PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 11, November 2006, Pages 3143–3151
S 0002-9939(06)08369-9
Article electronically published on May 18, 2006

SIMULTANEOUS NON-VANISHING OF TWISTS

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(Communicated by Wen-Ching Winnie Li)

ABSTRACT. Let f be a newform of even weight k, level M and character ψ and let g be a newform of even weight l, level N and character η. We give a generalization of a theorem of Elliott, regarding the average values of Dirichlet L-functions, in the context of twisted modular L-functions associated to f and g. Using this result, we find a lower bound in terms of Q for the number of primitive Dirichlet characters modulo prime $q \leq Q$ whose twisted product L-functions $L_{f,\chi}(s_0)L_{g,\chi}(s_0)$ are non-vanishing at a fixed point $s_0 = \sigma_0 + it_0$ with $\frac{1}{2} < \sigma_0 \leq 1$.

1. Introduction

Let $L_\chi(s) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ be the Dirichlet L-function associated to a Dirichlet character χ. In [E], Elliott proved the following.

Theorem. *Let $Q \geq 2$ be a real number, and $s_0 = \sigma_0 + it_0$ a complex number in the half-plane $\sigma_0 > \frac{1}{2}$. Then we have*

$$\sum_{p \leq Q} \sum_{\chi \neq \chi_0} |L_\chi(s_0)|^2 = \frac{Q^2}{2 \log Q} \zeta(2\sigma_0) + O\left(\frac{Q^2}{(\log Q)^2}\right)$$

as $Q \to \infty$. Here the inner sum is taken over all non-principal characters (mod p), *for each prime p, and the outer sum over all prime numbers not exceeding Q.*

Our first goal in this paper is to give a generalization of this theorem in the context of twisted modular L-functions. Let $S_k(\Gamma_0(M), \psi)$ be the space of holomorphic cusp forms of even weight k, level M and character ψ. For $f \in S_k(\Gamma_0(M), \psi)$, let

$$f(z) = \sum_{n=1}^{\infty} a_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}$$

be the Fourier expansion of f at $i\infty$. Let χ be a primitive Dirichlet character mod q with $(q, M) = 1$. Then the twisted L-function associated to f and χ is defined (for $Re(s) > 1$) by

$$L_{f,\chi}(s) = \sum_{n=1}^{\infty} \frac{a_f(n)\chi(n)}{n^s}.$$

Received by the editors August 16, 2004 and, in revised form, June 9, 2005.
2000 *Mathematics Subject Classification.* Primary 11F67.
This research was partially supported by NSERC.

Let

$$L_{\infty,k}(s) = (2\pi)^{-s}\Gamma(\frac{k-1}{2}+s)$$

and

$$\Lambda_{f,\chi}(s) = (q\sqrt{M})^s L_{\infty,k}(s) L_{f,\chi}(s).$$

Then it is known that $\Lambda_{f,\chi}(s)$ is entire, and if f is a newform (in Atkin-Lehner sense), it satisfies the functional equation

$$\Lambda_{f,\chi}(s) = \epsilon_{f,\chi}\Lambda_{\bar{f},\bar{\chi}}(1-s), \tag{1}$$

where $\bar{f}$ is the conjugate newform in $S_k(\Gamma_0(M),\bar{\psi})$. Here

$$\epsilon_{f,\chi} = \epsilon_f \psi(q)\chi(M)\tau(\chi)^2 q^{-1},$$

where $|\epsilon_f| = 1$ and $\tau(\chi)$ is the Gauss sum. Note that $|\epsilon_{f,\chi}| = 1$.

We recall that for $f \in S_k(\Gamma_0(M),\psi)$ and $g \in S_l(\Gamma_0(N),\eta)$ the Rankin-Selberg convolution L-function is defined (for $Re(s) > 1$) by

$$L(f\otimes g, s) = \sum_{n=1}^{\infty} \frac{a_f(n)\bar{b}_g(n)}{n^s}.$$

The following can be considered as a modular analogue of the above theorem of Elliott.

