AUTOMATIC CONTINUITY OF $\sigma$-DERIVATIONS ON $C^*$-ALGEBRAS

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(Communicated by Joseph A. Ball)

Abstract. Let $A$ be a $C^*$-algebra acting on a Hilbert space $H$, let $\sigma : A \to B(H)$ be a linear mapping and let $d : A \to B(H)$ be a $\sigma$-derivation. Generalizing the celebrated theorem of Sakai, we prove that if $\sigma$ is a continuous $*$-mapping, then $d$ is automatically continuous. In addition, we show the converse is true in the sense that if $d$ is a continuous $*$-$\sigma$-derivation, then there exists a continuous linear mapping $\Sigma : A \to B(H)$ such that $d$ is a $*$-$\Sigma$-derivation. The continuity of the so-called $*$-$(\sigma, \tau)$-derivations is also discussed.

1. Introduction

Let $A$ be a subalgebra of an algebra $B$, let $\mathcal{X}$ be a $B$-bimodule and let $\sigma : A \to B$ be a linear mapping. A linear mapping $d : A \to \mathcal{X}$ is called a $\sigma$-derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in A$ (see [3] and [4] and the references therein). If $B = \mathcal{X} = A$ and $\sigma$ is the identity map on $A$, then we reach to the usual notion of a derivation on the algebra $A$.


In this paper, we investigate the continuity of $\sigma$-derivations on $C^*$-algebras. Let $A$ be a $C^*$-algebra acting on a Hilbert space $H$. We prove that if $\sigma : A \to B(H)$ is a continuous $*$-linear mapping, then every $\sigma$-derivation from $A$ into $B(H)$ is automatically continuous, and so Sakai’s theorem [8] is generalized. In addition, we establish the converse in the sense that if $d : A \to B(H)$ is a continuous $*$-$\sigma$-derivation, then there exists a continuous mapping $\Sigma : A \to B(H)$ such that $d$ is a $*$-$\Sigma$-derivation. In the last section we discuss the continuity of the so-called $*$-$(\sigma, \tau)$-derivations.

The importance of our approach is that $\sigma$ is a linear mapping in general, not necessarily an algebra homomorphism. There are some applications of $\sigma$-derivations used to develop an approach to deformations of Lie algebras which have many applications in models of quantum phenomena and in analysis of complex systems; cf. [1].
For the definition and elementary properties of $C^*$-algebras we refer the reader to [5] and [6].

2. Elementary properties of $\sigma$-derivations

Throughout this section, $A$ is a subalgebra of an algebra $B$, $\mathcal{X}$ is a $B$-bimodule and $\sigma : A \to B$ is a linear mapping.

A linear mapping $d : A \to \mathcal{X}$ is called a $\sigma$-derivation if

$$d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$$

for all $a, b \in A$. Familiar examples are:

(i) every ordinary derivation $\delta$ of an algebra $A$ into an $A$-bimodule $\mathcal{X}$ is an $\iota$-derivation (where $\iota$ denotes the identity map on $A$);

(ii) every endomorphism $\phi$ on $A$ is a $\frac{1}{2}$-derivation;

(iii) for a given homomorphism $\sigma$ on $A$ and a fixed arbitrary element $x$ in an $A$-bimodule $\mathcal{X}$, the so-called $\sigma$-inner derivation is defined to be $d_{\sigma}(a) = x\sigma(a) - \sigma(a)x$.

An interesting link between $\sigma$-derivations and algebra homomorphisms follows.

Theorem 2.1. Let $\sigma : A \to B$ be a homomorphism and let $d : A \to \mathcal{X}$ be a $\sigma$-derivation. Then the following hold:

(i) $\mathcal{X}$ equipped with the module multiplications $a \cdot x = \sigma(a)x$ and $x \cdot a = x\sigma(a)$ is an $A$-bimodule denoted by $\overline{\mathcal{X}}$.

(ii) $d : A \to \overline{\mathcal{X}}$ is an ordinary derivation.

(iii) $C = A \oplus \overline{\mathcal{X}}$ equipped with the multiplication $(a, x)(b, y) = (ab, x \cdot b + a \cdot y)$ is an algebra, and $\varphi_d : A \to C$ defined by $\varphi_d(a) = (a, d(a))$ is an injective homomorphism.

