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THE REDUCED MINIMUM MODULUS OF DRAZIN INVERSES OF LINEAR OPERATORS ON HILBERT SPACES

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Abstract. In this article, we study the reduced minimum modulus of the Drazin inverse of an operator on a Hilbert space and give lower and upper bounds of the reduced minimum modulus of an operator and its Drazin inverse, respectively. Using these results, we obtain a characterization of the continuity of Drazin inverses of operators on a Hilbert space.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space. Denote by $\mathcal{B}(\mathcal{H})$ the Banach space of all bounded linear operators on $\mathcal{H}$. For an operator $T \in \mathcal{B}(\mathcal{H})$, the symbols $\mathcal{N}(T)$ and $\mathcal{R}(T)$ will denote the null space and the range space of T, respectively. The spectrum of T is denoted by $\sigma(T)$. Let $S_{\mathcal{M}} = \{x \in \mathcal{M} : \|x\| = 1\}$ if $\mathcal{M}$ is a subspace of $\mathcal{H}$ and let $\overline{\mathcal{K}}$ be the closure of a subset $\mathcal{K}$ of $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, if there exists an operator $T^D \in \mathcal{B}(\mathcal{H})$ satisfying the three operator equations

$$TT^D = T^D T, \quad T^D T T^D = T^D, \quad T^{k+1} T^D = T^k, \tag{1}$$

then T^D is called a Drazin inverse of T. For $T \in \mathcal{B}(\mathcal{H})$, if its Drazin inverse exists, then the Drazin inverse T^D of T is unique ([1]). The eigenprojection $T^\pi = I - T^D T$ of T corresponding to the eigenvalue 0 is the uniquely determined idempotent operator with

$$\mathcal{R}(TT^D) = \mathcal{R}(T^k) = \mathcal{R}(T^D) = \mathcal{N}(T^\pi) \quad \text{and} \quad \mathcal{N}(TT^D) = \mathcal{R}(T^\pi). \tag{2}$$

Recall that $\mathrm{asc}(T)$ ($\mathrm{des}(T)$), the ascent (descent) of $T \in \mathcal{B}(\mathcal{H})$, is the smallest non-negative integer n such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$ ($\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$), that is, $\mathrm{asc}(T) := \min\{k : \mathcal{N}(T^k) = \mathcal{N}(T^{k+1})\}$ ($\mathrm{des}(T) := \min\{k : \mathcal{R}(T^k) = \mathcal{R}(T^{k+1})\}$). If no such n exists, then $\mathrm{asc}(T) = \infty$ ($\mathrm{des}(T) = \infty$). It is well known that $\mathrm{des}(T) = \mathrm{asc}(T)$ if $\mathrm{asc}(T)$ and $\mathrm{des}(T)$ are finite ([9], [12]). An operator $T \in \mathcal{B}(\mathcal{H})$ has the Drazin inverse T^D if and only if T has finite ascent and descent, which is equivalent with the fact that 0 is a finite order pole of the resolvent operator $R_\lambda(T) = (\lambda I - T)^{-1}$, say of order k. In such a case, $\mathrm{ind}(T) = \mathrm{asc}(T) = \mathrm{des}(T) = k$, and 0 is not the accumulated point of $\sigma(T)$ ([6]). If $\dim\mathcal{H}$, the dimension of $\mathcal{H}$, is finite,

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then, for $T \in \mathcal{B}(\mathcal{H})$, it is clear that $\mathrm{asc}(T)$ and $\mathrm{des}(T)$ are finite. So all of the operators in $\mathcal{B}(\mathcal{H})$ are Drazin invertible when $\dim\mathcal{H}$ is finite. But if $\dim\mathcal{H}$ is infinite, then there exist operators which are not Drazin invertible. For example, let S be a unilateral shift in $\mathcal{B}(\mathcal{H})$. Since $\mathcal{R}(S^{k+1}) \subseteq \mathcal{R}(S^k)$ and $\mathcal{R}(S^{k+1}) \neq \mathcal{R}(S^k)$ for each positive integer n, $\mathrm{des}(S) = \infty$. Thus S is not Drazin invertible. Some details about Drazin inverses of operators appear in [1], [2], [4], [6], [7], [10]-[12] and [14]-[15].