Theorem 1.1. *Let $f \in S_k(\Gamma_0(M),\psi)$ and $g \in S_l(\Gamma_0(N),\eta)$ be newforms. Let $Q \geq 2$ and let $s_0 = \sigma_0 + it_0$ be a complex number with $\sigma_0 > \frac{1}{2}$. Then we have*

$$\sum_{\substack{q\leq Q,\, q \ prime\\ (q,MN)=1}} \sideset{}{^\star}\sum_{\chi (\mathrm{mod}\ q)} L_{f,\chi}(s_0)\overline{L_{g,\chi}(s_0)} = \frac{Q^2}{2\log Q}\frac{\phi(MN)}{MN} L(f\otimes g, 2\sigma_0) + O\left(\frac{Q^2}{(\log Q)^2}\right)$$

where the inner sum is taken over the primitive characters modulo prime q. The implied constant depends on f, g and s_0. Here, ϕ is the Euler function.

In proving Theorem 1.1, we first find an asymptotic formula for the values $L_{f,\chi}(s_0)\overline{L_{g,\chi}(s_0)}$ on average when χ varies on the set of primitive characters modulo a fixed positive integer q (see Proposition 2.5). This result generalizes a theorem of Stefanicki ([S], Theorem 2(a)).

Theorem 1.1 has an interesting application in the problem of non-vanishing of twisted L-functions inside the critical strip. In Proposition 3.1, by employing the large sieve inequality for characters, we establish an upper bound for the mean square of the values $|L_{f,\chi}(s_0)\overline{L_{g,\chi}(s_0)}|$. Together, Theorem 1.1 and Proposition 3.1 imply the following.

Theorem 1.2. *Let $f \in S_k(\Gamma_0(M),\psi)$ and $g \in S_l(\Gamma_0(N),\eta)$ be newforms. Let $s_0 = \sigma_0 + it_0$ be a fixed point in the strip $\frac{1}{2} < \sigma_0 \leq 1$. Then we have*

$$\#\{\chi |\ conductor(\chi)\ a\ prime \leq Q\ and\ L_{f,\chi}(s_0)L_{g,\chi}(s_0) \neq 0\} \gg \frac{Q^2}{(\log Q)^4}$$

as $Q \to \infty$. The implied constant depends on f, g and s_0.

This theorem should be compared to some non-vanishing results in the theory of automorphic forms. To explain the connection, let F be a number field, let S be a finite set of places of F, and let π be a unitary cuspidal automorphic representation of $\mathrm{GL}(n)$ over F. Let $s_0 = \sigma_0 + it_0$ be a fixed point in the complex plane. Then Rohrlich [R] proved that for $n = 1$ and 2 there are infinitely many primitive ray class characters χ of F such that χ is unramified at the places in S and $L(\pi \otimes \chi, s_0) \neq 0$. For $n \geq 3$, Barthel and Ramakrishnan [BR] proved that the same result remains true as long as π is tempered (i.e. satisfies the Ramanujan conjecture) and $\sigma_0 > 1 - \frac{2}{n+1}$ (see also [LRS] for a related result). For automorphic representations of GL(4) over $\mathbb{Q}$ (the case that is related to this paper) the result of Barthel and Ramakrishnan states that for $\sigma_0 > \frac{3}{5}$ there are infinitely many primitive Dirichlet characters such that $L(\pi \otimes \chi, s_0) \neq 0$. Note that our non-vanishing result (Theorem 1.2) surpasses the bound $\frac{3}{5}$. This is due to the fact that we are dealing with the product of two twisted GL(2) L-functions ($L_{f,\chi}(s)\overline{L_{g,\chi}(s)}$) and thus the Gauss sums associated to the functional equations of these two L-functions cancel each other (see Lemma 2.2). Therefore the contributions from the sums corresponding to $1 - s_0$ in Lemma 2.2 can be dealt with in ways similar to the sums corresponding to s_0. This enables us to prove a non-vanishing result in the half plane $\sigma_0 > \frac{1}{2}$. In fact a similar result should be true on the line $\sigma_0 = \frac{1}{2}$, however establishing such a result needs a more elaborate treatment of the error terms in Proposition 2.5.