(iv) If $A, B$ and $\mathcal{X}$ are normed, $\sigma$ is continuous, and $C$ is equipped with the norm $\|(a, x)\| = \|a\| + \sup\{|x|, \|a_1 \cdot x\|, \|a_2 \cdot x\|, \|\sigma(a_1 \cdot x)\|, \|\sigma(a_2 \cdot x)\| : a_1, a_2 \in A, \|a_1\| \leq 1, \|a_2\| \leq 1\}$, then $\varphi_d$ is continuous if and only if $d$ is continuous. Thus if every injective homomorphism of $A$ into a Banach algebra is continuous, then every $\sigma$-derivation of $A$ into a Banach $B$-bimodule is continuous.

Proof. Straightforward (see [6]). $\square$

Recall that if $Y$ and $Z$ are normed spaces and $T : Y \to Z$ is a linear mapping, then the set of all $z$ such that there is a sequence $\{y_n\}$ in $Y$ with $y_n \to 0$ and $Ty_n \to z$ is called the separating space $S(T)$ of $T$. Clearly,

$$S(T) = \bigcap_{n=1}^{\infty} \{T(y) : \|y\| < 1/n\}$$

is a closed linear space. If $Y$ and $Z$ are Banach spaces, by the closed graph theorem, $T$ is continuous if and only if $S(T) = \{0\}$.

Lemma 2.2. Let $d : A \to \mathcal{X}$ be a $\sigma$-derivation. Then

$$d(c)(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ca) - \sigma(c)\sigma(a))d(b)$$

for all $a, b, c \in A$. 

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Proof.
\[
d(c(ab)) = d(c)\sigma(ab) + \sigma(c)d(ab)
\]
\[
d(c)\sigma(ab) = d(cab) - \sigma(c)d(ab)
\]
\[
= (d(ca)\sigma(b) + \sigma(ca)d(b)) - \sigma(c)d(ab)
\]
\[
= (d(c)\sigma(a) + \sigma(c)d(a))\sigma(b) + \sigma(ca)d(b) - \sigma(c)d(ab)
\]
\[
= d(c)\sigma(a)\sigma(b) + \sigma(c)d(a)\sigma(b) + \sigma(ca)d(b) - \sigma(c)d(ab)
\]
\[
= d(c)\sigma(a)\sigma(b) + \sigma(c)d(a)\sigma(b) + \sigma(ca)d(b)
\]
\[
= d(c)\sigma(a)\sigma(b) + \sigma(c)d(a)\sigma(b) + \sigma(ca)d(b)
\]
\[
-\sigma(c)(d(a)\sigma(b) + \sigma(a)d(b)),
\]
whence
\[
d(c)((\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ca) - \sigma(c)\sigma(a))d(b).
\]

Lemma 2.3. Let \( \mathcal{A} \) and \( \mathcal{B} \) be normed algebras, let \( \sigma : \mathcal{A} \to \mathcal{B} \) be a continuous mapping and let \( d : \mathcal{A} \to \mathcal{B} \) be a \( \sigma \)-derivation. Then for each \( a \in \mathcal{S}(d) \) and \( b, c \in \mathcal{A} \) we have \( a(\sigma(bc) - \sigma(b)\sigma(c)) = 0 \).

Proof. For each \( a \in \mathcal{S}(d) \) there exists a sequence \( \{a_n\} \) such that \( a_n \to 0 \) and \( d(a_n) \to a \). By Lemma 2.2 we have
\[
d(a_n)(\sigma(bc) - \sigma(b)\sigma(c)) = (\sigma(a_nb) - \sigma(a_n)\sigma(b))d(c) \to 0.
\]
Thus \( a(\sigma(bc) - \sigma(b)\sigma(c)) = 0 \).