The reduced minimum modulus $\gamma(T)$ of T (see [8]) is defined by

$$\gamma(T) = \inf\{\|Tx\| : \mathrm{dist}(x, \mathcal{N}(T)) = 1\}.$$

Thus, from the definition of $\gamma(T)$, we deduce that $\|Tx\| \geq \gamma(T)\mathrm{dist}(x, \mathcal{N}(T))$ for any $x \in \mathcal{H}$. It is well known that $\mathcal{R}(T)$ is closed if and only if $\gamma(T) > 0$. In this case, $\gamma(T) = \|T^+\|^{-1}$, where T^+ is the Moore-Penrose inverse of T, is the unique solution of the four operator equations

$$TXT = T, \quad XTX = X, \quad TX = (TX)^*, \quad XT = (XT)^*$$

if there exists the Moore-Penrose inverse of T.

Let $\mathcal{M}$ and $\mathcal{N}$ be two closed linear subspaces of a Hilbert space $\mathcal{H}$. The gap between $\mathcal{M}$ and $\mathcal{N}$ (see [5], [8]) is defined by

$$\mathrm{gap}(\mathcal{M}, \mathcal{N}) = \max\{\delta(\mathcal{M}, \mathcal{N}), \delta(\mathcal{N}, \mathcal{M})\},$$

where $\delta(\mathcal{M}, \mathcal{N}) = \sup\{\mathrm{dist}(x, \mathcal{N}) : x \in \mathcal{M}, \| x \| \leq 1\}$.

In this note, we study the reduced minimum modulus $\gamma(T)$ of an operator $T \in \mathcal{B}(\mathcal{H})$ and the reduced minimum modulus $\gamma(T^D)$ of its Drazin inverse T^D, and give out their lower and upper bounds. Simultaneously, we obtain a characterization of the continuity of Drazin inverses.

2. Main results and proofs

As a motivation of this paper, we observe that there are many differences among the inverse, the Moore-Penrose inverse and the Drazin inverse of operators. In this section, we begin with some examples to show these differences. Let $T \in \mathcal{B}(\mathcal{H})$. It is known that T has the Moore-Penrose inverse T^+ if and only if $\mathcal{R}(T)$ is closed (that is, $\gamma(T) > 0$). But the operator T is not necessary to have the closed range if its Drazin inverse T^D exists. The following example can illustrate it.

Example 1. Let a linear operator T on the Hilbert space $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ with $\dim\mathcal{H} = \infty$ be given by the 3×3 operator matrices,

$$T = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & S \\ 0 & 0 & 0 \end{pmatrix},$$

where $A \in \mathcal{B}(\mathcal{H})$ is invertible and $S \in \mathcal{B}(\mathcal{H})$ is an arbitrary operator with $\mathcal{R}(S)$ not being closed. As we know, there exist many operators in $\mathcal{B}(\mathcal{H})$ with non- closed ranges. It is easy to verify that T^D exists and

$$T^D = \begin{pmatrix} A^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

But $\mathcal{R}(T) = \mathcal{H} \oplus \mathcal{R}(S) \oplus 0$ is not closed.

If $T \in \mathcal{B}(\mathcal{H})$ is invertible and $T_n \to T$ in norm as $n \to \infty$, then T_n^{-1} exists for large enough n and $T_n^{-1} \to T^{-1}$ as $n \to \infty$. The analogue is not available for the Drazin inverse.

Example 2. Let $A \in \mathcal{B}(\mathcal{H})$ be an invertible operator and let S be the unilateral shift operator defied by $Se_l = e_{l+1}$, where $\{e_l, l = 1, 2, 3, \ldots\}$ is an orthonormal basis for $\mathcal{H}$. Let T_n and T on the Hilbert space $\mathcal{H} \oplus \mathcal{H}$ be given by the 2×2 operator matrices

$$T_n = \begin{pmatrix} A & 0 \\ 0 & \frac{1}{n}S \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $T_n \to T$ in norm as $n \to \infty$, T^D exists and

