

## ATOMIZATION PROCESS FOR CONVOLUTION OPERATORS ON LOCALLY COMPACT GROUPS

CÉDRIC DELMONICO

(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. We develop for a large class of locally compact groups a method of approximation of convolution operators on  $G$  by finitely supported measures with control of the support and of the operator norm of the approximating measures.

### 1. INTRODUCTION

Let  $G$  be a locally compact group, let  $1 < p < \infty$  and let  $CV_p(G)$  be the Banach space of all  $p$ -convolution operators on  $L^p(G)$ . If  $\mu$  is a bounded measure on  $G$ , the corresponding  $p$ -convolution operator is denoted  $\lambda_G^p(f)(\varphi) = \varphi \star (\Delta^{-1/p} \check{f})$ .

If  $G$  is amenable, given  $T \in CV_p(G)$  and  $\Omega$  any open subset containing  $\text{supp } T$ , then the operator  $T$  is in the weak closure of the set of all operators  $\lambda_G^p(f)$ , where  $f$  is a continuous function with compact support ( $f \in C_{oo}(G)$ ) with  $\text{supp } f \subset \Omega$  and  $\|\lambda_G^p(f)\|_p \leq \|T\|_p$  (see [6, Thm. 5]). The following problem is discussed in the present paper: is it true that the operator  $T$  is in the weak closure of the set of all operators  $\lambda_G^p(\mu)$  where  $\mu$  is a finitely supported measure with  $\text{supp } \mu \subset \Omega$  and  $\|\lambda_G^p(\mu)\|_p \leq \|T\|_p$ ?

In [7, 8, 9], N. Lohoué investigated a similar question for  $G$  abelian. He used in a strong way the dual of  $G$ , and his approach only applies to commutative groups. Our techniques are different and apply to a large class of non-commutative groups, such as the Heisenberg group. For a given  $T \in CV_p(G)$ , we construct in an explicit way an approximating net of finitely supported measures converging to  $T$ . In a preprint of [6] C. Herz announced a similar result for locally compact amenable groups. In the final version of the paper this was suppressed.

This article represents a significant portion of the author's doctoral thesis [4]. The author would like to express his deep gratitude to his supervisor, Antoine Derighetti, for support, guidance and encouragement throughout his tenure as a graduate student at the University of Lausanne and afterward as a postdoc at the EPFL of Lausanne. The author would also like to thank Noël Lohoué for the valuable suggestions he gives when he visits us in Lausanne.

---

Received by the editors February 3, 2005 and, in revised form, May 17, 2005.

2000 *Mathematics Subject Classification*. Primary 43A05; Secondary 43A22, 47B38, 47B48.

*Key words and phrases*. Convolution operators, discrete measures, approximations, dense nets.

©2006 American Mathematical Society  
Reverts to public domain 28 years from publication

## 2. PRELIMINARIES

Let  $G$  be a locally compact group and  $\mathcal{H}$  the set of the closed subgroups of  $G$ . For each  $K$  compact subset of  $G$  and  $\mathcal{F}$  finite family of open subsets of  $G$ , we set  $\mathcal{U}(K, \mathcal{F}) = \{H \in \mathcal{H} : H \cap K = \emptyset, H \cap A \neq \emptyset \ \forall A \in \mathcal{F}\}$ . We provide  $\mathcal{H}$  with the topology generated by  $\mathcal{U}(K, \mathcal{F})$ . A net of closed subgroups which is converging to  $G$  in this topology is called a dense net. We use the terminology of [12]. For example, an increasing sequence  $(H_n)_{n \in \mathbb{N}}$  of closed subgroups with  $\bigcup_{n=1}^{\infty} H_n$  dense in  $G$  is a dense net. Let  $m_G$  denote a left Haar measure of  $G$ .

**Theorem 2.1.** *Let  $(H_\alpha)_{\alpha \in I}$  be a dense net of closed subgroups of  $G$ . For each  $\alpha \in I$ , there exists a left Haar measure  $m_\alpha$  of  $H_\alpha$  such that*

$$\lim_{\alpha \in I} \int_{H_\alpha} f(h) dm_\alpha(h) = \int_G f(x) dx, \text{ for all } f \in C_{oo}(G).$$

Moreover, if  $f \in C_{oo}(G)$ , then

$$\int_{H_\alpha} f(xh) dm_\alpha(h) \longrightarrow \int_G f(u) du$$

uniformly on the compact subsets of  $G$ .

This result is due to K. A. Ross & K. Stromberg [11, Thm. 2] for sequences and to J.-L. Rubio [12, Cor. 3] for nets.

The following lemma will be useful.

**Lemma 2.2.** *Let  $\Phi$  be a continuous map of  $G$  into  $(C_{oo}(G), \|\cdot\|_\infty)$  and  $K$  a compact subset of  $G$  such that  $\text{supp}(\Phi(y)) \subset K$ , for all  $y \in G$ . Then, for each  $\varepsilon > 0$  and  $L, V$  compact subsets of  $G$ , there is  $\alpha_0 \in I$  such that, for all  $\alpha \geq \alpha_0$ ,*

$$\left| \int_{H_\alpha} \Phi(y)(xh) dm_\alpha(h) - \int_G \Phi(y)(u) du \right| < \varepsilon,$$

for all  $x \in L$  and  $y \in V$ .