In the next two sections we prove the above theorems.

2. Proof of Theorem 1.1

Let $k \geq l$ and $s_0 = \sigma_0 + it_0$. We set

$$P_\chi(s_0) = L_{f,\chi}(s_0)\overline{L_{g,\chi}(s_0)}.$$

We first derive an asymptotic formula for $\sum_\chi P_\chi(s_0)$ as χ varies over the primitive characters mod q. Here we do not assume that q is a prime. Let

$$Z_{s_0}(x) = \frac{1}{2\pi i}\int_{(1)} L_{\infty,k}(s+s_0)L_{\infty,l}(s+\bar{s}_0)x^{-s}\frac{ds}{s}.$$

Writing the integral representations of the Γ functions in the expression for $Z_{s_0}(x)$ and interchanging the order of integration, we arrive at

$$Z_{s_0}(x) = (2\pi)^{-2\sigma_0}\int_0^\infty t_1^{\frac{k-1}{2}+s_0-1}e^{-t_1}\left(\int_{\frac{4\pi^2 x}{t_1}}^\infty t_2^{\frac{l-1}{2}+\bar{s}_0-1}e^{-t_2}dt_2\right)dt_1.$$

Note that this representation for $Z_{s_0}(x)$ shows that $|Z_{s_0}(x)| \leq Z_{\sigma_0}(x)$. Moreover by integration by parts we can find an expression for $Z_{s_0}(x)$ in terms of K-Bessel functions. Applying the standard bounds for K-Bessel functions yields

$$|Z_{s_0}(x)| \ll \begin{cases} 1, & x \leq 1, \\ x^{\frac{k}{4}+\frac{l}{4}+\sigma_0-\frac{5}{4}}e^{-4\pi\sqrt{x}}, & x > 1 \end{cases} \tag{2}$$

(see [A], Lemmas 6.2 and 6.3, for details).

We next represent $P_\chi(s_0)$ as a sum of two rapidly convergent series.

Lemma 2.1. *Let $f \in S_k(\Gamma_0(M), \psi)$ and $g \in S_l(\Gamma_0(N), \eta)$ be newforms. Suppose that χ is a primitive Dirichlet character modulo q with $(q, MN) = 1$. Then*

$$L_{\infty,k}(s_0)L_{\infty,l}(\bar{s}_0)P_\chi(s_0) = S_{f,g}(s_0) + \epsilon_{f,\chi}\epsilon_{\bar{g},\bar{\chi}}(q^2\sqrt{MN})^{1-2\sigma_0}\left(\frac{N}{M}\right)^{it_0}\hat{S}_{f,g}(1-s_0),$$

where

$$S_{f,g}(s_0)=\sum_{m,n\geq 1}\frac{a_f(m)\bar{a}_g(n)}{(mn)^{\sigma_0}}\left(\frac{n}{m}\right)^{it_0}Z_{s_0}(\frac{mn}{q^2\sqrt{MN}})\chi(m)\bar{\chi}(n)$$

and

$$\hat{S}_{f,g}(1-s_0)=\sum_{m,n\geq 1}\frac{\bar{a}_f(m)a_g(n)}{(mn)^{1-\sigma_0}}\left(\frac{m}{n}\right)^{it_0}Z_{1-s_0}(\frac{mn}{q^2\sqrt{MN}})\bar{\chi}(m)\chi(n).$$

Proof. We have

$$S_{f,g}(s_0)=\frac{1}{2\pi i}\int_{(1)}L_{\infty,k}(s+s_0)L_{f,\chi}(s+s_0)L_{\infty,l}(s+\bar{s}_0)L_{\bar{g},\bar{\chi}}(s+\bar{s}_0)(q^2\sqrt{MN})^s\frac{ds}{s}.$$