Remark 2.4. Recall that if \( E \) is a subset of an algebra \( \mathcal{B} \), the right annihilator \( ran(E) \) of \( E \) (resp. the left annihilator \( lan(E) \) of \( E \)) is defined to be \( \{ b \in \mathcal{B} : Eb = \{ 0 \} \} \) (resp. \( \{ b \in \mathcal{B} : bE = \{ 0 \} \} \)). The set \( ann(E) := ran(E) \cap lan(E) \) is called the annihilator of \( E \). The previous lemma shows that if \( \mathcal{A} \) and \( \mathcal{B} \) are Banach algebras, \( \sigma : \mathcal{A} \to \mathcal{B} \) is a continuous linear mapping, \( d : \mathcal{A} \to \mathcal{B} \) is a \( \sigma \)-derivation and \( ran(\mathcal{S}(d)) = \{ 0 \} \), then \( \sigma \) is an endomorphism, and if
\[
lan(\{ \sigma(bc) - \sigma(b)\sigma(c) : b, c \in \mathcal{A} \}) = \{ 0 \},
\]
then \( d \) is continuous.

Proposition 2.5. Suppose that \( \mathcal{A} \) is a Banach algebra, \( \mathcal{B} \) is a simple Banach algebra, \( \sigma : \mathcal{A} \to \mathcal{B} \) is a surjective continuous linear mapping and \( d : \mathcal{A} \to \mathcal{B} \) is a \( \sigma \)-derivation. Then \( d \) is continuous or \( \sigma \) is an endomorphism.

Proof. At first we show that \( \mathcal{S}(d) \) is a closed bi-ideal of \( \mathcal{B} \). To see this let \( b \in \mathcal{B} \) and \( a \in \mathcal{S}(d) \). Then there is a sequence \( \{a_n\} \) such that \( a_n \to 0 \) and \( d(a_n) \to a \). Since \( \sigma \) is surjective, there is an element \( c \in \mathcal{A} \) such that \( b = \sigma(c) \). Now we have \( ca_n \to 0 \) and \( d(ca_n) = d(c)\sigma(a_n) + \sigma(c)d(a_n) \to 0 \). This shows that \( ba \in \mathcal{S}(d) \). By the same way \( ab \in \mathcal{S}(d) \). Thus \( \mathcal{S}(d) \) is a bi-ideal.

\( \mathcal{S}(d) \) is \( \{ 0 \} \) or \( \mathcal{B} \). If \( \mathcal{S}(d) = \{ 0 \} \), then \( d \) is continuous, and if \( \mathcal{S}(d) = \mathcal{B} \), then \( \mathcal{S}(d) \) has zero right annihilator and so, by Remark 2.4, \( \sigma \) is an endomorphism.

The following example yields a continuous \( \sigma \)-derivation with a non-continuous linear mapping \( \sigma \).
Example 2.6. Let $A = C[0, 2]$, the $C^*$-algebra of all complex-valued continuous functions defined on the interval $[0, 2]$. Define the continuous function $h : [0, 2] \to \mathbb{C}$ by $h(t) = 0$ on $[0, 1]$ and $h(t) = t - 1$ on $[1, 2]$, define the linear mapping $\sigma : A \to A$ by

$$\sigma(f)(t) = \begin{cases} 
\alpha(f|_{[0,1/2]})(t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\
2(1-t)\alpha(f|_{[0,1/2]})(\frac{1}{2}) + (t - \frac{1}{2})f(1) & \text{if } \frac{1}{2} \leq t \leq 1, \\
\frac{1}{2}f(t) & \text{if } 1 \leq t \leq 2,
\end{cases}$$

where $\alpha$ is a discontinuous linear mapping on $C[0, 1/2]$, and define the linear mapping $d : A \to A$ by $d(f) = fh$. For all $t \in [0, 1]$, we have

$$d(fg)(t) = f(t)g(t)h(t) = 0$$

and for all $t \in [1, 2]$,

$$d(fg)(t) = f(t)g(t)h(t) = f(t)h(t)\frac{g(t)}{2}(t) + (\frac{t}{2})g(t)h(t) = d(f)(t)\sigma(g)(t) + \sigma(f)(t)d(g)(t) = (d(f)\sigma(g) + \sigma(f)d(g))(t).$$

In this example if we define $\Sigma$ on $A$ by $\Sigma(f) = \frac{f}{2}$, then $d$ is a $\Sigma$-derivation and $\Sigma$ is continuous. In the next section we will show that this is true in general, i.e., if $d$ is a continuous $\ast$-$\sigma$-derivation, then we can find a continuous linear mapping $\Sigma$ such that $d$ is a $\ast$-$\Sigma$-derivation.

