_{d'-1}, x_{d'})$ is a strict f-sequence of M, $y_i \in (x_1, \dots, x_{d'})R$ for $i \leqslant d'$ and $(x_1, \dots, x_{d'})M = (y_1, \dots, y_{d'-1}, x_{d'})M$. By Corollary 2.4, $(y_1, \dots, y_{d'-1}, x_{d'}^n)$ satisfies (MC) for $n \gg 0$. □

Note that in general, (MC) is not valid for M when $\dim M = 2$, but it is always valid for the rings of dimension 2. Therefore, to consider (MC) only for rings, we have the following statement which is stronger than that of Corollary 2.4.

Corollary 2.6. *Let* $\underline{x} = (x_1, \dots, x_d)$ *be a strict f-sequence of* R*. Set* $R_0 = R$ *and* $R_i = R/(x_1, \dots, x_i)R$ *for* $i = 1, \dots, d-2$*. Set* $n^*_{\underline{x}} = \sum_{i=1}^{d-2} \operatorname{Rl}(H^{d-i}_{\mathfrak{m}}(R_{i-1}))$*. Then* $(x_1, \dots x_{d-2}, x_{d-1}^m, x_d^n)$ *satisfies (MC) when* $mn > n^*_{\underline{x}}$*. In particular,* $(x_1, \dots, x_{d'-1}, x_{d'}^n)$ *satisfies (MC) when* $n > n^*_{\underline{x}}$.

Proof. We have as in the proof of Theorem 2.3 that

$$\begin{aligned}
q(x_1, \dots, x_d; R) &\geq q(x_{d-1}, x_d; R_{d-2}) - \sum_{i=1}^{d-2} \operatorname{Rl}(H^{d-i}_{\mathfrak{m}}(R_{i-1}))\\
&= q(x_{d-1}, x_d; R_{d-2}) - n^*_{\underline{x}}.
\end{aligned}$$

Since $\dim R_{d-2} = 2$, (MC) is valid for R_{d-2}. Hence $q(x_{d-1}, x_d; R_{d-2}) > 0$. So, by [CHL, Lemma 4.1(ii)] we have

$$q(x_{d-1}^n, x_d^m; R_{d-2}) \geq mn\, q(x_{d-1}, x_d; R_{d-2}) \geq mn.$$

Hence $q(x_1, \dots x_{d-2}, x_{d-1}^m, x_d^n; R) > 0$, i.e. $(x_1, \dots, x_{d-2}, x_{d-1}^m, x_d^n)$ satisfies (MC) when $mn > n^*_{\underline{x}}$. □

Remark. The number $n^*_{\underline{x}}$ in Corollary 2.6 does not depend on x_{d-2}, x_{d-1}, x_d. Moreover, Corollary 2.6 does not hold for modules. In fact, let $d \geq 2$ be an integer, let $R = k[[x_1, \dots, x_d]]$ be the ring of powers series in d variables over a field k and let $M = (x_1, \dots, x_d)R$. Then M is a Buchsbaum module, $(x_1, \dots, x_d)$ is a permutable strict f-sequence of M and $\sum_{i=1}^{d-2} \operatorname{Rl}(H^{d-i}_{\mathfrak{m}}(M_{i-1})) = 0$, where $M_0 = M$, $M_i = M/(x_1, \dots, x_i)M$ for $i \leqslant d-2$. However, $(x_1, \dots, x_d)$ does not satisfy (MC).

3. The strong uniform bound for the Monomial Conjecture

It has been shown that there always exists an integer $c > 0$ depending only on M such that $(x_1^{n_1}, \ldots, x_{d'}^{n_{d'}})$ satisfies (MC) for all s.o.p. $(x_1, \ldots, x_{d'})$ of M and all integers $n_1, \ldots, n_{d'} \geq c$; cf. [CHL, Theorem 3.3]. This number c is called a *uniform bound for (MC)* of M. In this section, we study a stronger kind of the uniform bound as follows.