Moving the line of integration to the left of zero and calculating the residue at $s=0$, along with application of (1) result in

$$\begin{aligned}S_{f,g}(s_0)=L_{\infty,k}(s_0)L_{\infty,l}(\bar{s}_0)P_\chi(s_0)+\epsilon_{f,\chi}\epsilon_{\bar{g},\bar{\chi}}(q^2\sqrt{MN})^{1-2\sigma_0}\left(\frac{N}{M}\right)^{it_0}\\ \times\frac{1}{2\pi i}\int_{(-1)}L_{\infty,k}(1-s-s_0)L_{\bar{f},\bar{\chi}}(1-s-s_0)\\ L_{\infty,l}(1-s-\bar{s}_0)L_{g,\chi}(1-s-\bar{s}_0)(q^2\sqrt{MN})^{-s}\frac{ds}{s}.\end{aligned}$$

Now changing s to $-s$ yields the result. □

From now on for simplicity we let $L_\infty(s_0)=L_{\infty,k}(s_0)L_{\infty,l}(\bar{s}_0)$. Next we average $P_\chi(s_0)$ over all primitive Dirichlet characters modulo q. We have

Lemma 2.2. *Let* $q\not\equiv 2 \pmod 4$ *and* $(q,MN)=1$. *Then*

$$\begin{aligned}&L_\infty(s_0)\sum_{\chi(\mathrm{mod}\ q)}^{\star}P_\chi(s_0)\\ &=\sum_{d|q}\mu(\frac{q}{d})\phi(d)\left(S^d_{f,g}(s_0)+\epsilon_f\epsilon_{\bar{g}}\psi\bar{\eta}(q)(q^2\sqrt{MN})^{1-2\sigma_0}\left(\frac{N}{M}\right)^{it_0}\hat{S}^d_{f,g}(1-s_0)\right),\end{aligned}$$

where μ *is the Möbius function,* ϕ *is the Euler function,*

$$S^d_{f,g}(s_0)=\sum_{\substack{m,n,\ (mn,q)=1\\ m\equiv n(\mathrm{mod}\ d)}}\frac{a_f(m)\bar{a}_g(n)}{(mn)^{\sigma_0}}\left(\frac{n}{m}\right)^{it_0}Z_{s_0}(\frac{mn}{q^2\sqrt{MN}}),$$

and

$$\hat{S}^d_{f,g}(1-s_0)=\sum_{\substack{m,n,\ (mn,q)=1\\ Nm\equiv Mn(\mathrm{mod}\ d)}}\frac{\bar{a}_f(m)a_g(n)}{(mn)^{1-\sigma_0}}\left(\frac{m}{n}\right)^{it_0}Z_{1-s_0}(\frac{mn}{q^2\sqrt{MN}}).$$

Proof. From Lemma 2.1 we have

$$\begin{aligned}&L_\infty(s_0)\sum_{\chi(\mathrm{mod}\ q)}^{\star}P_\chi(s_0)\\ &=\sum_{\chi(\mathrm{mod}\ q)}^{\star}\left(S_{f,g}(s_0)+\epsilon_f\epsilon_{\bar{g}}\psi\bar{\eta}(q)\chi(M)\bar{\chi}(N)(q^2\sqrt{MN})^{1-2\sigma_0}\left(\frac{N}{M}\right)^{it_0}\hat{S}_{f,g}(1-s_0)\right).\end{aligned}$$

Note that $\epsilon_{f,\chi}\epsilon_{\bar{g},\bar{\chi}} = \epsilon_f\epsilon_{\bar{g}}\psi\bar{\eta}(q)\chi(M)\bar{\chi}(N)$. To simplify the above expression, we need to evaluate $\sum^{\star}_{\chi (\mathrm{mod}\ q)}\chi(m)\bar{\chi}(n)$ and $\sum^{\star}_{\chi (\mathrm{mod}\ q)}\chi(Mn)\bar{\chi}(Nm)$. Let

$$h_{m,n}(q) = \sum^{\star}_{\chi (\mathrm{mod}\ q)} \chi(m)\bar{\chi}(n).$$

We have

$$\sum_{d|q} h_{m,n}(d) = \sum_{\chi (\mathrm{mod}\ q)} \chi(m)\bar{\chi}(n) = \begin{cases} \phi(q) & \text{if } m \equiv n \ (\mathrm{mod}\ q), \\ 0 & \text{otherwise.} \end{cases}$$