We also need to transfer convolution operators from groups to subgroups and from subgroups to groups.

If  $H$  is a closed subgroup of  $G$ , there is an isometry  $i$  of  $CV_p(H)$  into  $CV_p(G)$  (cf. [5, p. 76]) which satisfies, for all  $T \in CV_p(H)$  and  $r, s \in C_{oo}(G)$ ,

$$\langle i(T)r, s \rangle_{L^p(G), L^{p'}(G)} = \int_{G/H} \langle T \left( \frac{r}{q^{1/p}} \right)_{x,H}, \left( \frac{s}{q^{1/p'}} \right)_{x,H} \rangle_{L^p(H), L^{p'}(H)} d\dot{x},$$

where  $r_{x,H}(h) = r(xh)$ . The following properties are satisfied (cf. [1, 5]):

$$(2.1) \quad \| \| i(T) \| \|_p = \| \| T \| \|_p$$

and

$$(2.2) \quad \text{supp}(i(T)) = \text{supp}(T).$$

Let  $H$  be a closed subgroup of  $G$ ,  $f \in C_{oo}(G)$  and  $\dot{x} \in G/H$ . We define

$$T_{H,q}(f)(\dot{x}) = \int_H \frac{f(xh)}{q(xh)} dh,$$

where  $q(xh) = q(x)\Delta_H(h)\Delta_G(h^{-1})$  (cf. [10]).

For each  $T \in CV_p(G)$  and  $k, l \in C_{oo}(G)$ , J. Delaporte and A. Derighetti defined in [3]

$$(2.3) \quad \begin{aligned} & \langle \Lambda_{k,l}^{(q)}(T)\varphi, \psi \rangle_{L^p(H), L^{p'}(H)} \\ &= \langle T(\tau_p(q^{1/p}(k \star_H \tau_p \varphi))), \tau_{p'}(q^{1/p'}(l \star_H \tau_{p'} \psi)) \rangle_{L^p(G), L^{p'}(G)}, \end{aligned}$$

for all  $\varphi, \psi \in C_{oo}(G)$ . Then  $\Lambda_{k,l}^{(q)}$  is a linear map of  $CV_p(G)$  into  $CV_p(H)$  with

$$(2.4) \quad |||\Lambda_{k,l}^{(q)}(T)||| \leq \|T_H(|k|)\|_p \|T_H(|l|)\|_{p'},$$

$$(2.5) \quad \Lambda_{k,l}^{(q)}(PM_p(G)) \subset PM_p(H),$$

$$(2.6) \quad \text{supp}(\Lambda_{k,l}^{(q)}(T)) \subset H \cap (\text{supp}(k)^{-1} \text{supp}(T) \text{supp}(l)).$$

In the present paper, we always suppose  $q = 1$  (i.e.  $\text{Res}_H(\Delta_G) = \Delta_H$ ). This property is verified for normal closed subgroups or more generally for neutral closed subgroups of  $G$ . In this case, we set  $\Lambda_{k,l}^{(q)} = \Lambda_{k,l}$  and  $T_{H,q} = T_H$ .

**Lemma 2.3.** *Let  $1 < p < \infty$ ,  $G$  a locally compact group,  $H$  a closed subgroup of  $G$ ,  $f \in C_{oo}(G)$ ,  $k, l \in C_{oo}^+(G)$ ,  $\varphi \in L^p(G)$  and  $\psi \in L^{p'}(G)$ . Then, we have*

$$\begin{aligned} & \langle i(\Lambda_{k,l}(\lambda_G^p(f)))\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} - \langle \lambda_G^p(f)\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \\ &= \int_G f(z) \left( \int_G \left( \int_G \Delta_G^{1/p}(u) k(z^{-1}yu) \varphi(u) \right. \right. \\ & \quad \left. \left. \left( T_H(\Delta_G^{1/p'}(yl)\psi)(\pi(u)) - \overline{\int_G \Delta_G^{1/p'}(v) l(yv) \psi(v) dv} \right) du \right) dy \right) dz \\ & \quad + \langle f \star k \star \tau_p \varphi, l \star \tau_{p'} \psi - \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)} \\ & \quad + \langle f \star (k \star \tau_p \varphi - \tau_p \varphi), \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)}, \end{aligned}$$

where  $\pi$  is the canonical map of  $G$  onto  $G/H$ .

*Proof.* Let  $\beta$  be a Bruhat function of  $(G, H)$ . On the one hand, we have

$$\begin{aligned} & \langle i(\Lambda_{k,l}(\lambda_G^p(f)))\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \\ &= \int_{G/H} \langle \Lambda_{k,l}(\lambda_G^p(f))\varphi_{x,H}, \psi_{x,H} \rangle_{L^p(H), L^{p'}(H)} dx \\ &= \int_G \beta(x) \langle \lambda_G^p(f)\tau_p(k \star_H \tau_p(\varphi_{x,H})), \tau_{p'}(l \star_H \tau_{p'}(\psi_{x,H})) \rangle_{L^p(G), L^{p'}(G)} dx \\ &= \int_G f(z) \left( \int_G \left( \int_G \beta(x) T_H \left( \Delta_G^{1/p} (z^{-1}yk) \varphi \left( \overline{T_H(\Delta_G^{1/p'}(yl)\psi) \circ \pi} \right) \right) (\pi(x)) dx \right) dy \right) dz \\ &= \int_G f(z) \left( \int_G \left( \int_G \left( \Delta_G^{1/p} (z^{-1}yk) \varphi \left( \overline{T_H(\Delta_G^{1/p'}(yl)\psi) \circ \pi} \right) \right) (u) du \right) dy \right) dz \\ &= \int_G f(z) \left( \int_G \left( \int_G \Delta_G^{1/p}(u) k(z^{-1}yu) \varphi(u) \overline{T_H(\Delta_G^{1/p'}(yl)\psi)(\pi(u))} du \right) dy \right) dz. \end{aligned}$$

