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A NOTE ON THE ENGULFING PROPERTY AND THE $\Gamma^{1+\alpha}$-REGULARITY OF CONVEX FUNCTIONS IN CARNOT GROUPS

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Abstract. We study the engulfing property for convex functions in Carnot groups. As an application we show that the horizontal gradient of functions with this property is Hölder continuous.

1. Introduction

A celebrated result of Caffarelli [3] establishes that every strictly convex, generalized solution u of the Monge-Ampère equation in $\mathbb{R}^n$, $\det D^2u = \mu$ must be in the class $C^{1,\alpha}_{loc}(\mathbb{R}^n)$ for some $\alpha \in (0,1)$, provided the Borel measure μ satisfies a suitable doubling property (see also the monograph by Gutierrez [15] and references therein). This result is at the core of the geometrical approach to the regularity theory for Monge-Ampère. The Aleksandrov-Bakelman-Pucci (ABP) maximum principle is a crucial tool in this approach. It serves as a link between the measure theoretic and geometric aspects of the solution u. As an instance of its applications, it can be proved that the mentioned doubling property of μ is equivalent to a geometric property of the sections of u known as the $E(\mathbb{R}^n, K)$ *engulfing property* (see Caffarelli [2], Gutiérrez and Huang [16], Gutierrez [15], and Forzani and Maldonado [11]).

Recently there has been considerable interest in the study of fully non-linear equations in Carnot groups. In particular, several interesting results concerning Monge-Ampère-type equations and notions of Hessian measures have been proved by Gutierrez and Montanari [17], [18], Garofalo and Tournier [14], Danielli *et al.* [4], [6], Lu *et al.* [22]. At the moment it seems very difficult to obtain sub-Riemannian analogues of Euclidean regularity results such as Caffarelli's [3]. There are two different kinds of obstacles met when attempting to achieve this goal. On one hand, presently it is not clear which equation or which Hessian measure is the sub-Riemannian *correct* analogue of the Euclidean objects. On the other hand, proving regularity of the gradient of solutions of subelliptic equations is generally quite difficult because of the non-commutativity of the vector fields involved. We believe that a more geometric point of view would greatly benefit the analytic intuition in

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this area. For instance, one needs a better understanding of the properties of the horizontal Gauss map in order to even formulate an appropriate ABP principle.

The purpose of this note is to show that, independently of the ABP principle, one can still introduce appropriate geometric objects, i.e. a sub-Riemannian analogue of the engulfing property $E(G, K)$ (see Definition 3.4 below), to obtain regularity in the Folland-Stein class $\Gamma^{1+\alpha}$ of convex functions in Carnot groups.

The main result of this paper is the following.

Theorem 1.1. *Let G be a Carnot group and consider a strictly convex, everywhere differentiable function u which satisfies the engulfing property $E(G, K)$. If $X_1, ..., X_m$ denote a left-invariant basis for the first layer of the stratification of the Lie algebra of G, then we have that $X_i u$, $i = 1, ..., m$, is $1/K$-Hölder continuous with respect to any left-invariant homogeneous pseudo-norm in G.*

Unlike the proofs in [3] and [15] in the Euclidean setting, our proof for Theorem 1.1 will be a constructive one. This will allow us to quantify the Hölder exponent for $X_i u$. The key point in our argument is a reduction of the general discussion to the one-dimensional case (see Lemma 3.5 below). Our approach is strongly influenced by the techniques used by Forzani and Maldonado in [10], [11], and [12].

To conclude this Introduction, we recall that in the Euclidean setting the engulfing property is equivalent to the doubling property for the Hessian measure. If an analogue equivalence were to hold in the setting of Carnot groups, then Theorem 1.1 would immediately yield a regularity result for solutions to subelliptic Monge-Ampère-type equations similar to the one proved in [3]. From this point of view, the main contribution of this note is to shift the focus of the regularity theory more towards the study of the geometry of solutions.