Now applying the Möbius inversion formula ([M], Section 1.1) on the above identity yields

$$\sum^{\star}_{\chi (\mathrm{mod}\ q)} \chi(m)\bar{\chi}(n) = h_{m,n}(q) = \sum_{d|(q,m-n)} \mu(\frac{q}{d})\phi(d).$$

Applying this and a similar identity for $\sum^{\star}_{\chi (\mathrm{mod}\ q)}\chi(Mn)\bar{\chi}(Nm)$ in the expression for $L_\infty(s_0)\sum^{\star}_{\chi (\mathrm{mod}\ q)}P_\chi(s_0)$ at the beginning of the proof imply the result. □

Next we find an asymptotic for the terms in $S^d_{f,g}(s_0)$ corresponding to $m = n$. To explain our result we need to introduce a notation. We know that for any prime p, $a_f(p) = \alpha_{f,1}(p)+\alpha_{f,2}(p)$ and $a_g(p) = \alpha_{g,1}(p)+\alpha_{g,2}(p)$, where $\alpha_{f,1}(p)\alpha_{f,2}(p) = \psi(p)$ and $\alpha_{g,1}(p)\alpha_{g,2}(p) = \eta(p)$. Let

$$R_q(s) = \prod_{p|q}\left(1 - \frac{\psi\bar{\eta}(p)}{p^{2s}}\right)^{-1}\prod_{i=1}^{2}\prod_{j=1}^{2}\left(1 - \frac{\alpha_{f,i}(p)\bar{\alpha}_{g,j}(p)}{p^s}\right).$$

Lemma 2.3. *Let f, g and s_0 be as Theorem 1.1. Then*

$$\sum_{n,(n,q)=1} \frac{a_f(n)\bar{a}_g(n)}{n^{2\sigma_0}} Z_{s_0}(\frac{n^2}{q^2\sqrt{MN}}) \sim L_\infty(s_0)L(f\otimes g, 2\sigma_0)R_q(2\sigma_0)$$

as $q \to \infty$.

Proof. From the definition of $Z_{s_0}(x)$, we have

$$\sum_{n,(n,q)=1} \frac{a_f(n)\bar{a}_g(n)}{n^{2\sigma_0}} Z_{s_0}(\frac{n^2}{q^2\sqrt{MN}})$$

$$= \frac{1}{2\pi i}\int_{(1)} L_{\infty,k}(s+s_0)L_{\infty,l}(s+\bar{s}_0)L(f\otimes g, 2s+2\sigma_0)R_q(2s+2\sigma_0)(q^2\sqrt{MN})^s\frac{ds}{s}.$$

Moving the line of integration to the left of zero implies the result. □

The next lemma gives an estimation for the off-diagonal terms in $S^d_{f,g}(s_0)$.

Lemma 2.4. *Let $\epsilon > 0$ be arbitrary. Then*

$$\sum_{d|q}\mu(\frac{q}{d})\phi(d) \sum_{\substack{m,n,\ (mn,q)=1 \\ m\equiv n (\mathrm{mod}\ d), m\neq n}} \frac{a_f(m)\bar{a}_g(n)}{(mn)^{\sigma_0}}\Big(\frac{n}{m}\Big)^{it_0} Z_{s_0}(\frac{mn}{q^2\sqrt{MN}}) = O(q^{2-2\sigma_0+\epsilon}).$$

The implied constant depends on f, g, s_0 and ϵ.

Proof. We closely follow Section 3.2 of [S]. First of all we recall Rankin-Shiu's estimate for the sum of Fourier coefficients of modular forms. Let $d \neq 1$ and $(n, d) = 1$. Then for a newform g we have

$$\sum_{\substack{n \leq x \\ n \equiv m (\bmod\ d)}} |a_g(n)| \ll \frac{x}{\phi(d)} (\log x)^{-\epsilon_1}$$

as $x \to \infty$, where $x > d^\alpha$ for $1 < \alpha < 2$. Here, $0 < \epsilon_1 \leq \delta \simeq 0.06...$ is arbitrary, and the bound is uniform in m, d and α (see [S], page 5 for details). We use this together with Rankin's estimate [RA1]

$$\sum_{m \leq x} |a_f(m)| \ll x (\log x)^{-\delta}$$

to bound the inner sum in the statement of the lemma. Let $1 < \alpha < \frac{10}{9}$ be a fixed number. We only need to find estimates for the following ranges of m and n:

(i) $n > d^\alpha$.