On the other hand,

$$\langle \lambda_G^p(f)\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} = \langle f \star \tau_p \varphi, \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)}$$

and

$$\begin{aligned} & \int_G f(z) \left( \int_G \left( \int_G \Delta_G^{1/p}(u) k(z^{-1}yu) \varphi(u) \left( \int_G \Delta_G^{1/p'}(v) l(yv) \psi(v) dv \right) du \right) dy \right) dz \\ &= \langle f \star k \star \tau_p \varphi, l \star \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)}. \end{aligned}$$

Finally,

$$\begin{aligned} & \langle i(\Lambda_{k,l}(\lambda_G^p(f)))\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} - \langle \lambda_G^p(f)\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \\ &= \langle i(\Lambda_{k,l}(\lambda_G^p(f)))\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} - \langle f \star k \star \tau_p \varphi, l \star \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)} \\ & \quad + \langle f \star k \star \tau_p \varphi, l \star \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)} - \langle f \star k \star \tau_p \varphi, \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)} \\ & \quad + \langle f \star k \star \tau_p \varphi, \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)} - \langle \lambda_G^p(f)\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \\ &= \int_G f(z) \left( \int_G \left( \int_G \Delta_G^{1/p}(u) k(z^{-1}yu) \varphi(u) \right. \right. \\ & \quad \left. \left. \left( \overline{(\mathcal{T}_H(\Delta_G^{1/p'}(y,l)\psi)(\pi(u)) - \int_G \Delta_G^{1/p'}(v) l(yv) \psi(v) dv)} \right) du \right) dy \right) dz \\ & \quad + \langle f \star k \star \tau_p \varphi, l \star \tau_{p'} \psi - \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)} \\ & \quad + \langle f \star (k \star \tau_p \varphi - \tau_p \varphi), \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)}. \end{aligned}$$

□

### 3. ATOMIZATION PROCESS FOR AN INDUCTIVE LIMIT OF DISCRETE GROUPS

Let  $1 < p < \infty$ ,  $G$  a locally compact group and  $(H_\alpha)_{\alpha \in I}$  a dense net of closed subgroups<sup>1</sup> of  $G$ .

**Theorem 3.1.** *Let  $(H_\alpha)_{\alpha \in I}$  be a dense net of closed subgroups of  $G$ . Then, there is a net  $(\mathcal{L}_\beta)_{\beta \in \mathcal{I}}$  of uniformly bounded endomorphisms of  $CV_p(G)$  such that:*

- (i) *for every  $f \in C_{oo}(G)$ ,  $\lambda_G^p(f)$  is a cluster point of the net  $(\mathcal{L}_\beta(\lambda_G^p(f)))_{\beta \in \mathcal{I}}$  in the weak operator topology;*
- (ii) *for each  $\beta = (\varepsilon, k, l, \alpha) \in \mathcal{I}$ ,  $\|\mathcal{L}_\beta\| \leq 1 + \varepsilon$ ;*
- (iii) *for every neighborhood  $U$  of  $\text{supp } f$ , there is  $\beta_0 \in \mathcal{I}$  with  $\text{supp } \mathcal{L}_\beta(\lambda_G^p(f)) \subset U \cap H_\alpha$ , for  $\beta = (\varepsilon, k, l, \alpha) \geq \beta_0$ .*

*Proof.* Let

$$A = \left\{ k \in C_{oo}^+(G) : \int_G k(u) du = 1 \right\}$$

<sup>1</sup>We suppose  $\text{Res}_{H_\alpha}(\Delta_G) = \Delta_{H_\alpha}$ .

and

$$\mathcal{I} = \left\{ (\varepsilon, k, l, \alpha) : 0 < \varepsilon < 1, k, l \in A, \text{ supp } k, \text{ supp } l \text{ neigh. of } e \text{ in } G, \right. \\ \left. \left| \int_{H_\alpha} k(xh) dm_\alpha(h) - 1 \right| < \varepsilon \text{ for all } x \in \text{ supp } k, \right. \\ \left. \left| \int_{H_\alpha} l(xh) dm_\alpha(h) - 1 \right| < \varepsilon \text{ for all } x \in \text{ supp } l \right\}.$$

For  $\beta = (\varepsilon, k, l, \alpha), \beta' = (\varepsilon', k', l', \alpha') \in \mathcal{I}$ , we define the relation  $\beta \leq \beta'$  if and only if  $\varepsilon' \leq \varepsilon, \alpha \leq \alpha', \text{ supp } k' \subset \text{ supp } k$  and  $\text{ supp } l' \subset \text{ supp } l$ . Then,  $(\mathcal{I}, \leq)$  is a directed set.