2. Carnot groups

Let G be an analytic and simply connected Lie group such that its Lie algebra $\mathcal{G}$ admits a stratification $\mathcal{G} = V^1 \oplus V^2 \oplus ... \oplus V^r$, where $[V^1, V^j] = V^{j+1}$, if $j = 1, ..., r-1$, and $[V^k, V^r] = 0$, $k = 1, ..., r$. Such groups are called *stratified nilpotent Lie groups* in [7], [9], and [25]. We will call them more briefly *Carnot groups*. For $k = 1, ..., r$, we let $m_k = \dim(V^k)$ and denote by $X_{1,k}, ..., X_{m_k,k}$ a basis of V^k. The exponential map $\exp : \mathcal{G} \to G$ is a global diffeomorphism, and we use exponential coordinates in G, i.e., if

$$x = \exp\left(\sum_{k=1}^{r}\sum_{i=1}^{m_k} x_{i,k} X_{i,k}\right),$$

then we will write $x = (x_{i,k})_{i=1,...,m_k}^{k=1,...,r}$. Define non-isotropic dilations as $\delta_s(x) = (s^k x_{i,k})$, for $s > 0$. Throughout the paper, for any $x \in G$, we denote by x_1 the projection of x onto the first layer V^1, i.e. $x_1 = (x_{1,1}, ..., x_{m_1,1})$. We set $H(0) = V^1$, and for any $x \in G$ we let $H(x) = xH(0) = \text{span}[X_{1,1}, ..., X_{m_1,1}](x)$. The distribution $H(x)$ is called *the horizontal subbundle*.

The vectors $X_{1,1}....X_{m_1,1}$ and their commutators span all the Lie algebra $\mathcal{G}$, and consequently verify Hörmander's finite rank condition ([19]). Following [23], one can define a left invariant control metric $d_C(x, y)$ associated to the distribution $X_{1,1}....X_{m_1,1}$, which is called *the Carnot-Carathéodory metric*. If $x \in G$ and $r > 0$, we will denote by $B_C(x, r) = \{y \in G \mid d_C(x, y) < r\}$ the metric balls in the control metric d_C.

The Carnot-Carathéodory metric is equivalent to a more explicitly defined pseudo-distance function, that we will call (improperly) *gauge distance*, defined as

$$|x|_G^{2r!} = \sum_{k=1}^{r}\sum_{i=1}^{m_k} |x_{i,k}|^{\frac{2r!}{k}} \text{ and } d(x,y) = |y^{-1}x|_G.$$

From the results of Nagel, Stein and Wainger in [23], there exists a constant $a = a(G) > 1$ such that

$$a^{-1}d_C(x,y) \leq d(x,y) \leq a d_C(x,y), \tag{2.1}$$

for any $x, y \in G$. Both d_C and d give rise to the same notions of differentiability, and rectifiable curves. If we denote by $B(x,r)$ the metric balls in the pseudo distance $d(x,y)$, then $|B(x,r)| = c_G r^Q$, where $Q = \sum_{k=1}^{r}\sum_{i=1}^{m_k} km_k$ denotes the *homogeneous dimension* of the group G.

If u is a function defined on G we will use the notation $Xu = (X_{1,1}u, ..., X_{m_1,1}u)$. If $x_0 \in G$, $0 < \alpha < 1$, and f is a function defined in a neighborhood Ω of x_0, we can define the Folland-Stein Hölder norm of f at x_0 as

$$||f||_{\Gamma^\alpha(x_0)} = \sup_{x \neq x_0, x \in \Omega} \frac{|f(x) - f(x_0)|}{|x_0^{-1}x|_G^\alpha}.$$

If $\Omega \subset G$ is an open set and $0 < \alpha < 1$, then we define

$$\Gamma^\alpha(\Omega) = \{f : \Omega \to \mathbb{R} \mid ||f||_{\Gamma^\alpha(x)} \leq C(f, \Omega, \alpha) < \infty \text{ for any } x \in \Omega\}.$$

The local version of the Hölder space is defined as follows:

$$\Gamma^\alpha_{\text{loc}}(\Omega) = \{f : \Omega \to \mathbb{R} \mid af \in \Gamma^\alpha(\Omega), \text{ for some } a \in C_0^\infty(\Omega)\}.$$

We also have the analogue of the Euclidean $C^{1,\alpha}$ class, i.e.

$$\Gamma^{1+\alpha}_{\text{loc}}(\Omega) = \{f : \Omega \to \mathbb{R} \mid X_{i,1}f \in \Gamma^\alpha_{\text{loc}}(\Omega) \text{ for any index } i = 0, ..., m_1\}.$$

We will measure Hölder continuity of functions by means of the Campanato $1+\alpha$ class (see p. 142 in [9] for a more general definition).