(ii) $d^{\frac{4}{5}\alpha} \leq n < m \leq d^\alpha$.

(iii) $n < d^{\frac{4}{5}\alpha}$ and $d \leq m \leq d^\alpha$.

Now we estimate the inner sum in the statement of the lemma in each case.

(i) We assume $n > d^\alpha$. By employing Rankin-Shiu's and Rankin's estimates, bounds for $Z_{\sigma_0}(x)$ and partial summation we have

$$\sum_{\substack{m \geq \frac{q^2\sqrt{MN}}{d^\alpha} \\ (m,q)=1}} \frac{|a_f(m)|}{m^{\sigma_0}} \sum_{\substack{n > d^\alpha, (n,q)=1 \\ n \equiv m (\bmod\ d)}} \frac{|a_g(n)|}{n^{\sigma_0}} Z_{\sigma_0}\left(\frac{mn}{q^2\sqrt{MN}}\right) \ll \frac{1}{\phi(d)} (q^2\sqrt{MN})^{1-\sigma_0}$$

and

$$\sum_{\substack{m < \frac{q^2\sqrt{MN}}{d^\alpha} \\ (m,q)=1}} \frac{|a_f(m)|}{m^{\sigma_0}} \sum_{\substack{n > d^\alpha, (n,q)=1 \\ n \equiv m (\bmod\ d)}} \frac{|a_g(n)|}{n^{\sigma_0}} Z_{\sigma_0}\left(\frac{mn}{q^2\sqrt{MN}}\right)$$
$$\ll \frac{1}{\phi(d)} (q^2\sqrt{MN})^{1-\sigma_0} (\log(d^{-\alpha} q^2 \sqrt{MN}))^{1-\delta}.$$

(ii) Next we consider the range $d^{\frac{4}{5}\alpha} \leq n < m \leq d^\alpha$. We recall from the Rankin-Selberg theory [RA2] the asymptotic formula

$$\sum_{n \leq x} |a_g(n)|^2 = c_g x + O(x^{\frac{3}{5}}),$$

where c_g is a constant depending only on g. By employing the Cauchy-Schwarz inequality and the above asymptotic we have

$$\sum_{n \leq x} |a_f(n + dt) a_g(n)| \ll x$$

uniformly for $x \gg (dt)^{\frac{3}{5}}$. Now by writing $m = n + dt$ for $1 \leq t \leq d^{\alpha-1}$, and applying partial summation, we get the following estimation of the inner sum:

$$\sum_{1\leq t\leq d^{\alpha-1}} \sum_{d^{\frac{4}{5}\alpha}<n\leq \frac{q^2\sqrt{MN}}{dt}} \frac{|a_f(n+dt)a_g(n)|}{(n+dt)^{\sigma_0}n^{\sigma_0}} Z_{\sigma_0}(\frac{(n+dt)n}{q^2\sqrt{MN}})$$

$$\ll \sum_{1\leq t\leq d^{\alpha-1}} \frac{(q^2\sqrt{MN})^{1-\sigma_0}}{dt} \ll \frac{1}{d}(q^2\sqrt{MN})^{1-\sigma_0}\log d.$$

A similar estimation is true for the range $\frac{q^2\sqrt{MN}}{dt} < n \leq d^{\alpha}$.