For each  $\beta = (\varepsilon, k, l, \alpha) \in \mathcal{I}$ , we define the endomorphism of  $CV_p(G)$ ,

$$\mathcal{L}_\beta = i_\alpha \circ \Lambda_{\alpha, k, l},$$

where  $i_\alpha$  is the canonical isometry of  $CV_p(H_\alpha)$  into  $CV_p(G)$  and  $\Lambda_{\alpha, k, l}$  the map of  $CV_p(G)$  into  $CV_p(H_\alpha)$  defined by (2.3).

Let  $\beta = (\varepsilon, k, l, \alpha) \in I, \varphi \in L^p(G)$  and  $\psi \in L^{p'}(G)$ . From (2.1),

$$\left| \langle i_\alpha(\Lambda_{\alpha, k, l}(T))\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right| \leq \|T\|_p \|T_{H_\alpha}(k)\|_p \|T_{H_\alpha}(l)\|_{p'} \|\varphi\|_p \|\psi\|_{p'}.$$

But, the net  $(T_{H_\alpha}(k))_{\alpha \in I}$  is converging to  $\int_G k(u) du = 1$  uniformly on  $\pi_\alpha(\text{ supp } k)$ , and  $\|T_{H_\alpha}(k)\|_p^p = \int_{G/H_\alpha} |T_{H_\alpha}(k)(\dot{x})|^p d\dot{x} \leq \|T_{H_\alpha}(k)\|_\infty^{p-1}$ .

Moreover, for each  $\dot{x} \in \pi_n(\text{ supp } k), |T_{H_\alpha}(k)(\dot{x})| < \varepsilon + 1$ . Then,  $\|T_{H_\alpha}(k)\|_p \leq (\varepsilon + 1)^{1/p'}$  and  $\|T_{H_\alpha}(l)\|_{p'} \leq (\varepsilon + 1)^{1/p}$ . Therefore

$$\left| \langle i_\alpha(\Lambda_{\alpha, k, l}(T))\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right| \leq \|T\|_p (\varepsilon + 1) \|\varphi\|_p \|\psi\|_{p'},$$

and (ii) is verified.

From (2.2) and (2.6), for each  $\beta \in \mathcal{I}$ ,

$$\begin{aligned} \text{supp}(\mathcal{L}_\beta(T)) &= \text{supp}(i_\alpha(\Lambda_{\alpha, k, l}(T))) \subset i_\alpha(\text{supp}(\Lambda_{\alpha, k, l}(T))) \\ &= \text{supp}(\Lambda_{\alpha, k, l}(T)) \subset H_\alpha \cap (\text{supp}(k)^{-1} \text{supp}(T) \text{supp}(l)). \end{aligned}$$

Then, (iii) is verified.

Let  $f \in C_{oo}(G), W \subset G$  a neighborhood of  $\text{supp}(f)^{-1}, \varepsilon_1 > 0, \varphi, \psi \in C_{oo}(G)$  and  $\beta = (\varepsilon, k, l, \alpha_0) \in \mathcal{I}$ .

Let  $0 < \varepsilon \leq \min\{\varepsilon_1, \varepsilon_2\}$ .

There is a compact neighborhood  $V$  of  $e$  with  $V = V^{-1}$  and  $V \text{supp}(f)V \subset W$ . There are  $k', l' \in C_{oo}(G)$  with  $\|k'\|_1 = \|l'\|_1 = 1$  such that  $\text{supp}(k')$  and  $\text{supp}(l')$  are neighborhoods of  $e, \text{supp}(k') \subset \text{supp}(k) \cap V, \text{supp}(l') \subset \text{supp}(l) \cap V,$

$$\|k' \star \tau_p \varphi - \tau_p \varphi\|_p < \frac{\varepsilon}{1 + 3\|f\|_1 \|\psi\|_{p'}}$$

and

$$\|l' \star \tau_{p'} \psi - \tau_{p'} \psi\|_{p'} < \frac{\varepsilon}{1 + 3\|f\|_1 \|\varphi\|_p}.$$

There is  $\alpha_1 \in I$  such that, for all  $\alpha \geq \alpha_1, |T_{H_\alpha}(k')(\dot{x}) - 1| < \varepsilon,$  for each  $\dot{x} \in \pi_\alpha(\text{supp}(k'))$  and  $|T_{H_\alpha}(l')(\dot{x}) - 1| < \varepsilon,$  for each  $\dot{x} \in \pi_\alpha(\text{supp}(l'))$ .

By Lemma 2.2, there is  $\alpha_2 \in I$  such that, for all  $\alpha \geq \alpha_2$ ,

$$\left| \int_{H_\alpha} (\Delta_G^{1/p'}(y)l\psi)(uh)dm_\alpha(h) - \int_G (\Delta_G^{1/p'}(y)l\psi)(v) dv \right| < \frac{\varepsilon}{1 + 3\|f\|_1\|\tau_p\varphi\|_1},$$

for each  $y \in \text{supp}(f)V \text{supp}(\varphi)^{-1}$  and for each  $u \in \text{supp}(\varphi)$ .

Let  $\alpha' \geq \alpha_0, \alpha_1, \alpha_2$ . We define  $\beta' = (\varepsilon, k', l', \alpha')$ . We have  $\beta' \in \mathcal{I}$  and  $\beta' \geq \beta$ .