$$T^D = \begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

But, from $\mathcal{R}(T_n) = \mathcal{R}(A) \oplus \mathcal{R}(S)$, $\mathcal{R}(T_n^k) = \mathcal{R}(A^k) \oplus \mathcal{R}(S^k) = \mathcal{R}(A) \oplus \mathcal{R}(S^k)$, $\mathcal{R}(S^{k+1}) \subseteq \mathcal{R}(S^k)$ and $\mathcal{R}(S^{k+1}) \neq \mathcal{R}(S^k)$, we know $\mathcal{R}(T_n^{k+1}) \subseteq \mathcal{R}(T_n^k)$ and $\mathcal{R}(T_n^{k+1}) \neq \mathcal{R}(T_n^k)$ for all $n = 1, 2, 3, \ldots$, so $\mathrm{des}(S) = \infty$. Hence T_n^D does not exist for all $n = 1, 2, 3, \ldots$.

Although T_n^D and T^D exist for all n, $T_n \to T$ as $n \to \infty$ does not imply that $T_n^D \to T^D$ as $n \to \infty$.

Example 3. Let us consider the operator matrices

$$T_n = \begin{pmatrix} \frac{1}{n}I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix}$$

on the Hilbert space $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$. Then T_n and T are not invertible and $T_n \to T$ in norm for large enough n, T_n^D and T^D exist and

$$T_n^D = \begin{pmatrix} nI & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T^D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

But T_n^D does not converge to T^D as $n \to \infty$.

Next, we will give some lemmas which will be useful later.

Lemma 1. *For an operator $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{R}(T)$ is closed if and only if there exists $X \in \mathcal{B}(\mathcal{H})$ such that $TXT = T$.*

Lemma 2. *Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse T^D and ind$(T) = k$. Then $\mathcal{R}(T^k)$ is closed and $\mathcal{R}(T^D) = \mathcal{R}(T^k)$.*

Proof. From $T^D T T^D = T^D$ and $TT^D = T^D T$, we have $T^D = T^k (T^D)^{k+1}$. So $\mathcal{R}(T^D) \subseteq \mathcal{R}(T^k)$. On the other hand, from $TT^D = T^D T$ and $T^{k+1}T^D = T^k$, we can get $T^k = T^D T^{k+1}$ and $\mathcal{R}(T^k) \subseteq \mathcal{R}(T^D)$. Therefore, $\mathcal{R}(T^D) = \mathcal{R}(T^k)$.

Since T^D is the Drazin inverse of T, $T^D T T^D = T^D$. By Lemma 1, $\mathcal{R}(T^D)$ is closed. □

The following result gives a characterization of the spectral radius $r_\sigma(T^D)$ of the Drazin inverse T^D of T.

Lemma 3 ([6])**.** *Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse T^D and ind$(T) = k$. Then dist$(0, \sigma(T) \backslash \{0\}) = (r_\sigma(T^D))^{-1}$, where $r_\sigma(A)$ denotes the spectral radius of A.*

Proof. Since T has the Drazin inverse T^D, then $\mathcal{R}(T^k) = \mathcal{R}(T^D)$ is closed. From $TT^k = T^{k+2}T^D = T^kT^2T^D$, $\mathcal{R}(T^k)$ is an invariant subspace of T. So T and T^D have the operator matrices

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \text{ and } T^D = \begin{pmatrix} T_{11}^{-1} & \sum_{i=0}^{k-1} T_{11}^{i-k-1}T_{12}T_{22}^{k-1-i} \\ 0 & 0 \end{pmatrix} \tag{3}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{R}(T^k) \oplus \mathcal{R}(T^k)^{\perp}$, where T_{11} is invertible and T_{22} is k-nilpotent (see [6], theorem 2.5). So $\sigma(T) = \sigma(T_{11}) \cup \{0\}$ and $\sigma(T^D) = \sigma(T_{11}^{-1}) \cup \{0\}$, since $\sigma(T_{22}) = \{0\}$ is a singleton. Hence

$$\sigma(T^D) = \{\lambda^+ : \lambda \in \sigma(T)\},$$

where

$$\lambda^+ = \begin{cases} \lambda^{-1} & \text{if } \lambda \neq 0, \\ 0 & \text{if } \lambda = 0. \end{cases}$$

By the definition of $\mathrm{dist}(x, \mathcal{M})$, it is easy to get

$$\mathrm{dist}(0, \sigma(T)\backslash\{0\}) = (r_\sigma(T^D))^{-1}.$$

□

For the reduced minimum modulus $\gamma(T^D)$ of T^D, we have the following estimation.