3. $\sigma$-Derivations on $C^*$-Algebras

In this section we establish several significant theorems concerning the continuity of $\sigma$-derivations on $C^*$-algebras. Throughout this section, $A$ denotes a $C^*$-algebra acting on a Hilbert space $H$, i.e., a closed $\ast$-subalgebra of the algebra $B(H)$ consisting of all bounded linear mappings on $H$. In addition, we assume that $\sigma$ and $d$ are linear mappings of $A$ into $B(H)$.

Our first result states that when we deal with a continuous $\sigma$-derivation we may assume that $\sigma$ is continuous.

Lemma 3.1. Let $d : A \to B$ be a continuous $\sigma$-derivation. Then $S(\sigma) \subseteq \text{ann}(d(A))$.

Proof. Assume $A \in S(\sigma)$. Then there is a sequence $\{A_n\}$ in $A$ such that $A_n \to 0$ and $\sigma(A_n) \to A$. For each $B \in A$ we have

$$d(A_nB) = d(A_n)\sigma(B) + \sigma(A_n)d(B) \to d(0)\sigma(B) + Ad(B).$$

Since $d(A_nB) \to d(0)$ and $d(0) = 0$, we obtain $Ad(B) = 0$. Similarly, we can show that $d(B)A = 0$. \qed

Theorem 3.2. Let $\sigma$ be a linear mapping and let $d$ be a continuous $\ast$-$\sigma$-derivation. Then there is a continuous linear mapping $\Sigma : A \to B(H)$ such that $d$ is a $\ast$-$\Sigma$-derivation.
Thus we can write

\[ \text{Proof.} \]

Thus we have proved that

\[ A \]

other hand, for each

\[ A \]

we can write

\[ \text{d} \]

and

\[ P \]

\[ K \]

\[ = ( ) \]

Moreover, \( \Sigma \) is continuous on \( A \). Let \( A_n \in A \) with \( A_n \to 0 \) and \( \Sigma(A_n) \to A \). Then for each \( \ell \in L_0 \) there is a \( B \in A \) and there is an \( h \in H \) such that \( \ell = d(B)(h) \). Thus we can write \( A(\ell) = A(d(B)(h)) = (Ad(B))(h) = 0 \), since \( A \in \text{ann}(d(A)) \). We therefore have \( A = 0 \) on \( L_0 \) and so \( A = 0 \) on \( L \), since \( A \) is continuous. On the other hand, for each \( k \in K \) we have \( 0 = \Sigma(A_n)(k) \to A(k) \) and so \( A = 0 \) on \( K \). Thus we have proved that \( A = 0 \) on \( H \). This shows that \( S(\Sigma) = \{0\} \) and so \( \Sigma \) is continuous on \( A \).

\[ \text{Theorem 3.3.} \]

Let \( \sigma \) be a \( * \)-linear mapping and let \( d \) be a continuous \( \sigma \)-derivation. Then there is a continuous linear mapping \( \Sigma : A \to B(H) \) such that \( d \) is a \( \Sigma \)-derivation.

\[ \text{Proof.} \]

Define the \( \sigma \)-derivation \( d^* : A \to B(H) \) by \( d^*(A) = d(A^*)^\ast \). Let \( L_0 = (\bigcup_{A \in A} d(A)(H)) \cup (\bigcup_{A \in A} d^*(A)(H)) \) and let \( L \) be the closed linear span of \( L_0 \). Then we can write \( H = L \oplus K \), where \( K = L^\perp \). Thus \( 0 = \langle d(A^*)(h), k \rangle = \langle h, d^*(A)k \rangle \) and \( 0 = \langle d^*(A^*)(h), k \rangle = \langle h, d(A)k \rangle \) for all \( h \in H, k \in K, A \in A \) such that \( K = (\bigcap_{A \in A} \text{ker} d(A)) \cap (\bigcap_{A \in A} \text{ker} d^*(A)) \). Now define \( \Sigma \) on \( A \) by \( \Sigma(A) = \sigma(A)P \) where \( P \) denotes the corresponding projection to \( L \).

Using the same argument in the proof of Theorem 3.2 one can show that both \( d \) and \( d^* \) are \( \Sigma \)-derivations (note that \( d^* \) is a \( \sigma \)-derivation).