(iii) Finally we consider m and n's in the range $n < d^{\frac{4}{5}\alpha}$ and $d \leq m \leq d^{\alpha}$. Note that since $1 < \alpha < \frac{10}{9}$ and $\sigma_0 > \frac{1}{2}$, we have $\frac{1}{2} - \frac{9}{10}\alpha(1-\sigma_0) > 0$. We choose an ϵ such that $0 < \epsilon < \frac{1}{2} - \frac{9}{10}\alpha(1-\sigma_0)$. By Deligne's bound for the Fourier coefficients of newforms we have

$$|a_f(m)a_g(n)| \ll_{\epsilon} d^{\epsilon}.$$

By applying this bound we have

$$\sum_{\substack{n<d^{\frac{4}{5}\alpha},d\leq m\leq d^{\alpha}\\ m\equiv n \pmod d}} \frac{|a_f(m)a_g(n)|}{(mn)^{\sigma_0}} Z_{\sigma_0}(\frac{mn}{q^2\sqrt{MN}}) \ll d^{\frac{9}{5}\alpha(1-\sigma_0)-1+\epsilon} \ll d^{-\epsilon}.$$

Applying estimates of (i), (ii) and (iii) to the inner sum in the statement of the lemma implies the result. □

Combining the results of Lemmas 2.2, 2.3 and 2.4 we have the following.

Proposition 2.5. *Let f, g and s_0 be as Theorem* 1.1 *and let $\epsilon > 0$ be arbitrary. Let $q \not\equiv 2 \pmod 4$ and $(q, MN) = 1$. We have*

$$\sum_{\chi \pmod q}^{\star} P_{\chi}(s_0) = L(f\otimes g, 2\sigma_0)R_q(2\sigma_0)\sum_{d|q}\mu(\frac{q}{d})\phi(d) + O(q^{2-2\sigma_0+\epsilon}).$$

The implied constant depends on f, g, s_0 and ϵ.

Proof. First of all note that a result similar to Lemma 2.4 is also true for the off-diagonal terms in $\hat{S}^d_{f,g}(1-s_0)$, and in this case the corresponding sum is bounded by $q^{2\sigma_0+\epsilon}$. For the diagonal terms in $\hat{S}^d_{f,g}(1-s_0)$ (ones corresponding to $Nm = Mn$), by applying Deligne's bound for Fourier coefficients of new forms we have

$$\sum_{\substack{m,n,(mn,q)=1\\ Nm=Mn}} \frac{|a_f(m)a_g(n)|}{(mn)^{1-\sigma_0}} Z_{1-\sigma_0}(\frac{mn}{q^2\sqrt{MN}})$$

$$\ll_{M,N} \sum_{n,(n,q)=1} \frac{n^{\epsilon}}{n^{2(1-\sigma_0)}} Z_{1-\sigma_0}(\frac{Mn^2}{q^2N\sqrt{MN}}). \ll q^{2\sigma_0-1+\epsilon}.$$

Applying these estimates together with Lemmas 2.3 and 2.4 in Lemma 2.2 will imply the desired result. □

Proof of Theorem 1.1. We sum the asymptotic formula given in Lemma 2.5 over primes $q \leq Q$ where $(q, MN) = 1$. Note that for q prime, $R_q(2\sigma_0) = 1 + O(q^{-2\sigma_0})$ and $\sum_{d|q}\mu(\frac{q}{d})\phi(d) = q-2$. Now the result follows from the prime number theorem. □

3. Proof of Theorem 1.2

First of all in Lemma 2.1 let

$$b_u = \sum_{u=mn} a_f(m)\bar{a}_g(n)\chi(m)\bar{\chi}(n)\left(\frac{n}{m}\right)^{it_0}.$$

Note that by Deligne's bound for Fourier coefficients of newforms we have

$$|b_u| \ll_\epsilon u^\epsilon.$$

So by Lemma 2.1 for given X we have

$$\begin{aligned} L_\infty(s_0)P_\chi(s_0) &= \sum_{u\le X}\frac{b_u}{u^{\sigma_0}}Z_{s_0}(\frac{u}{q^2\sqrt{MN}}) + \sum_{u\ge X}\frac{b_u}{u^{\sigma_0}}Z_{s_0}(\frac{u}{q^2\sqrt{MN}}) \\ &+ \epsilon_{f,\chi}\epsilon_{\bar{g},\bar{\chi}}(q^2\sqrt{MN})^{1-2\sigma_0}\left(\frac{N}{M}\right)^{it_0}\sum_{u\le X}\frac{\bar{b}_u}{u^{1-\sigma_0}}Z_{1-s_0}(\frac{u}{q^2\sqrt{MN}}) \\ &+ \epsilon_{f,\chi}\epsilon_{\bar{g},\bar{\chi}}(q^2\sqrt{MN})^{1-2\sigma_0}\left(\frac{N}{M}\right)^{it_0}\sum_{u\ge X}\frac{\bar{b}_u}{u^{1-\sigma_0}}Z_{1-s_0}(\frac{u}{q^2\sqrt{MN}}) \\ &= L_1(s_0)+L_2(s_0)+L_3(s_0)+L_4(s_0). \end{aligned}$$