Theorem 4. *Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse $T^D \neq 0$ and ind$(T) = k$. Then*

$$\frac{1}{\|T\|} \leq \gamma(T^D) \leq \frac{\|T^DT\|^2}{\|T^DT^2\|}.$$

Proof. For any $x \in \mathcal{H}, y \in \mathcal{N}(T^D)$, since $TT^D = T^DT$, we have

$$\|T^DT(x)\| = \|T^DT(x-y)\| \leq \|T^DT\|\|x-y\|$$

and

$$\mathrm{dist}(x, \mathcal{N}(T^D)) \leq \|x - (I - T^DT)x\| = \|T^DTx\|.$$

Thus

$$\|T\|\|T^Dx\| \geq \|T^DTx\| \geq \mathrm{dist}(x, \mathcal{N}(T^D)) \geq \frac{\|T^DTx\|}{\|T^DT\|}. \tag{4}$$

Combining the definition of $\gamma(T)$ with (4), we have $\gamma(T^D) \geq \|T\|^{-1}$ and

$$\|T^Dx\| \geq \gamma(T^D)\mathrm{dist}(x, \mathcal{N}(T^D)) \geq \gamma(T^D)\frac{\|T^DTx\|}{\|T^DT\|}. \tag{5}$$

Replacing x by $Tz, \forall z \in \mathcal{H}$ in (5), we have $\|T^DTz\| \geq \gamma(T^D)\frac{\|T^DT^2z\|}{\|T^DT\|}$, that is, $\gamma(T^D) \leq \frac{\|T^DT\|^2}{\|T^DT^2\|}$. □

By the definition of the gap between two subspaces, we get the following results.

Theorem 5. *Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse T^D and ind$(T) = k$. If $\mathcal{N}(T) \neq \mathcal{N}(T^D)$, then*

$$\delta(\overline{\mathcal{R}(T)}, \mathcal{R}(T^D)) = 1, \delta(\mathcal{R}(T^D), \overline{\mathcal{R}(T)}) = 0 \tag{6}$$

and

$$\delta(\mathcal{N}(T^D), \mathcal{N}(T)) = 1, \delta(\mathcal{N}(T), \mathcal{N}(T^D)) = 0. \tag{7}$$

Proof. Since T has the Drazin inverse T^D and $\mathrm{ind}(T) = k$, we have

$$\mathcal{R}(T^D) = \mathcal{R}(T^k) \subseteq \mathcal{R}(T) \subseteq \overline{\mathcal{R}(T)}.$$

Since $T^D = (T^D)^2 T$ and $T^k = T^{k+1}T^D$,

$$\mathcal{N}(T) \subseteq \mathcal{N}(T^D) \subseteq \mathcal{N}(T^k). \tag{8}$$

If $\mathcal{N}(T) \neq \mathcal{N}(T^D)$, then $\mathrm{ind}(T) = k > 1$ and $\mathcal{R}(T^D) \neq \mathcal{R}(T)$. Since $\mathcal{R}(T^D) \subseteq \overline{\mathcal{R}(T)}$, by definition of the gap function δ between two subspaces, we have

$$\delta(\mathcal{R}(T^D), \overline{\mathcal{R}(T)}) = 0.$$

Since $\mathcal{R}(T^D) \neq \mathcal{R}(T)$, there exists $x \in \overline{\mathcal{R}(T)} \cap \mathcal{R}(T^D)^\perp$ with $\|x\| = 1$ such that

$$\mathrm{dist}(x, \mathcal{R}(T^D))^2 = \|x\|^2 - \|P_{\mathcal{R}(T^D)}x\|^2 = 1.$$

Hence

$$\delta(\overline{\mathcal{R}(T)}, \mathcal{R}(T^D)) = 1.$$

By the similar argument, we can obtain formula (7). Hence the proof of this part is omitted. □