Definition 2.1. Let $B(x,R) \subset G$. If $\alpha \in (0,1)$, we set $C^{1+\alpha}_{\infty,1}(x)$ to be the space of all functions $u \in L^\infty_{\text{loc}}(B(x,R))$ such that

$$[u]_{\infty,1+\alpha}(x) = \sup_{0<r<R} \inf_{P \in \mathbb{R}^{m_1}} r^{-1-\alpha} \sup_{0<|y|<r} |u(xy) - u(x) - \langle P, y_1\rangle| < \infty.$$

If $\Omega \subset G$ is an open set, we define $C^{1+\alpha}_{\infty,1}(\Omega)$ to be the set of functions u such that $[u]_{\infty,1+\alpha,\Omega} = \sup_{B(x,R)\subset\Omega}[u]_{\infty,1+\alpha}(x) < \infty$.

The relation between $C^{1+\alpha}_{\infty,1}$ and the Folland-Stein Hölder space $\Gamma^{1+\alpha}$ is given in the following theorem (see [9], [20] and [21]).

Theorem 2.2. *Let $\alpha \in (0,1)$. If $u \in C^{1+\alpha}_{\infty,1}(B(x_0,r))$, then $Xu \in \Gamma^\alpha(B(x_0,r))$.*

The next two lemmata allow for a weaker form of the Morrey-Campanato norm, involving only horizontal directions. First we recall a result of Folland-Stein [9, Lemma 1.40].

Lemma 2.3. *There exists $C > 0$ and $N \in \mathbb{N}$ such that for all $y \in G$ one can find $x_1, ..., x_n \in \exp H(0)$ such that $y = x_1...x_n$, $n \leq N$, and $|x_i| \leq C|y|$.*

We also observe explicitly that if $x_1, ..., x_n \in \exp H(0)$, then

$$\pi_1(x_1...x_n) = x_1 + ... + x_n. \tag{2.2}$$

Lemma 2.4. *Let $\Omega \subset G$ be an open set, $u \in L^\infty_{loc}(\Omega)$ and $\alpha \in (0,1)$. If there exists $C = C(u,\Omega) > 0$ such that for any $B(x,R) \subset \Omega$ the function u satisfies*

$$\sup_{0<r<R} \inf_{P\in\mathbb{R}^{m_1}} r^{-1-\alpha} \sup_{y\in\exp H(0), 0<|y|<r} |u(xy) - u(x) - \langle P, y\rangle| < \infty,$$

then $Xu \in \Gamma^\alpha(\Omega)$.

Proof. Let $x, y \in G$ and u as in the hypothesis. In view of (2.2) we have that

$$\begin{aligned} u(xy) - \left[u(x) - \left\langle P, \pi_1(y)\right\rangle\right] &= u(xx_1...x_n) - \left[u(x) - \left\langle P, \pi_1(x_1...x_n)\right\rangle\right] \\ &= \sum_{k=1}^{n} \left(u(xx_1...x_k) - u(xx_1...x_{k-1}) - \left\langle P, x_k\right\rangle\right), \end{aligned} \tag{2.3}$$

for all $P \in \mathbb{R}^{m_1}$. If we consider the supremum in y of the left-hand side and bound it with the supremum over all choices $x_1, ..., x_n$ as in Lemma 2.3, we obtain

$$\begin{aligned} &\inf_{P\in\mathbb{R}^{m_1}} \sup_{0<|y|<R} \left|u(xy) - \left[u(x) - \left\langle P, \pi_1(y)\right\rangle\right]\right| \\ &\leq \inf_{P\in\mathbb{R}^{m_1}} \sum_{k=1}^{n} \sup_{x_k\in\exp H(0), 0<|x_k|<R} \left|u(xx_1...x_k) - \left[u(xx_1...x_{k-1}) - \left\langle P, x_k\right\rangle\right]\right|. \end{aligned} \tag{2.4}$$

In view of the hypotheses there exists a constant $C = C(u,\Omega) > 0$ such that the left-hand side of (2.4) is bounded by $CR^{1+\alpha}$. We have thus proved that

$$u \in C^{1+\alpha}_{\infty,1}(x)$$

and that $\sup_{B(x,R)\subset\Omega} [u]_{\infty,1+\alpha}(x) < C$. The result now follows from Theorem 2.2. □

3. Convex functions and the engulfing property

Roughly speaking, the notion of convex functions in the subelliptic setting takes into account both the behavior of a function along the horizontal directions and the non-integrable structure of the horizontal subbundle.