We have  $\left\{ y \in G : \exists u \in \text{supp}(\varphi), \exists z \in \text{supp}(f) \text{ with } k(z^{-1}yu) \neq 0 \right\} \subset \text{supp}(f)V \text{supp}(\varphi)^{-1}$ . Let us make some estimations:

$$\begin{aligned} & \int_G |f(z)| \left( \int_G 1_A(y) \left( \int_G |\Delta_G^{1/p}(u)k(z^{-1}yu)\varphi(u)| du \right) dy \right) dz \\ &= \int_G |f(z)| \left( \int_G \left( \int_G |\Delta_G^{1/p}(u)k(yu)\varphi(u)| du \right) dy \right) dz \\ &= \int_G |f(z)| dz \int_G |k| \star |\tau_p\varphi|(y) dy = \|f\|_1\|\varphi\|_1. \end{aligned}$$

Then,

$$\begin{aligned} & \left| \int_G f(z) \left( \int_G \left( \int_G \Delta_G^{1/p}(u) k(z^{-1}yu) \varphi(u) \right. \right. \right. \\ & \quad \left. \left. \left. \left( T_{H_\alpha}(\Delta_G^{1/p'}(y)l\psi)(\pi_\alpha(u)) - \int_G \Delta_G^{1/p'}(v) l(yv) \psi(v) dv \right) du \right) dy \right) dz \right| \\ & \leq \int_G |f(z)| \left( \int_G 1_A(y) \left( \int_G |\Delta_G^{1/p}(u)k(z^{-1}yu)\varphi(u)| \right. \right. \\ & \quad \left. \left. \left| T_{H_\alpha}(\Delta_G^{1/p'}(y)l\psi)(\pi_\alpha(u)) - \int_G \Delta_G^{1/p'}(v) l(yv) \psi(v) dv \right| du \right) dy \right) dz \\ & < \int_G |f(z)| \left( \int_G 1_A(y) \left( \int_G |\Delta_G^{1/p}(u)k(z^{-1}yu)\varphi(u)| du \right) dy \right) dz \frac{\varepsilon}{1 + 3\|f\|_1\|\varphi\|_1} \\ & < \frac{\varepsilon}{3}. \end{aligned}$$

Moreover,

$$\begin{aligned} & \left| \langle f \star k \star \tau_p\varphi, l \star \tau_{p'}\psi - \tau_{p'}\psi \rangle_{L^p(G), L^{p'}(G)} \right| \\ & < \|f \star k \star \tau_p\varphi\|_p \|l \star \tau_{p'}\psi - \tau_{p'}\psi\|_{p'} < \|f\|_1 \|k \star \tau_p\varphi\|_p \|l \star \tau_{p'}\psi - \tau_{p'}\psi\|_{p'} \\ & < \|f\|_1 \|k\|_1 \|\tau_p\varphi\|_p \frac{\varepsilon}{1 + 3\|f\|_1\|\varphi\|_p} < \frac{\varepsilon}{3} \end{aligned}$$

and

$$\begin{aligned} & \left| \langle f \star (k \star \tau_p\varphi - \tau_p\varphi), \tau_{p'}\psi \rangle_{L^p(G), L^{p'}(G)} \right| \\ & < \|f \star (k \star \tau_p\varphi - \tau_p\varphi)\|_p \|\tau_{p'}\psi\|_{p'} < \|f\|_1 \|k \star \tau_p\varphi - \tau_p\varphi\|_p \|\psi\|_{p'} \\ & < \|f\|_1 \|\psi\|_{p'} \frac{\varepsilon}{1 + 3\|f\|_1\|\psi\|_{p'}} < \frac{\varepsilon}{3}. \end{aligned}$$

Finally, from Lemma 2.3, we have

$$\left| \langle \mathcal{L}_\beta(\lambda_G^p(f))\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} - \langle \lambda_G^p(f)\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right| < \varepsilon.$$

□

**Corollary 3.2.** *Let  $1 < p < \infty$ ,  $G$  a locally compact group,  $(H_\alpha)_{\alpha \in I}$  a dense net of discrete subgroups of  $G$ . Then, for each  $f \in C_{oo}(G)$ , there is a net  $(\mu_\beta)$  of finitely supported measures of  $G$  such that:*

- (i) *for every  $f \in C_{oo}(G)$ ,  $\lambda_G^p(f)$  is a cluster point of the net  $(\lambda_G^p(\mu_\beta))_{\beta \in I}$  in the weak operator topology;*
- (ii) *for each  $\beta = (\varepsilon, k, l, \alpha) \in I$ ,  $\|\lambda_G^p(\mu_\beta)\| \leq \|\lambda_G^p(f)\|_p + \varepsilon$ ;*
- (iii) *for every neighborhood  $U$  of  $\text{supp } f$ , there is  $\beta_0 \in I$  with  $\text{supp } \mathcal{L}_\beta(\lambda_G^p(f)) \subset U$ , for  $\beta \geq \beta_0$ .*

*Proof.* By Theorem 3.1, we have  $\text{supp } \mathcal{L}_\beta(\lambda_G^p(f)) \subset \text{supp } \Omega_{k,l}(\lambda_G^p(f))$ , where  $\beta = (\varepsilon, k, l, \alpha)$ . But  $\text{supp } \Omega_{k,l}(\lambda_G^p(f))$  is a compact subset of the discrete group  $H_\alpha$  and therefore is a finite set. Finally, there is a finitely supported measure  $\mu_\beta$  with  $\mathcal{L}_\beta(\lambda_G^p(f)) = \lambda_G^p(\mu_\beta)$ . □