Remark. (1) Theorem 3 gives us an example to show that, for two closed subspaces $\mathcal{M}$ and $\mathcal{N}$, $\delta(\mathcal{M}, \mathcal{N})$ is not symmetric, that is, in general $\delta(\mathcal{M}, \mathcal{N}) \neq \delta(\mathcal{N}, \mathcal{M})$. But $\delta(\mathcal{M}, \mathcal{N}) = \delta(\mathcal{N}^\perp, \mathcal{M}^\perp)$ since $\mathrm{dist}(x, \mathcal{M}) = \sup_{y \in S_{\mathcal{M}^\perp}} |\langle x, y\rangle|$ (see [8]). In fact,

$$\begin{aligned}
\delta(\mathcal{M}, \mathcal{N}) &= \sup_{x \in S_{\mathcal{M}}} \mathrm{dist}(x, \mathcal{N}) \\
&= \sup_{x \in S_{\mathcal{M}}} \sup_{y \in S_{\mathcal{N}^\perp}} |\langle x, y\rangle| \\
&= \sup_{y \in S_{\mathcal{N}^\perp}} \sup_{x \in S_{\mathcal{M}}} |\langle x, y\rangle| \\
&= \sup_{y \in S_{\mathcal{N}^\perp}} \mathrm{dist}(y, \mathcal{M}^\perp) \\
&= \delta(\mathcal{N}^\perp, \mathcal{M}^\perp).
\end{aligned}$$

(2) Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse T^D and $\mathrm{ind}(T) = k$. If T and T^D have the operator matrices (3), respectively, and $x_0 + x \in \mathcal{N}(T)$, where $x_0 \in \mathcal{R}(T^k)$ and $x \in \mathcal{R}(T^k)^\perp$, then

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x \end{pmatrix} = 0.$$

So we have

$$\mathcal{N}(T) = \{ \begin{pmatrix} -T_{11}^{-1}T_{12}x \\ x \end{pmatrix} : x \in \mathcal{N}(T_{22}) \}.$$

Similarly, if $y_0 + y \in \mathcal{N}(T^D)$, where $y_0 \in \mathcal{R}(T^k)$ and $y \in \mathcal{R}(T^k)^\perp$, then

$$\begin{pmatrix} T_{11}^{-1} & \sum_{i=0}^{k-1} T_{11}^{i-k-1}T_{12}T_{22}^{k-1-i} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ y \end{pmatrix} = 0.$$

So we have

$$\mathcal{N}(T^D) = \{ \begin{pmatrix} -\sum_{i=0}^{k-1} T_{11}^{i-k}T_{12}T_{22}^{k-1-i}y \\ y \end{pmatrix} : y \in \mathcal{R}(T^k)^\perp \}.$$

In general, $\mathcal{N}(T_{22}) \subseteq \mathcal{N}(T^k)^\perp$; this shows that $\mathcal{N}(T) \subseteq \mathcal{N}(T^D)$. Moreover, $\mathcal{N}(T) = \mathcal{N}(T^D)$ if and only if $\mathcal{N}(T_{22}) = \mathcal{N}(T^k)^\perp$. In this case, $\mathrm{ind}(T) = 1$, that is, $\mathcal{N}(T_{22}) = \mathcal{N}(T)^\perp$.

Theorem 6. *Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse T^D and ind$(T) = k$. If $\mathcal{N}(T) \neq \mathcal{N}(T^D)$, then*

$$\gamma(T) \leq \|T - T^D\|.$$

Proof. If $\mathcal{R}(T)$ is not closed, then $\gamma(T) = 0$. It is clear that $\gamma(T) \leq \|T - T^D\|$.

Next, we assume that $\mathcal{R}(T)$ is closed.

For any $z \in \mathcal{N}(T)$ and $u \in \mathcal{R}(T)$ with $\|u\| = 1$, there exists $x \in \mathcal{H}$ such that $Tx = u$ and

$$\begin{aligned} \operatorname{dist}(u, \mathcal{R}(T^D)) &\leq \|Tx - T^D(x - z)\| \\ &= \|T(x - z) - T^D(x - z)\| \\ &\leq \|T - T^D\|\|x - z\|. \end{aligned}$$

So $\delta(\mathcal{R}(T), \mathcal{R}(T^D)) \leq \|T - T^D\|\|x - z\|$. Since $\|Tx\| \geq \gamma(T)\operatorname{dist}(x, \mathcal{N}(T))$ and $\mathcal{R}(T)$ is closed,

$$\operatorname{dist}(x, \mathcal{N}(T)) \leq \frac{\|Tx\|}{\gamma(T)} = \frac{1}{\gamma(T)}.$$

That is, $1 = \delta(\mathcal{R}(T), \mathcal{R}(T^D)) \leq \|T - T^D\| \frac{1}{\gamma(T)}$ by Theorem 5. The proof is completed. □

The following theorem gives a relation of the reduced minimum modulus of the operators T and T^D.