Definition 3.1. Let $c > 0$ be an integer. c is called a *strong uniform bound for (MC)* of M if $(x_1, \ldots, x_{d'-1}, x_{d'}^n)$ satisfies (MC) for all s.o.p. $(x_1, \ldots, x_{d'})$ of M and all integers $n \geq c$.

By using strict f-sequences, we have the following reduction.

Lemma 3.2. *Let $c > 0$ be an integer. The following statements are equivalent:*
(i) *c is a strong uniform bound for (MC) of M.*
(ii) *For every strict f-sequence s.o.p. $(x_1, \ldots, x_{d'})$ of M, the s.o.p. $(x_1, \ldots, x_{d'-1}, x_{d'}^n)$ satisfies (MC) for all integers $n \geq c$.*

Proof. It is enough to prove (ii)⇒(i). Let $(x_1, \ldots, x_{d'})$ be an s.o.p. of M and $n \geq c$ an integer. Without any loss of generality, we can assume that $\operatorname{Ann} M = 0$. As in the proof of Corollary 2.5, we can choose a strict f-sequence $(y_1, \ldots, y_{d'-1})$ of M such that

$$(y_1, \ldots, y_{d'-1}, x_{d'}^n)R = (x_1, \ldots, x_{d'-1}, x_{d'}^n)R.$$

It is clear that $(y_1, \ldots, y_{d'-1}, x_{d'})$ is a strict f-sequence of M. So, $(y_1, \ldots, y_{d'-1}, x_{d'}^n)$ satisfies (MC) by the hypothesis (ii). By the same arguments as in the last part of the proof of Lemma 2.2, it follows that $(x_1, \ldots, x_{d'-1}, x_{d'}^n)$ satisfies (MC). $\square$

For each s.o.p. $\underline{x} = (x_1, \ldots, x_{d'})$ of M and each d'−tuple of positive integers $\underline{n} = (n_1, \ldots, n_{d'})$, we set $\underline{x}(\underline{n}) = (x_1^{n_1}, \ldots, x_{d'}^{n_{d'}})$ and

$$I(\underline{x}(\underline{n}); M) = \ell(M/(\underline{x}(\underline{n})M)) - e(\underline{x}(\underline{n}); M),$$
$$J(\underline{x}(\underline{n}); M) = e(\underline{x}(\underline{n}); M) - \ell(M/Q(\underline{x}(\underline{n}); M)),$$

where $Q(\underline{x}; M) = \bigcup_{t>0} (x_1^{t+1}, \ldots, x_{d'}^{t+1})M :_M x_1^t \ldots x_{d'}^t$. It is known that in general $I(\underline{x}(\underline{n}); M)$ and $J(\underline{x}(\underline{n}); M)$, considered as functions in $n_1, \ldots, n_{d'}$, are not polynomials when $n_1, \ldots, n_{d'} \gg 0$, but they always take non-negative values and are bounded above by polynomials. In particular, the least degree of all polynomials in $\underline{n}$ bounding above the function $I(\underline{x}(\underline{n}); M)$ (resp. $J(\underline{x}(\underline{n}); M)$) does not depend on the choice of $\underline{x}$ (cf. [C], [CM]). This least degree is denoted by $p(M)$ (resp. $pf(M)$). Following N. T. Cuong [C], $p(M)$ is called *the polynomial type* of M. If we stipulate that the degree of zero is $-\infty$, then M is Cohen-Macaulay if and only if $p(M) = -\infty$, and M is generalized Cohen-Macaulay if and only if $p(M) \leqslant 0$. In general, $pf(M) \leqslant p(M)$, and if $\dim M \geq 2$, then $pf(M) \leqslant \dim M - 2$ (cf. [CM]).

The following fact has been proved in [CN, Corollary 3.5].

Lemma 3.3. *If $pf(M) \leqslant 0$, then M has a strong uniform bound for (MC).*

In the case $pf(M) > 0$, we have the following result.