Convex functions in the Heisenberg group setting were first introduced by Luis Caffarelli (in unpublished work from 1996). This notion did not really surface in the literature until 2002, when it was independently formulated and studied, in the more general setting of Carnot groups, in [4] and in [22]. Essentially, a convex function in G is a function whose restriction to horizontal lines through any fixed point are Euclidean convex functions of one variable.

Definition 3.1. A proper function $u : G \to \mathbb{R}$ is convex if for any $x \in G$ and $s \in [0,1]$ one has

$$u(x\delta_s(y)) \leq (1-s)u(x) + su(xy) \text{ for all } y \in \exp H(0).$$

As a consequence of the work in [4] and [22] one has that convex functions are Lipschitz continuous with respect to the Gauge pseudo-distance $d(\cdot,\cdot)$, hence differentiable almost everywhere along the horizontal directions $X_{1,1}, ..., X_{m_1,1}$ (see for instance [24] and [13]).

For any $x \in G$ and $y \in \exp H(0)$ we define the function $\phi_{x,y} : \mathbb{R} \to \mathbb{R}$ as

$$\phi_{x,y}(s) = u(x\delta_s(y)). \tag{3.1}$$

In [22, Lemma 4.1] it is shown that u is convex if and only if $\phi_{x,y}$ is convex for all choices of $x \in G$ and $y \in \exp H(0)$. If u is twice differentiable along the horizontal directions at a point $x\delta_s(y)$, then a direct computation (see for instance [4, Proposition 5.2 and 5.4]) yields

$$\phi'_{x,y}(s) = \langle Xu(x\delta_s(y)), y\rangle \text{ and } \phi''_{x,y}(s) = \langle D^2u(x\delta_s(y))y, y\rangle, \tag{3.2}$$

where D^2u is the *horizontal Hessian*, i.e. the symmetric part of the $m_1 \times m_1$ matrix whose (i,j) entry is $X_{i,1}X_{j,1}u$.

Definition 3.2. Let $u : G \to \mathbb{R}$ be a convex function. Let $x \in G$, $P \in \mathbb{R}^{m_1}$ and $R > 0$. The section $S(x, P, R)$ is given by

$$\left\{xy \text{ such that } y \in \exp H(0) \text{ and } u(xy) < u(x) + \left\langle P, y\right\rangle + R\right\}.$$

If P is omitted, then we are assuming u, horizontally differentiable at x and $S(x, R)$, is implicitly defined by the choice $P = Xu(x)$.

Definition 3.3. Let $u : G \to \mathbb{R}$ be a convex function. Let $x \in G$, $y \in \exp H(0)$ and $t \in \mathbb{R}$ such that $x\delta_t(y)$ is a differentiability point for u. For $R > 0$ we define the section

$$\begin{aligned} S_{\phi_{x,y}}(t, R) = &\quad \left\{s \in \mathbb{R} \mid u(x\delta_s(y)) < u(x\delta_t(y)) + \left\langle Xu(x\delta_t(y)), y\right\rangle (s-t) + R\right\} \\ = &\quad \left\{s \in \mathbb{R} \mid \phi_{x,y}(s) < \phi_{x,y}(t) + \phi'_{x,y}(t)(s-t) + R\right\}. \end{aligned} \tag{3.3}$$

Next we introduce the Carnot group analogue of the *engulfing property* introduced by Caffarelli.

Definition 3.4. We say that the differentiable, convex function $u : G \to \mathbb{R}$ satisfies the engulfing property $E(G, K)$ if there exists $K \geq 1$ such that for any $x \in G$ and $R > 0$, if $xy \in S(x, R)$, then $x \in S(xy, KR)$.

If $G = \mathbb{R}^n$ is the usual Euclidean space, then this definition is the classical one (see for instance [15, Section 3.3.2]). In this setting, examples of convex functions verifying the engulfing property include those functions u such that $0 < \lambda \leq \det D^2u \leq \Lambda$, in the Aleksandrov sense, for some constants λ, Λ. The archetypal example is the function $u(x) = |x|^2/2$, whose sections coincide with the Euclidean balls. More examples come from functions u verifying $\det D^2u = p$, where p is a (non-negative) polynomial. In this case the constant K depends on n and the degree of p, but not on its coefficients (see [15]). Non-Euclidean examples are constructed in Corollary 3.8.