**Corollary 3.3.** *Let  $1 < p < \infty$ ,  $G$  a locally compact group with  $(H_\alpha)_{\alpha \in I}$  a dense net of discrete subgroups of  $G$  and  $T \in CV_p(G)$  with compact support. Then, there exists a net of finitely supported measures which converges to  $T$  with control of the support and the operator norms of the approximating measures.*

*Proof.* There exists a net  $(f_\alpha)_{\alpha \in I}$  of positive compactly supported functions of  $G$  with  $\|f_\alpha\|_1 = 1$  such that  $\lambda_G^p(\tau_p(T(\Delta_G^{1/p'} f_\alpha)))$  converges weakly to  $T$  with control of the support and of the operator norm (see [6, Prop. 9]). □

*Remark 3.4.*

- (1) If  $G$  is amenable and has a dense net of discrete subgroups, every  $T \in CV_p(G)$  can be approximated similarly by finitely supported measures with control of the support and of the operator norm. A similar result was formulated in a preprint of [6] but was suppressed in the published version.
- (2) The amenability is not needed for the Corollaries 3.2 and 3.3.

**Example 3.5.** The atomization process is available for the following groups:

- (1)  $\mathbb{R}$  with the subgroups  $H_n = 2^{-n}\mathbb{Z}$ .
- (2)  $\mathbb{T}$  with the subgroups  $H_n = \mathbb{Z}/2^{-n}\mathbb{Z}$ .
- (3) The group of the symmetries of the plane  $0(2)$  with the subgroups  $H_n$  generated by the symmetries of the regular polygons with  $2^n$  sides.
- (4) The Heisenberg groups with the subgroups  $H_n$  of the matrices
 
$$\begin{pmatrix} 1 & a2^{-n} & b2^{-2n} \\ 0 & 1 & c2^{-n} \\ 0 & 0 & 1 \end{pmatrix}, \text{ where } a, b, c \in \mathbb{Z}.$$

This also applies to finite products of groups admitting a dense net of discrete subgroups:

- (5) Elementary groups,  $\mathbb{R}^l \times \mathbb{T}^m \times \mathbb{Z}^n \times F$  with  $l, m, n \in \mathbb{N}$ , where  $F$  is finite.
- (6) Locally compact abelian groups with no small subgroups [2, Prop. 7.9].

4. INDUCTIVE LIMITS

Let  $G$  be a locally compact group and  $(G^{(m)})_{m=1}^\infty$  a dense sequence of closed subgroups of  $G$ . We suppose that for each  $m \in \mathbb{N}$ , there is a dense sequence  $(G_n^{(m)})_{n=1}^\infty$  of discrete subgroups of  $G^{(m)}$ .

With this hypothesis, we have:

**Theorem 4.1.** *There is a net  $(\mathcal{L}_\gamma)_{\gamma \in \mathcal{K}}$  of uniformly bounded endomorphisms of  $CV_p(G)$  such that:*

- (i) for every  $f \in C_{oo}(G)$ ,  $\lambda_G^p(f)$  is a cluster point of the net  $(\mathcal{L}_\gamma(\lambda_G^p(f)))_{\gamma \in \mathcal{K}}$  in the weak operator topology;
- (ii) for each  $\gamma = (\varepsilon, k, l, m, n) \in \mathcal{K}$ ,  $\|\mathcal{L}_\gamma\| \leq 1 + \varepsilon$ ;
- (iii) for every neighborhood  $U$  of  $\text{supp } f$ , there is  $\gamma_0 \in \mathcal{K}$  with  $\text{supp } \mathcal{L}_\gamma(\lambda_G^p(f)) \subset U \cap G_n^{(m)}$ , for every  $\gamma = (\varepsilon, k, l, m, n) \geq \gamma_0$ .

*Proof.* For  $m, n \in \mathbb{N}$ , the maps  $i_{m,n} : G_n^{(m)} \rightarrow G^{(m)}$ ,  $i_m : G^{(m)} \rightarrow G$  and  $\check{i}_{m,n} = i_n \circ i_{m,n} : G_n^m \rightarrow G$  denote the canonical inclusions;  $\pi_m : G \rightarrow G/G^{(m)}$  and  $\pi_{m,n} : G^{(m)} \rightarrow G^{(m)}/G_n^{(m)}$  denote the projections.

For  $k, l \in C_{oo}(G)$ ,  $\Lambda_{m,n,k,l}$ ,  $\Lambda_{m,k,l}$  and  $\check{\Lambda}_{m,n,k,l}$  denote the associated<sup>2</sup> maps of these inclusions.

Let  $f \in C_{oo}(G)$ ,  $\varphi \in L^p(G)$  and  $\psi \in L^{p'}(G)$ .