Theorem 7. *Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse T^D and ind$(T) = k$. Then*

$$\frac{1}{\|T^{k+1}\|}(\gamma(T) - \|T^k - T\|) \leq \gamma(T^D) \leq \|(T^D)^{k+1}\|\gamma(T^k).$$

Proof. From $T^D T T^D = T^D$ and $TT^D = T^D T$, we have $T^D = (T^D)^{k+1} T^k$. Equation (8) shows that $\mathcal{N}(T)^\perp \supseteq \mathcal{N}(T^D)^\perp \supseteq \mathcal{N}(T^k)^\perp$. Therefore, by the definition of $\gamma(T^D)$, we have

$$\begin{aligned} \gamma(T^D) &= \inf\{\|T^D x\| : \operatorname{dist}(x, \mathcal{N}(T^D)) = 1\} \\ &= \inf\{\|T^D x\| : x \in S_{\mathcal{N}(T^D)^\perp}\} \\ &\leq \inf\{\|T^D x\| : x \in S_{\mathcal{N}(T^k)^\perp}\} \\ &= \inf\{\|(T^D)^{k+1} T^k x\| : x \in S_{\mathcal{N}(T^k)^\perp}\} \\ &\leq \|(T^D)^{k+1}\| \inf\{\|T^k x\| : x \in S_{\mathcal{N}(T^k)^\perp}\} \\ &\leq \|(T^D)^{k+1}\|\gamma(T^k). \end{aligned}$$

On the other hand, observing that $T^k = T^{k+1} T^D$, we get

$$\begin{aligned} \gamma(T^D) &= \inf\{\|T^D x\| : x \in S_{\mathcal{N}(T^D)^\perp}\} \\ &\geq \tfrac{1}{\|T^{k+1}\|} \inf\{\|T^k x\| : x \in S_{\mathcal{N}(T^D)^\perp}\} \\ &\geq \tfrac{1}{\|T^{k+1}\|} \inf\{\|T^k x\| : x \in S_{\mathcal{N}(T)^\perp}\} \\ &\geq \tfrac{1}{\|T^{k+1}\|} \inf\{\|Tx\| - \|T^k x - Tx\| : x \in S_{\mathcal{N}(T)^\perp}\} \\ &\geq \tfrac{1}{\|T^{k+1}\|}(\gamma(T) - \|T^k - T\|). \end{aligned}$$

Hence

$$\frac{1}{\|T^{k+1}\|}(\gamma(T) - \|T^k - T\|) \leq \gamma(T^D) \leq \|(T^D)^{k+1}\|\gamma(T^k).$$

□

Combining the proceeding theorem with Theorem 4, we obtain the following result.

Corollary 8. *Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse $T^D \neq 0$ and ind$(T) = k$. Then*

$$\gamma(T^k) \geq \frac{1}{\|T\|\|(T^D)^{k+1}\|}.$$

Note that $\mathcal{N}(T^D) = \mathcal{N}(T^D T T^D) \supseteq \mathcal{N}(TT^D) \supseteq \mathcal{N}(T^D)$. So $\mathcal{N}(T^D) = \mathcal{N}(TT^D)$ and $\gamma(TT^D) = \inf\{\|TT^D x\| : x \in S_{\mathcal{N}(TT^D)^\perp}\} \leq \|T\| \inf\{\|T^D x\| : x \in S_{\mathcal{N}(T^D)^\perp}\} = \|T\|\gamma(T^D)$. Thus we have the following corollary.

Corollary 9. *Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse T^D and ind$(T) = k$. Then*

$$\frac{\gamma(TT^D)}{\|T\|} \leq \gamma(T^D) \leq \|(T^D)^{k+1}\|\gamma(T^k).$$

Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. If $P_\mathcal{M}$ and $P_\mathcal{N}$ are orthogonal projections onto $\mathcal{M}$ and $\mathcal{N}$, respectively, we have the following lemma for the gap between the two subspaces $\mathcal{M}$ and $\mathcal{N}$.