The following two results are crucial steps in the proof of Theorem 1.1. The first one allows us to reduce matters to the one-dimensional situation. The second one provides quantitative information about the engulfing property in dimension one.

Lemma 3.5. *If $u : G \to \mathbb{R}$ is a differentiable convex function, $x \in G$, $y \in \exp H(0)$ and $\phi_{x,y}$ is defined as in* (3.1)*, then $u \in E(G, K)$ if and only if $\phi_{x,y} \in E(\mathbb{R}, K)$.*

Proof. Assume that $\phi_{x,y}$ satisfies the engulfing property $E(\mathbb{R}, K)$. Choose $xy \in S(x, R)$, so that $y \in \exp H(0)$ and

$$u(xy) < u(x) + \left\langle Xu(x),\ y \right\rangle + R. \tag{3.4}$$

The latter can also be read in terms of $\phi_{x,y}$ as $1 \in S_{\phi_{x,y}}(0, R)$. Consequently we have $0 \in S_{\phi_{x,y}}(1, KR)$, which in turn implies

$$u(x) < u(xy) + \left\langle Xu(xy),\ y \right\rangle (-1) + KR. \tag{3.5}$$

Observe that $(xy)^{-1}x = y^{-1}$ and that, since $y \in \exp H(0)$, $y^{-1} = -y$. From (3.5) we conclude $x = xyy^{-1} \in S(xy, KR)$.

Vice versa, if $s \in S_{\phi_{x,y}}(t, R)$, then $u(x\delta_s y) < u(x\delta_t y) + \langle Xu(x), y\rangle(s - t) + R$, that is, $x\delta_s y \in S(x\delta_t y, R)$. Assuming $u \in E(G, K)$ we have $x\delta_t y \in S(x\delta_s y, KR)$ which immediately yields $t \in S_{\phi_{x,y}}(s, KR)$, concluding the proof. □

Lemma 3.6. *If $\phi \in E(\mathbb{R}, K)$, $\phi(0) = 0$, then for any $t \in \mathbb{R}$ one has*

$$\frac{1}{K}\Big(\phi(t) - \phi'(0)t\Big) \le \phi'(t)t - \phi(t) \le K\Big(\phi(t) - \phi'(0)t\Big).$$

Proof. Let $t \in \mathbb{R}$ and set $R = \phi'(t)t - \phi(t)$. Since ϕ is convex we have that $R \ge 0$. For any $\epsilon > 0$ we then obtain $0 = \phi(0) < \phi(t) + \phi'(t)(-t) + R + \epsilon$, that is, $0 \in S(t, R + \epsilon)$. The engulfing property $E(\mathbb{R}, K)$ yields

$$\phi(t) < \phi(0) + \phi'(0)t + K(R + \epsilon),$$

which in turn implies

$$\frac{\phi(t) - \phi'(0)t}{K} < \phi'(t)t - \phi(t) + \epsilon, \tag{3.6}$$

for all $\epsilon > 0$ and $t \in \mathbb{R}$. To prove the second inequality in (3.6) we let $\tilde{R} = -\phi'(0)t - \phi(0) + \phi(t)$ and observe that by convexity $\tilde{R} \ge 0$. Consequently we can write $\phi(t) < \phi(0) + \phi'(0)t + \tilde{R} + \epsilon$, i.e., $t \in S(0, R + \epsilon)$. We invoke $\phi \in E(\mathbb{R}, K)$ and obtain

$$\phi(0) < \phi(t) - \phi'(t)t + K[-\phi'(0)t - \phi(0) + \phi(t) + \epsilon].$$

The latter immediately yields

$$\frac{\phi'(t)t - \phi(t)}{K} < \phi(t) - \phi'(0)t + \epsilon. \tag{3.7}$$

The lemma follows from (3.6) and (3.7) by letting $\epsilon \to 0$. □

In [12], Forzani and the second-named author proved that the comparison property in the previous lemma actually characterizes the engulfing property: If $\phi : \mathbb{R} \to \mathbb{R}$ is a convex differentiable function, then $\phi \in E(\mathbb{R}, K)$ if and only if

$$\begin{aligned}\frac{K+1}{K}&\left(\phi(y) - \phi(x) - \phi'(x)(y - x)\right)\\ &\le \left(\phi'(x) - \phi'(y)\right)(x - y)\\ &\le (K+1)\left(\phi(y) - \phi(x) - \phi'(x)(y - x)\right),\end{aligned}$$

for every $x, y \in \mathbb{R}$. Also, if ϕ is strictly convex, then $\phi \in E(\mathbb{R}, K)$ if and only if ϕ' is a quasi-symmetric mapping in $\mathbb{R}$ (see [10]).