We consider the set  $\mathcal{K}$  of all  $\gamma = (\varepsilon, k, l, m, n)$  with:

- (a)  $\varepsilon > 0$ ,  $m, n \in \mathbb{N}$ ,  $k, l \in C_{oo}(G)$  with  $\|k\|_1 = \|l\|_1 = 1$ ;
- (b)  $\text{supp}(k)$  and  $\text{supp}(l)$  are neighborhoods of  $e$  in  $G$ ;
- (c)  $\left| \int_{G^{(m)}} k(xh) dm_{G^{(m)}}(h) - 1 \right| < \frac{\varepsilon}{2}$ , for all  $x \in \text{supp}(k)$ ;
- (d)  $\left| \int_{G^{(m)}} l(xh) dm_{G^{(m)}}(h) - 1 \right| < \frac{\varepsilon}{2}$ , for all  $x \in \text{supp}(l)$ ;
- (e)  $\left| \int_{G_n^{(m)}} k(xh) dm_{G_n^{(m)}}(h) - \int_{G^{(m)}} k(u) dm_{G^{(m)}}(u) \right| < \varepsilon$ , for all  $x \in \text{supp}(k) \cap G^{(m)}$ ;
- (f)  $\left| \int_{G_n^{(m)}} l(xh) dm_{G_n^{(m)}}(h) - \int_{G^{(m)}} l(u) dm_{G^{(m)}}(u) \right| < \varepsilon$ , for all  $x \in \text{supp}(l) \cap G^{(m)}$ ;
- (g)  $\left| \langle \check{i}_{m,n}(\Lambda_{m,n,k,l}(\lambda_{G^{(m)}}^p(f)))\varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} - \langle \lambda_{G^{(m)}}^p(f)\varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \right| \leq \varepsilon$ .

Lemma 2.2 and Theorem 3.1 imply that  $\mathcal{K}$  is non-empty. Moreover with the relation  $(\varepsilon, k, l, m, n) \leq (\varepsilon', k', l', m', n')$  if and only if  $\varepsilon' \leq \varepsilon$ ,  $\text{supp}(k') \subset \text{supp}(k)$ ,  $\text{supp}(l') \subset \text{supp}(l)$ ,  $m \leq m'$  and  $n \leq n'$ ,  $\mathcal{K}$  is a directed set.

For each  $\gamma \in \mathcal{K}$ , we define

$$\mathcal{L}_\gamma = \check{i}_{m,n} \circ \check{\Lambda}_{m,n,k,l}.$$

The points (ii) and (iii) are straightforward from the proof of Theorem 3.1.

Let  $\varepsilon > 0$  and  $\gamma_0 = (\varepsilon_0, k_0, l_0, m_0, n_0) \in \mathcal{K}$ .

<sup>2</sup>See (2.3).

I) There are  $k, l \in C_{oo}(G)$  with  $\|k\|_1 = \|l\|_1 = 1$ ,  $\text{supp}(k) \subset \text{supp}(k_0)$ ,  $\text{supp}(l) \subset \text{supp}(l_0)$  and

$$\begin{aligned} \|\tau_p \varphi - k \star_{G^{(m)}} \tau_p \varphi\|_p &< \frac{\epsilon}{16(1 + \|f\|_1 \|\psi\|_{p'})}, \\ \|\tau_{p'} \psi - l \star_{G^{(m)}} \tau_{p'} \psi\|_p &< \frac{\epsilon}{16(1 + \|f\|_1 \|\varphi\|_p)}. \end{aligned}$$

Therefore, for each  $m \in \mathbb{N}$ ,

$$\begin{aligned} &\left| \langle \lambda_{G^{(m)}}^p(f) \varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \right. \\ &\quad \left. - \langle f \star_{G^{(m)}} k \star_{G^{(m)}} \tau_p \varphi, l \star_{G^{(m)}} \tau_{p'} \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \right| < \frac{\epsilon}{8}. \end{aligned}$$

Moreover, by an indirect use of Theorem 2.1, there is  $m_1 \geq m_0$  with, for each  $m \geq m_1$ ,

$$\begin{aligned} &\left| \langle f \star_{G^{(m)}} k \star_{G^{(m)}} \tau_p \varphi, l \star_{G^{(m)}} \tau_{p'} \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \right. \\ &\quad \left. - \langle i_m \left( \Lambda_{m,k,l}(\lambda_G^p(f)) \right) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right| < \frac{\epsilon}{8}. \end{aligned}$$

In conclusion, for all  $m \geq m_1$ , we have

$$\begin{aligned} &\left| \langle i_m \left( \Lambda_{m,k,l}(\lambda_G^p(f)) \right) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} - \langle \lambda_{G^{(m)}}^p(f) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right| \\ &\leq \left| \langle i_m \left( \Lambda_{m,k,l}(\lambda_G^p(f)) \right) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right. \\ &\quad \left. - \langle f \star_{G^{(m)}} k \star_{G^{(m)}} \tau_p \varphi, l \star_{G^{(m)}} \tau_{p'} \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \right| \\ &+ \left| \langle f \star_{G^{(m)}} k \star_{G^{(m)}} \tau_p \varphi, l \star_{G^{(m)}} \tau_{p'} \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \right. \\ &\quad \left. - \langle \lambda_{G^{(m)}}^p(f) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right| < \frac{\epsilon}{4}. \end{aligned}$$

With similar arguments, we can choose  $k, l \in C_{oo}(G)$  with  $\|k\|_1 = \|l\|_1 = 1$  and  $m_1 \in \mathbb{N}$  such that, for each  $m \geq m_1$  and  $n \in \mathbb{N}$ , we also have

$$\begin{aligned} &\left| \langle \check{i}_{m,n} \left( \check{\Lambda}_{m,n,k,l}(\lambda_G^p(f)) \right) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right. \\ &\quad \left. - \langle i_{m,n} \left( \Lambda_{m,n,k,l}(\lambda_{G^{(m)}}^p(f)) \right) \varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \right| < \frac{\epsilon}{4}. \end{aligned}$$