Lemma 10 ([5], [8])**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. If $P_\mathcal{M}$ and $P_\mathcal{N}$ are orthogonal projections onto $\mathcal{M}$ and $\mathcal{N}$, respectively, then*

$$\begin{aligned} gap(\mathcal{M},\mathcal{N}) &= \max\{\|P_\mathcal{M}(I-P_\mathcal{N})\|, \|P_\mathcal{N}(I-P_\mathcal{M})\|\} \\ &= gap(\mathcal{M}^\perp,\mathcal{N}^\perp), \end{aligned}$$

$$\|(I-P_\mathcal{N})P_\mathcal{M}\| \leq \|I-P_\mathcal{N}\|\|P_\mathcal{M}\| gap(\mathcal{R}(P_\mathcal{M}),\mathcal{R}(P_\mathcal{N})) \tag{9}$$

and

$$\|P_\mathcal{N}(I-P_\mathcal{M})\| \leq \|P_\mathcal{N}\|\|I-P_\mathcal{M}\| gap(\mathcal{N}(P_\mathcal{M}),\mathcal{N}(P_\mathcal{N})). \tag{10}$$

Proof. (1) See corollary 7 in [5].

(2) Let $y \in \mathcal{N}$. Then, for any $x \in \mathcal{H}, (I-P_\mathcal{N})P_\mathcal{M}x = (I-P_\mathcal{N})(P_\mathcal{M}x - y)$, which implies

$$\begin{aligned} \|(I-P_\mathcal{N})P_\mathcal{M}x\| &\leq \|(I-P_\mathcal{N})\|\mathrm{dist}(P_\mathcal{M}x,\mathcal{N}) \\ &\leq \|(I-P_\mathcal{N})\|\mathrm{gap}(\mathcal{R}(P_\mathcal{M}),\mathcal{R}(P_\mathcal{N}))\|P_\mathcal{M}x\| \\ &\leq \|(I-P_\mathcal{N})\|\|P_\mathcal{M}\|\mathrm{gap}(\mathcal{R}(P_\mathcal{M}),\mathcal{R}(P_\mathcal{N}))\|x\|. \end{aligned}$$

Hence formula (9) holds.

Similarly, noting that $\mathcal{N}(P_\mathcal{M}) = \mathcal{R}(I-P_\mathcal{M}), \mathcal{N}(P_\mathcal{N}) = \mathcal{R}(I-P_\mathcal{N})$, we can show

$$\|P_\mathcal{N}(I-P_\mathcal{M})\| \leq \|P_\mathcal{N}\|\|I-P_\mathcal{M}\|\mathrm{gap}(\mathcal{N}(P_\mathcal{M}),\mathcal{N}(P_\mathcal{N})).$$

□

Remark. From the above proof, the inequalities (9) and (10) are also true when $P_\mathcal{M}$ and $P_\mathcal{N}$ are projections on $\mathcal{M}$ and $\mathcal{N}$, respectively, but not orthogonal projections.

To discuss the continuity of Drazin inverse in the rest of this paper, we need a lemma (see [8], I, Theorem 6.35).

Lemma 11 ([8])**.** *Let P', Q' be two idempotents on $\mathcal{H}$ and let $\mathcal{M} = \mathcal{R}(P'), \mathcal{N} = \mathcal{R}(Q')$. Let P, Q be the orthogonal projections on $\mathcal{M}, \mathcal{N}$, respectively. Then $\|P-Q\| \leq \|P'-Q'\|$.*

In the corollary 1 of [11], Y. Wei and G. Chen give sufficient and necessary conditions for the continuity of Moore-Penrose inverses in a Hilbert space. With the application of the characterization of gap between two subspaces, we give a similar theorem for Drazin inverses.