Theorem 3.4. *Let $k \geq 0$ be an integer. Then there exists a finitely generated module M over a regular local ring R such that $pf(M) = k$ and M does not have a strong uniform bound for (MC).*

Proof. Let $d \geq k+2$ be an integer. Let $R = k[[x_1, \ldots, x_d]]$ be the ring of power series in d variables over a field k. Let $\mathfrak{m} = (x_1, \ldots, x_d)R$ and let $M = (x_1, \ldots, x_{d-k})R$. From the exact sequence

$$0 \longrightarrow M \longrightarrow R \longrightarrow R/M \longrightarrow 0$$

we can check that $H^{k+1}_{\mathfrak{m}}(M) \cong H^k_{\mathfrak{m}}(R/M)$ and $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq k+1$, $i \neq d$. Note that

$$\dim R/\operatorname{Ann}(H^{k+1}_{\mathfrak{m}}(M)) = \dim R/\operatorname{Ann}(H^k_{\mathfrak{m}}(R/M)) = k.$$

So, $p(M) = \max_{i<d} \dim R/\operatorname{Ann}(H^i_{\mathfrak{m}}(M)) = k$ by [C, Theorem 1.2], where $p(M)$ is the polynomial type of M. Since $\operatorname{depth} M = k+1 > k = p(M)$, we get by [CM, Proposition 3.5, (ii)] that $pf(M) = k$. For the rest of the statement, it is clear that $(x_1, \ldots, x_d)$ is an s.o.p. of M. Since $k > 0$, we can easily check that $(x_1, \ldots, x_{d-1}, x_d^n)$ does not satisfy (MC) for all $n > 0$. So M does not have a strong uniform bound for (MC). □

Remark. It would be of interest to prove a weaker statement of Monomial Conjecture: *Every local ring has a strong uniform bound for (MC).*

4. A characterization of Cohen-Macaulay canonical modules

The notion of Cohen-Macaulay canonical modules was introduced by P. Schenzel [Sc] as follows.

Definition 4.1. Assume that R admits a dualizing complex. M is called a *Cohen-Macaulay canonical module* if the canonical module $K(M)$ of M is Cohen-Macaulay.

It is clear that if M is a Cohen-Macaulay canonical module with $\dim M = d$, then the canonical module $K(M)$ of M is a maximal Cohen-Macaulay module. In this case, (MC) is valid for R. Therefore it is of interest to find conditions for a module to be Cohen-Macaulay canonical. Schenzel [Sc, Theorem 1.1] gave a characterization of Cohen-Macaulay canonical domains in term of a so-called birational Macaulayfication. There is also a characterization for unmixed Cohen-Macaulay canonical modules M of $\operatorname{depth} M = d' - 1$ in terms of the Cohen-Macaulayness of the $d' - 1^{th}$ deficiency module $K^{d'-1}(M)$ of M; cf. [Sc]. However, as far as the author knows, there is no characterization of Cohen-Macaulay canonical modules M in terms of intrinsic data of M. Such characterization is given below.

From now on, for each finitely generated $R-$module N with $\dim N = d'$ and each strict f-sequence $\underline{x} = (x_1, \ldots, x_{d'})$ of N, we set $N_{\underline{x},0} = N$ and $N_{\underline{x},i} = N/(x_1, \ldots, x_i)N$ for $i = 1, \ldots, d'-2$. When R admits a dualizing complex, we denote by $K^i(N)$ the i^{th} deficiency module of N for $i = 1, \ldots, d'-1$, and $K(N) = K^{d'}(N)$ the canonical module of N; see [Sc, Definition 2.1].

Theorem 4.2. *Assume that R admits a dualizing complex. The following statements are equivalent:*

(i) *M is a Cohen-Macaulay canonical module.*

(ii) $\sum_{i=1}^{d'-2} \operatorname{Rl}(H^{d'-i}_{\mathfrak{m}}(M_{\underline{x},i-1})) = 0$ *for all strict f-sequences s.o.p. $\underline{x}$ of M.*

(iii) $\sum_{i=1}^{d'-2} \operatorname{Rl}(H^{d'-i}_{\mathfrak{m}}(M_{\underline{x},i-1})) = 0$ *for a strict f-sequence s.o.p. $\underline{x}$ of M.*