We conclude this section by constructing more examples of functions satisfying the engulfing property. Let us recall the following important result.

Proposition 3.7. *If $\phi : \mathbb{R} \to \mathbb{R}$ is a differentiable, strictly convex function such that $\mu(a,b) = \phi'(b) - \phi'(a)$ (that is, $\mu = \phi''$ in the weak sense) is a doubling measure, then $\phi \in E(\mathbb{R}, K)$, where K depends only on the doubling constants. Moreover, if $m \le \mu \le M$, for some positive constants m and M, then $K = K(M/m)$.*[1]

This proposition is due to Gutiérrez and Huang [16], and it holds in any dimension. Moreover, its converse is also true, i.e. *the engulfing property of ϕ is equivalent to the doubling property of its Hessian measure* (see [11]).

Using (3.2), Lemma 3.5, and applying Proposition 3.7 to the restrictions $\phi_{x,y}$ of a strictly convex function u to horizontal lines we immediately obtain

Corollary 3.8. *Let $u : G \to \mathbb{R}$ be a strictly convex function. If the horizontal restrictions $\phi_{x,y}$ give rise to doubling measures as in Proposition 3.7 with doubling constants uniform in $x \in G$ and unit $y \in \mathbb{R}^m$, then $u \in E(G, K)$ for some choice of K depending on the doubling constants. In particular, if there exists $C > 0$ such that the minimum eigenvalue λ and the maximum eigenvalue Λ of the horizontal Hessian D^2u satisfy $C^{-1} < \lambda \le \Lambda < C$, then $u \in E(G, K)$ for some $K = K(\Lambda/\lambda)$.*

Remark 3.9. We do not know if the hypotheses $C^{-1} < \lambda\Lambda < C$ would guarantee the engulfing property for strictly convex functions. If true, this would provide a link between the *horizontal* Monge-Ampère measure $detD^2u$ and the geometry of u in the spirit of the Gutiérrez-Huang result. On the other hand, in view of the counterexamples from [5], $\mu = detD^2u$ might not be the appropriate Hessian measure to be considered.

4. Proof of Theorem 1.1

Lemma 3.6 yields

$$\left(1 + \frac{1}{K}\right)\frac{1}{t} \le \left[\frac{\phi'(t) - \phi'(0)}{\phi(t) - \phi'(0)t}\right] \le \left(1 + K\right)\frac{1}{t}, \tag{4.1}$$

for all $t > 0$. The first inequality implies that

$$t \mapsto \frac{\phi(t) - \phi'(0)t}{t^{1+\frac{1}{K}}}$$

is non-decreasing in $(0, \infty)$. Hence, for any choice of $t_0 > 0$ we deduce that

$$\sup_{0<t<t_0} \frac{\phi(t) - \phi'(0)t}{t^{1+\frac{1}{K}}} = \frac{\phi(t_0) - \phi'(0)t_0}{t_0^{1+\frac{1}{K}}}.$$

It is an easy exercise to check that if $\phi \in E(\mathbb{R}, K)$ and $\varphi(t) := \phi(-t)$, then $\varphi \in E(\mathbb{R}, K)$. Therefore, we obtain

$$\sup_{0<|t|<t_0} \frac{\left|\phi(t) - \phi'(0)t\right|}{|t|^{1+\frac{1}{K}}} = \max\left(\frac{\phi(t_0) - \phi'(0)t_0}{t_0^{1+\frac{1}{K}}}, \frac{\phi(-t_0) + \phi'(0)t_0}{t_0^{1+\frac{1}{K}}}\right) =: C. \tag{4.2}$$

[1] For the exact dependence see [12, Section 7].

Reading (4.2) in terms of the function u we have

$$\sup_{y\in\exp H(0),0<|y|<1} \frac{\left|u(xy)-u(x)-<Xu(x),y>\right|}{|y|^{1+\frac{1}{K}}} \leq C. \tag{4.3}$$

The proof follows from Lemma 2.4 and (4.3). □

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