From here we have

(3)

$$|L_\infty(s_0)|^2\sum_{\substack{q\le Q\\(q,MN)=1}}\frac{q}{\phi(q)}\sideset{}{^\star}\sum_{\chi(\mathrm{mod}\ q)}|P_\chi(s_0)|^2 \ll |L_\infty(s_0)|^2\sum_{i=1}^{4}\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^\star}\sum_{\chi(\mathrm{mod}\ q)}|L_i(s_0)|^2.$$

Now let $X=\sqrt{MN}Q^2(\log Q)^2$. Then by employing bound (2) for $Z_{s_0}(x)$, we have

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^\star}\sum_{\chi(\mathrm{mod}\ q)}|L_i(s_0)|^2 \ll Q^{-19} \tag{4}$$

for $i=2$, 4. We know that by the large sieve inequality for characters we have

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^\star}\sum_{\chi(\mathrm{mod}\ q)}\left|\sum_{u=1}^{X}a_u\chi(u)\right|^2 \le (X+3Q^2)\sum_{u=1}^{X}|a_u|^2$$

(see [D], page 160, Theorem 4). Let

$$a_u = u^{-\sigma_0}Z_{s_0}(\frac{u}{q^2\sqrt{MN}})\sum_{\substack{u=mn\\(n,q)=1}}a_f(m)\bar{a}_g(n)\bar{\chi}(n^2)\Big(\frac{n}{m}\Big)^{it_0}.$$

By Deligne's bound and (2) we have

$$|a_u| \ll_\epsilon u^{-\sigma_0+\epsilon}.$$

This bound together with the large sieve inequality imply that for $i=1$

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^\star}\sum_{\chi(\mathrm{mod}\ q)}|L_i(s_0)|^2 = \sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^\star}\sum_{\chi(\mathrm{mod}\ q)}\left|\sum_{u=1}^{X}a_u\chi(u)\right|^2 \ll Q^2(\log Q)^2. \tag{5}$$

The same bound is valid for $i=3$. Now applying (4) and (5) in (3) implies the following.

Proposition 3.1. *Let f, g and s_0 be as Theorem* 1.1. *We have*

$$\sum_{\substack{q\le Q\\(q,MN)=1}} \frac{q}{\phi(q)} \sideset{}{^\star}\sum_{\chi \,(\mathrm{mod}\ q)} |P_\chi(s_0)|^2 \ll Q^2(\log Q)^2.$$

The implied constant depends on f, g and s_0.

Proof of Theorem 1.2. By Cauchy-Schwarz inequality we have

$$\#\{\chi|\ \mathrm{conductor}(\chi) \text{ a prime} \le Q \text{ and } P_\chi(s_0)\neq 0\} \ge \frac{\left|\displaystyle\sum_{\substack{q\le Q,\, q \text{ prime}\\(q,MN)=1}} \sideset{}{^\star}\sum_{\chi \,(\mathrm{mod}\ q)} P_\chi(s_0)\right|^2}{\displaystyle\sum_{\substack{q\le Q\\(q,MN)=1}} \frac{q}{\phi(q)} \sideset{}{^\star}\sum_{\chi \,(\mathrm{mod}\ q)} |P_\chi(s_0)|^2}.$$

The result follows from this inequality, Theorem 1.1 and Proposition 3.1. □

Acknowledgment

The author would like to thank the referee for helpful comments and suggestions. I would also like to thank Hershy Kisilevsky for several helpful discussions related to this work.

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