\( \Sigma \) is continuous on \( A \). To see this, assume that \( A_n \in A \), \( A_n \to 0 \) and \( \Sigma(A_n) \to A \). Then for each \( \ell \in L_0 \) there is a \( B \in A \) and there is an \( h \in H \) such that \( \ell = d(B)(h) \) or \( \ell = d^*(B)(h) \). Thus we can write \( A(\ell) = A(d(B)(h)) = (Ad(B))(h) = 0 \) or \( A(\ell) = A(d^*(B))(h) = (Ad^*(B))(h) = 0 \), since \( A \in \text{ann}(d(A)) \cap \text{ann}(d^*(A)) \). We therefore have \( A = 0 \) on \( L_0 \) and so \( A = 0 \) on \( L \), since \( A \) is continuous. On the other hand, for each \( k \in K \) we have \( 0 = \Sigma(A_n)(k) \to A(k) \) and so \( A = 0 \) on \( K \). Thus we have proved that \( A = 0 \) on \( H \). This shows that \( S(\Sigma) = \{0\} \) and so \( \Sigma \) is continuous on \( A \).

The next two propositions allow us to assume that \( \sigma \) is a homomorphism when we discuss the continuity of \( \sigma \)-derivations.

\[ \text{Proposition 3.4.} \]

Let \( \sigma \) be a continuous \( * \)-linear mapping and let \( d \) be a \( \sigma \)-derivation. Then there is a continuous \( * \)-homomorphism \( \Sigma : A \to B(H) \) and a
\[\Sigma\text{-derivation } D : A \to B(H) \text{ such that } D \text{ is continuous if and only if so is } d.\]

Moreover, if \(d\) preserves \(*\), then so does \(D\).

**Proof.** By Lemma 2.3 for each \(A \in S(d)\) and \(B, C \in A\) we have

\[
(*) \quad A(\sigma(BC) - \sigma(B)\sigma(C)) = 0
\]

Now let \(L_0 = \bigcup_{B,C \in A}(\sigma(BC) - \sigma(B)\sigma(C))(H)\) and let \(L\) be the closed linear span of \(L_0\). Then (*) implies that \(A(L) = 0\) for each \(A \in S(d)\).

We can write \(H = L \oplus K\), where \(K = L^\perp\). For each \(B, C \in A, h \in H\) and \(k \in K\) we have

\[
0 = \langle (\sigma(BC) - \sigma(B)\sigma(C))(h), k \rangle \\
= \langle h, (\sigma(BC) - \sigma(B)\sigma(C))^*(k) \rangle \\
= \langle h, (\sigma(C^*B^*) - \sigma(C)\sigma(B^*))(k) \rangle.
\]

Since \(A\) is a \(*\)-subalgebra of \(B(H)\), we infer that \((\sigma(BC) - \sigma(B)\sigma(C))(k) = 0\) for each \(B, C \in A\) and \(k \in K\). This shows that \(K = \bigcap_{B,C \in A}\ker(\sigma(BC) - \sigma(B)\sigma(C))\).

Now let \(P = P_K\) be the projection corresponding to \(K\). At first we show that \(\sigma(A)P = P\sigma(A)\) for all \(A \in A\). For each \(A, B, C \in A\) and \(k \in K\) we have

\[
(\sigma(BC) - \sigma(B)\sigma(C))\sigma(A)(k) = (\sigma(BC)\sigma(A) - \sigma(B)\sigma(C)\sigma(A))(k) \\
= (\sigma(BCA) - \sigma(BCA))(k) \\
= 0.
\]

This shows that \(\sigma(A)(K) \subseteq K\) and so \(\sigma(A)P = P\sigma(A)\).

By using Lemma 2.2 we get

\[
0 = d(B)(\sigma(CA) - \sigma(C)\sigma(A))(k) = (\sigma(BC) - \sigma(B)\sigma(C))d(A)(k)
\]

for all \(k \in K\). This implies that \(d(A)(k) \in K\) for all \(k \in K\). Hence \(d(A)(K) \subseteq K\) and so \(d(A)P = Pd(A)\).