II) From Theorem 3.1, there exists  $m \in \mathbb{N}$  with  $(\epsilon_0, k, l, m, n) \geq (\epsilon_0, k, l, m_1, n_0)$ , such that

$$\begin{aligned} &\left| \langle i_m \left( \Lambda_{m,k,l}(\lambda_G^p(f)) \right) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right. \\ &\quad \left. - \langle \lambda_G^p(f) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right| < \frac{\epsilon}{4}. \end{aligned}$$

But,  $(G_n^{(m)})_{n=1}^\infty$  is a dense sequence of  $G^{(m)}$ . Therefore, by Theorem 3.1, there exists  $n \geq n_0$  such that

$$\begin{aligned} &\left| \langle i_{m,n} \left( \Lambda_{m,n,k,l}(\lambda_{G^{(m)}}^p(f)) \right) \varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \right. \\ &\quad \left. - \langle \lambda_{G^{(m)}}^p(f) \varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \right| < \frac{\epsilon}{4}. \end{aligned}$$

III) Finally, let  $\gamma = (\varepsilon_0, k, l, m, n)$ . We have  $\gamma \geq \gamma_0$  and

$$\begin{aligned} & \left| \langle \check{i}_{m,n} \left( \check{\Lambda}_{m,n,k,l}(\lambda_G^p(f)) \right) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} - \langle \lambda_G^p(f) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right| \\ & \leq \left| \langle \check{i}_{m,n} \left( \check{\Lambda}_{m,n,k,l}(\lambda_G^p(f)) \right) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right. \\ & \quad \left. - \langle i_{m,n} \left( \Lambda_{m,n,k,l}(\lambda_{G^{(m)}}^p(f)) \right) \varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \right| \\ & + \left| \langle i_{m,n} \left( \Lambda_{m,n,k,l}(\lambda_{G^{(m)}}^p(f)) \right) \varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \right. \\ & \quad \left. - \langle \lambda_{G^{(m)}}^p(f) \varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \right| \\ & + \left| \langle \lambda_{G^{(m)}}^p(f) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right. \\ & \quad \left. - \langle i_m \left( \Lambda_{m,k,l}(\lambda_G^p(f)) \right) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right| \\ & + \left| \langle i_m \left( \Lambda_{m,k,l}(\lambda_G^p(f)) \right) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right. \\ & \quad \left. - \langle \lambda_G^p(f) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right| < \epsilon. \end{aligned}$$

□

**Example 4.2.** Consequently, we have an atomization process for abelian metrizable locally connected groups [2, Prop. 8.17].

#### REFERENCES

1. Jean-Philippe Anker, *Applications de la  $p$ -induction en analyse harmonique*, Comment. Math. Helv. **58** (1983), no. 4, 622–645. MR0728457 (85c:22006)
2. David L. Armacost, *The structure of locally compact abelian groups*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 68, Marcel Dekker, Inc., New York, 1981. MR0637201 (83h:22010)
3. Jacques Delaporte and Antoine Derighetti,  *$p$ -pseudomeasures and closed subgroups*, Monatsh. Math. **119** (1995), no. 1-2, 37–47. MR1315682 (97a:43001)
4. Cédric Delmonico, *Opérateurs de convolution et homomorphismes de groupes localement compacts*, Ph.D. thesis, Université de Lausanne, 2003.
5. Antoine Derighetti, *Relations entre les convoluteurs d'un groupe localement compact et ceux d'un sous-groupe fermé*, Bull. Sci. Math. (2) **106** (1982), no. 1, 69–84. MR0654425 (83j:43008)
6. Carl Herz, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier (Grenoble) **23** (1973), no. 3, 91–123. MR0355482 (50:7956)
7. Noël Lohoué, *Sur le critère de S. Bochner dans les algèbres  $B_p(R^n)$  et l'approximation bornée des convoluteurs*, C. R. Acad. Sci. Paris Sér. A-B **271** (1970), A247–A250. MR0273300 (42:8180)
8. ———, *Algèbres  $A_p(G)$  et convoluteurs de  $L^p(G)$* , Doctorat d'état, Université Paris-Sud, Centre d'Orsay, 1971.
9. ———, *La synthèse des convoluteurs sur un groupe abélien localement compact*, C. R. Acad. Sci. Paris Sér. A-B **272** (1971), A27–A29. MR0275062 (43:820)
10. Hans Reiter and Jan D. Stegeman, *Classical harmonic analysis and locally compact groups*, second ed., The Clarendon Press, Oxford University Press, New York, 2000. MR1802924 (2002d:43005)

11. Kenneth A. Ross and Karl Stromberg, *Jessen's theorem on Riemann sums for locally compact groups*, Pacific J. Math. **20** (1967), 135–147. MR0207889 (34:7702)
12. Jose L. Rubio, *Nets of subgroups in locally compact groups*, Comment. Math. Prace Mat. **20** (1977/78), no. 2, 453–466. MR0519384 (80c:43002)

SCHOOL OF BASIC SCIENCES, ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE, STATION 8,  
CH-1005 LAUSANNE, SWITZERLAND

*E-mail address:* `cedric.delmonico@a3.epfl.ch`