Proof. (i)$\Rightarrow$(ii). We prove by induction on d'. There is nothing to prove for the case $d' \leqslant 2$. Let $d' \geq 3$. Let $\underline{x} = (x_1, \dots, x_{d'})$ be an arbitrary strict f-sequence of M. Then $\ell(0 :_M x_1) < \infty$. Therefore, from the exact sequence

$$0 \longrightarrow M/0 :_M x_1 \xrightarrow{.x_1} M \longrightarrow M/x_1M \longrightarrow 0$$

we get the exact sequences

$$(*) \qquad 0 \longrightarrow K^{i+1}(M)/x_1K^{i+1}(M) \longrightarrow K^i(M/x_1M) \longrightarrow 0 :_{K^i(M)} x_1 \longrightarrow 0$$

for $i = 0, 1, \dots, d'-1$. Since $K(M)$ is Cohen-Macaulay by the assumption (i) and x_1 is a regular element of $K(M)$, it follows that $K(M)/x_1K(M)$ is Cohen-Macaulay. Moreover, since $\operatorname{depth}(K(M/x_1M)) > 0$, we have $H^0_{\mathfrak{m}}(K(M/x_1M)) = 0$. So, we get by the exact sequence (*) with respect to $i = d' - 1$ that $H^0_{\mathfrak{m}}(0 :_{K^{d'-1}(M)} x_1) = 0$. Since x_1 is a strict f-element of M, we can check that $\ell(0 :_{K^{d'-1}(M)} x_1) < \infty$. Hence

$$0 :_{K^{d'-1}(M)} x_1 = H^0_{\mathfrak{m}}(0 :_{K^{d'-1}(M)} x_1) = 0.$$

So $K(M)/x_1K(M) \cong K(M/x_1M)$ by the exact sequence (*) with respect to $i = d' - 1$. Hence $K(M/x_1M)$ is Cohen-Macaulay, i.e. M/x_1M is Cohen-Macaulay canonical. Since $\underline{x}' = (x_2, \dots, x_{d'})$ is a strict f-sequence of M/x_1M, we get by the induction hypothesis that

$$\sum_{i=1}^{(d'-1)-2} \operatorname{Rl}(H^{(d'-1)-i}_{\mathfrak{m}}((M/x_1M)_{\underline{x}',i-1})) = 0,$$

i.e.

$$\sum_{i=2}^{d'-2} \operatorname{Rl}(H^{d'-i}_{\mathfrak{m}}(M_{\underline{x},i-1})) = 0.$$

Since $0 :_{K^{d'-1}(M)} x_1 = 0$, we obtain $\mathfrak{m} \notin \operatorname{Ass} K^{d'-1}(M)$. Hence $\operatorname{Rl}(H^{d'-1}_{\mathfrak{m}}(M)) = 0$ and hence $\sum_{i=1}^{d'-2} \operatorname{Rl}(H^{d'-i}_{\mathfrak{m}}(M_{\underline{x},i-1})) = 0$.

(ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(i). Let $\underline{x} = (x_1, \dots, x_{d'})$ be a strict f-sequence of M which satisfies the condition as in (iii). We prove statement (i) by induction on d'. The case $d' \leqslant 2$ is already true. Let $d' \geq 3$. Note that $\underline{x}' = (x_2, \dots, x_{d'})$ is a strict f-sequence of M/x_1M. Moreover, by (iii),

$$\sum_{i=2}^{d'-2} \operatorname{Rl}(H^{d'-i}_{\mathfrak{m}}(M_{\underline{x},i-1})) = 0,$$

i.e.

$$\sum_{i=1}^{(d'-1)-2} \operatorname{Rl}(H^{(d'-1)-i}_{\mathfrak{m}}((M/x_1M)_{\underline{x}',i-1})) = 0.$$

So we can apply the induction hypothesis on M/x_1M, and we get that $K(M/x_1M)$ is Cohen-Macaulay. It follows by (iii) that $\operatorname{Rl}(H^{d'-1}_{\mathfrak{m}}(M)) = 0$. Therefore $\mathfrak{m} \notin \operatorname{Att}(H^{d'-1}_{\mathfrak{m}}(M))$. Moreover, $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att} H^{d'-1}_{\mathfrak{m}}(M) \setminus \{\mathfrak{m}\}$. These facts imply that $0 :_{K^{d'-1}_M} x_1 = 0$. So, we have by the exact sequence (*) with respect to $i = d'-1$ that $K(M)/x_1K(M) \cong K(M/x_1M)$. Since $K(M/x_1M)$ is Cohen-Macaulay and x_1 is a regular element of $K(M)$, we get that $K(M)$ is Cohen-Macaulay, i.e. M is Cohen-Macaulay canonical. $\square$

The following result gives a sufficient condition for a ring R to satisfy (MC).