Theorem 12. *Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse T^D, $T_n \to T$ as $n \to \infty$, and let $T_n \in \mathcal{B}(\mathcal{H})$ with Drazin inverse T_n^D as n large enough. Then the following statements are equivalent:*

(1) *$T_n^D \to T^D$ as $n \to \infty$.*

(2) *gap*$(\mathcal{R}(T_n^\pi), \mathcal{R}(T^\pi)) \to 0$ *and gap*$(\mathcal{N}(T_n^\pi), \mathcal{N}(T^\pi)) \to 0$ *as* n *large enough, where* $T_n^\pi = I - T_n^D T_n$ *and* $T^\pi = I - T^D T$.

Proof. (1) $\Rightarrow$ (2) Let $O(T_n^\pi)$ and $O(T^\pi)$ be the orthogonal projections of $\mathcal{H}$ onto $\mathcal{R}(T_n^\pi)$ and $\mathcal{R}(T^\pi)$, respectively. By Lemma 11, we have

$$\|O(T_n^\pi) - O(T^\pi)\| \le \|T_n^\pi - T^\pi\|.$$

Since $T, T_n, T_n^D \in \mathcal{B}(\mathcal{H}), T_n \to T$ and $T_n^D \to T^D$ as $n \to \infty$, we have

$$\|T_n^\pi - T^\pi\| = \|T_n^D T_n - T^D T\| \le \|T_n\|\|T_n^D - T^D\| + \|T^D\|\|T_n - T\| \to 0$$

as $n \to \infty$. Hence $O(T_n^\pi) \to O(T^\pi)$ as $n \to \infty$. From Lemma 10, we get

$$\mathrm{gap}(\mathcal{R}(T_n^\pi), \mathcal{R}(T^\pi)) = \max\{\|O(T_n^\pi)(I - O(T^\pi))\|, \|O(T^\pi)(I - O(T_n^\pi))\|\} \to 0$$

as $n \to \infty$. Since $O(T_n^{\pi *}) \to O(T^{\pi *})$ as $n \to \infty$,

$$\mathrm{gap}(\mathcal{N}(T_n^\pi), \mathcal{N}(T^\pi)) = \mathrm{gap}(\mathcal{R}(T_n^{\pi *})^\perp, \mathcal{R}(T^{\pi *})^\perp) = \mathrm{gap}(\mathcal{R}(T_n^{\pi *}), \mathcal{R}(T^{\pi *})) \to 0$$

as $n \to \infty$.

(2)$\Rightarrow$ (1) Since

$$\|(I - T^\pi)T_n^\pi\| \le \|I - T^\pi\|\|T_n^\pi\|\mathrm{gap}(\mathcal{R}(T_n^\pi), \mathcal{R}(T^\pi)) \to 0$$

as $n \to \infty$ and

$$\|T^\pi(I - T_n^\pi)\| \le \|T^\pi\|\|I - T_n^\pi\|\mathrm{gap}(\mathcal{N}(T_n^\pi), \mathcal{N}(T^\pi)) \to 0$$

as $n \to \infty$, we have

$$\|T_n^D T_n - TT^D\| = \|T_n^\pi - T^\pi\| \le \|T_n^\pi - T^\pi T_n^\pi\| + \|T^\pi T_n^\pi - T^\pi\| \to 0$$

as $n \to \infty$. Therefore,

$$\begin{aligned} T_n^D - T^D &= T_n^D TT^D + T_n^D(I - TT^D) - (I - T_n^D T_n)T^D - T_n^D T_n T^D \\ &= T_n^D(T - T_n)T^D + T_n^D(I - TT^D) - (I - T_n^D T_n)T^D \\ &= T_n^D(T - T_n)T^D + T_n^D(T_n^D T_n - TT^D) - (T^D T - T_n^D T_n)T^D \\ &\to 0, \text{ as } n \to \infty. \end{aligned}$$

Thus $T_n^D \to T^D$ in $\mathcal{B}(\mathcal{H})$. □

Remark. (1) Example 2 shows that, in general, it does not imply that T_n is Drazin invertible as n large enough if $T_n \to T$ as $n \to \infty$ and T is Drazin invertible. So in (1) of Theorem 12 the condition that T_n has the Drazin inverse T_n^D for n large enough is necessary.

(2) Theorem 12 shows there is a close relation between the continuality of the Drazin inverse and the gaps of the ranges and the gaps of the null-spaces of the eigenprojections T_n^π and T^π of T_n and T corresponding to the eigenvalue 0, respectively.

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