Now put \(\Sigma(A) = \sigma(A)P\) and \(D(A) = d(A)P\) for all \(A \in A\). First, \(\Sigma\) is a \(*\)-homomorphism. For \(k \in K\) we have

\[
\Sigma(AB)(k) = \sigma(AB)P(k) \\
= \sigma(AB)(k) \\
= \sigma(A)\sigma(B)(k) \\
= \sigma(A)\sigma(B)P^2(k) \\
= \sigma(A)P\sigma(B)P(k) \\
= \Sigma(A)\Sigma(B)(k).
\]

Also, for \(\ell \in L\),

\[
\Sigma(AB)(\ell) = \sigma(AB)P(\ell) = 0 = \sigma(A)P\sigma(B)P(\ell) = \Sigma(A)\Sigma(B)(\ell).
\]

Moreover, \(\Sigma(A^*) = \sigma(A^*)P = \sigma(A)^*P = (P\sigma(A))^* = (\sigma(A)P)^* = \Sigma(A)^*(A \in A)\).
Second, $D$ is a $\Sigma$-derivation, since for $A, B \in \mathcal{A}$ and $k \in \mathcal{K}$ we have
\[
D(AB)(k) = d(AB)P(k) = d(AB)(k)
\]
\[
= d(A)\sigma(B)(k) + \sigma(A)d(B)(k)
\]
\[
= d(A)\sigma(B)P^2(k) + \sigma(A)d(B)P^2(k)
\]
\[
= d(A)P\sigma(B)P(k) + \sigma(A)Pd(B)P(k)
\]
\[
= (D(A)\Sigma(B) + \Sigma(A)d(B))(k).
\]
Also, for $\ell \in \mathcal{L}$,
\[
D(AB)(\ell) = d(AB)P(\ell)
\]
\[
= 0
\]
\[
= d(A)P\sigma(B)P(\ell) + \sigma(A)Pd(B)P(\ell)
\]
\[
= (D(A)\Sigma(B) + \Sigma(A)d(B))(\ell).
\]
Moreover, $D$ is continuous if and only if $d$ is also. To show this let $D$ be continuous and $A \in \mathcal{S}(D)$. Then there is a sequence $\{A_n\}$ in $\mathcal{A}$ such that $A_n \to 0$ and $d(A_n) \to A$. By the first paragraph of the proof we know that $A(\mathcal{L}) = 0$, and for $k \in \mathcal{K}$ we have
\[
A(k) = \lim_{n \to \infty} d(A_n)(k) = \lim_{n \to \infty} d(A_n)P(k) = \lim_{n \to \infty} D(A_n)(k) = 0,
\]
since $D$ is continuous.

On the other hand, if $d$ is continuous and $A \in \mathcal{S}(D)$, then there is a sequence $\{A_n\}$ in $\mathcal{A}$ such that $A_n \to 0$ and $D(A_n) \to A$. Obviously for $\ell \in \mathcal{L}$ we have
\[
A(\ell) = \lim_{n \to \infty} D(A_n)(\ell) = \lim_{n \to \infty} d(A_n)P(\ell) = 0,
\]
and for $k \in \mathcal{K}$,
\[
A(k) = \lim_{n \to \infty} D(A_n)(k) = \lim_{n \to \infty} d(A_n)P(k) = \lim_{n \to \infty} d(A_n)(k) = 0,
\]
since $d$ is continuous.

Similarly, one can show that $\Sigma$ is continuous. If $d$ is a $\ast$-$\sigma$-derivation, then the relation $d(A)P = Pd(A)$ implies that $D$ is also a $\ast$-$\Sigma$-derivation. \hfill $\square$

Remark 3.5. If $\mathcal{A}$ is a von Neumann algebra and both $\sigma$ and $d$ are mappings of $\mathcal{A}$ into $\mathcal{A}$, then, due to the fact that the range projection of every element of $\mathcal{A}$ is in $\mathcal{A}$, we conclude that the projection $P$ constructed in the proof of Proposition 3.4 belongs to $\mathcal{A}$ and so the ranges of the derivation $D$ and the linear mapping $\Sigma$ are contained in $\mathcal{A}$.

Proposition 3.6. Let $\sigma$ be a continuous linear mapping and let $d$ be a $\ast$-$\sigma$-derivation. Then there is a continuous $\ast$-homomorphism $\Sigma : \mathcal{A} \to B(\mathcal{H})$ and a $\ast$-$\Sigma$-derivation $D : \mathcal{A} \to B(\mathcal{H})$ such that $D$ is continuous if and only if $d$ is also.