Corollary 4.3. *If there exists a finitely generated $R-$module M with $\dim M = d$, and a strict f-sequence s.o.p. $\underline{x}$ of M such that $\mathfrak{m} \notin \operatorname{Att} H^{d-i}_{\mathfrak{m}}(M_{\underline{x},i-1})$ for all $i = 1, \dots, d-2$, then (MC) is valid for R.*

Proof. It follows by the assumption that $\sum_{i=1}^{d-2} \operatorname{Rl}(H^{d-i}_{\mathfrak{m}}(M_{\underline{x},i-1})) = 0$. Therefore $\sum_{i=1}^{d-2} \operatorname{Rl}(H^{d-i}_{\mathfrak{m}\widehat{R}}(\widehat{M}_{\underline{x},i-1})) = 0$. So, $\widehat{M}$ is Cohen-Macaulay canonical by Theorem 4.2. Hence (MC) is valid for $\widehat{R}$ and for R. □

The following result has been proved by Schenzel [Sc, Theorem 4.3] by using some knowledge on spectral sequences. Here we use strict f-sequences to give an elementary proof.

Corollary 4.4. *Suppose that R has a dualizing complex. If* $\operatorname{depth}(K^i(M)) \geq i-1$ *for all $2 \leqslant i < d'$, then M is Cohen-Macaulay canonical.*

Proof. Let $\underline{x} = (x_1, \dots, x_{d'})$ be a strict f-sequence of M. By Theorem 4.2, it is enough to prove $\sum_{i=1}^{d-2} \operatorname{Rl}(H^{d-i}_{\mathfrak{m}}(M_{\underline{x},i-1})) = 0$. We prove this by induction on d'. The case $d' \leqslant 2$ is trivial. Let $d' \geq 3$. As in the proof of Theorem 4.2, we have the exact sequences

$$0 \longrightarrow K^{i+1}(M)/x_1K^{i+1}(M) \longrightarrow K^i(M/x_1M) \longrightarrow 0 :_{K^i(M)} x_1 \longrightarrow 0$$

for $i = 0, 1, \dots, d'-1$. Let $2 \leqslant i \leqslant d'-1$. Since $\operatorname{depth}(K^i(M)) > 0$, we have $0 :_{K^i(M)} x_1 = 0$. So, by the above exact sequences, $\operatorname{depth} K^i(M/x_1M) \geq i-1$ for all $2 \leqslant i < d'-1$. Therefore $\sum_{i=1}^{(d'-1)-2} \operatorname{Rl}(H^{(d'-1)-i}_{\mathfrak{m}}((M/x_1M)_{\underline{x}',i-1})) = 0$ by the induction hypothesis, where $\underline{x}' = (x_2, \dots, x_{d'})$. So, $\sum_{i=2}^{d'-2} \operatorname{Rl}(H^{d'-i}_{\mathfrak{m}}(M_{\underline{x},i-1})) = 0$. Since $\operatorname{depth}(K^{d'-1}(M)) > 0$, we get $\operatorname{Rl}(H^{d'-1}_{\mathfrak{m}}(M)) = 0$, and the result follows. □

By Theorem 4.2 and by induction on d', we can easily prove the following result. Note that this result has been proved in [Sc, Theorem 4.6, (iii)] in case M is umixed.

Corollary 4.5. *Assume that R admits a dualizing complex and* $\operatorname{depth} M = d'-1$. *Then M is Cohen-Macaulay canonical if and only if $K^{d'-1}(M)$ is Cohen-Macaulay of dimension at least $d'-2$.*

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