Proof. Define $\sigma^* : \mathcal{A} \to B(\mathcal{H})$ by $\sigma^*(A) = \sigma(A^*)^\ast$. Then $d$ is a $\sigma^*$-derivation. Clearly $d$ is a $\tau$-derivation where $\tau = \frac{\sigma + \sigma^*}{2}$ is a continuous $\ast$-linear mapping. By Proposition 3.4 there exist a continuous $\ast$-homomorphism $\Sigma$ and a $\ast$-$\Sigma$-derivation $D$ such that $D$ is continuous if and only if $d$ is also. \hfill $\square$

The following theorem is an extension of Sakai’s theorem [8] to $\sigma$-derivations.
Theorem 3.7. Let $\sigma$ be a continuous $*$-linear mapping. Then every $\sigma$-derivation $d$ is automatically continuous.

Proof. By Proposition 3.4 we may assume that $\sigma$ is a continuous $*$-homomorphism. Theorem 2.1 implies that $d$ is an ordinary derivation from $A$ into $B(\mathcal{H})$. By the main theorem of [7] we conclude that $d$ is continuous. \hfill $\square$

Using the same argument as in the proof of Proposition 3.6 we can establish the following theorem:

Theorem 3.8. Let $\sigma$ be a continuous linear mapping. Then every $*$-$\sigma$-derivation is automatically continuous.

4. $(\sigma, \tau)$-derivations on $C^*$-algebras

In [3] the authors considered the notion of $(\sigma, \tau)$-derivation. Assume that $A$ is a $*$-subalgebra of a $*$-algebra $B$ and $\sigma, \tau : A \to B$ are $*$-linear mappings. A linear mapping $d : A \to B$ is called a $*$-$(\sigma, \tau)$-derivation if $d$ preserves $*$ and $d(ab) = d(a)\sigma(b) + \tau(a)d(b)$ for all $a, b \in A$. Obviously, a $*$-$\sigma$-derivation is a $*$-$(\sigma, \sigma)$-derivation.

Theorem 4.1. Let $A$ be a $*$-subalgebra of a $*$-algebra $B$ and $\sigma, \tau : A \to B$ are $*$-linear mappings. Then every $*$-$(\sigma, \tau)$-derivation $d : A \to B$ is a $*$-$(\frac{\sigma + \tau}{2}, \frac{\sigma + \tau}{2})$-derivation.

Proof. First we show that each $*$-$(\sigma, \tau)$-derivation is a $*$-$(\tau, \sigma)$-derivation. We have $d(ab) = d(b^*a^*)^* = (d(b)^*\sigma(a)^*)^* + \tau(b)^*d(a)^* = d(a)\tau(b) + \sigma(a)d(b)$.

Now we conclude that $d(ab) = \frac{1}{2}d(ab) + \frac{1}{2}d(ab) = \frac{1}{2}(d(a)\sigma(b) + \tau(a)d(b)) + \frac{1}{2}(d(a)\tau(b) + \sigma(a)d(b)) = d(a)\frac{\tau + \sigma}{2}(b) + \frac{\tau + \sigma}{2}(a)d(b)$. \hfill $\square$

The previous theorem enables us to focus on $\sigma$-derivations while we deal with $*$-algebras. In particular, by using Theorem 4.1 we obtain the following generalizations of Theorem 3.2 and Theorem 3.7.

Theorem 4.2. Let $\sigma$ and $\tau$ be continuous $*$-linear mappings from a $C^*$-algebra $A$ acting on a Hilbert space $\mathcal{H}$ into $B(\mathcal{H})$ and let $d : A \to B(\mathcal{H})$ be a continuous $*$-$(\sigma, \tau)$-derivation. Then there is a continuous linear mapping $\Sigma : A \to B(\mathcal{H})$ such that $d$ is a $*$-$\Sigma$-derivation.

Theorem 4.3. If $\sigma$ and $\tau$ are continuous $*$-linear mappings from a $C^*$-algebra $A$ acting on a Hilbert space $\mathcal{H}$ into $B(\mathcal{H})$, then every $*$-$(\sigma, \tau)$-derivation $d : A \to B(\mathcal{H})$ is automatically continuous.

Acknowledgement

The authors would like to sincerely thank the referee for his/her valuable comments and useful suggestions.

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