THE REDUCED MINIMUM MODULUS OF DRAZIN INVERSES OF LINEAR OPERATORS ON HILBERT SPACES

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ABSTRACT. In this article, we study the reduced minimum modulus of the Drazin inverse of an operator on a Hilbert space and give lower and upper bounds of the reduced minimum modulus of an operator and its Drazin inverse, respectively. Using these results, we obtain a characterization of the continuity of Drazin inverses of operators on a Hilbert space.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space. Denote by $\mathcal{B}(\mathcal{H})$ the Banach space of all bounded linear operators on $\mathcal{H}$. For an operator $T \in \mathcal{B}(\mathcal{H})$, the symbols $\mathcal{N}(T)$ and $\mathcal{R}(T)$ will denote the null space and the range space of $T$, respectively. The spectrum of $T$ is denoted by $\sigma(T)$. Let $S_M = \{x \in M : \|x\| = 1\}$ if $M$ is a subspace of $\mathcal{H}$ and let $\mathcal{K}$ be the closure of a subset $K$ of $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, if there exists an operator $T^D \in \mathcal{B}(\mathcal{H})$ satisfying the three operator equations

$$(1) \quad TT^D = T^D T, \quad T^D TT^D = T^D, \quad T^{k+1}T^D = T^k,$$

then $T^D$ is called a Drazin inverse of $T$. For $T \in \mathcal{B}(\mathcal{H})$, if its Drazin inverse exists, then the Drazin inverse $T^D$ of $T$ is unique ([11]). The eigenprojection $T^\pi = I - T^D T$ of $T$ corresponding to the eigenvalue 0 is the uniquely determined idempotent operator with

$$(2) \quad \mathcal{R}(TT^D) = \mathcal{R}(T^k) = \mathcal{R}(T^D) = \mathcal{N}(T^\pi) \quad \text{and} \quad \mathcal{N}(TT^D) = \mathcal{R}(T^\pi).$$

Recall that $\text{asc}(T)$ (des$(T)$), the ascent (descent) of $T \in \mathcal{B}(\mathcal{H})$, is the smallest non-negative integer $n$ such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$ (if $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$), that is,

$\text{asc}(T) := \min\{k : \mathcal{N}(T^k) = \mathcal{N}(T^{k+1})\}$ (des$(T) := \min\{k : \mathcal{R}(T^k) = \mathcal{R}(T^{k+1})\}$). If no such $n$ exists, then $\text{asc}(T) = \infty$ (des$(T) = \infty$). It is well known that des$(T) = \text{asc}(T)$ if $\text{asc}(T)$ and des$(T)$ are finite ([9], [12]). An operator $T \in \mathcal{B}(\mathcal{H})$ has the Drazin inverse $T^D$ if and only if $T$ has finite ascent and descent, which is equivalent with the fact that 0 is a finite order pole of the resolvent operator $R_\lambda(T) = (\lambda I - T)^{-1}$, say of order $k$. In such a case, ind$(T) = \text{asc}(T) = \text{des}(T) = k$, and 0 is not the accumulated point of $\sigma(T)$ ([3]). If $\dim \mathcal{H}$, the dimension of $\mathcal{H}$, is finite,
then, for $T \in \mathcal{B}(\mathcal{H})$, it is clear that \( \text{asc}(T) \) and \( \text{des}(T) \) are finite. So all of the operators in $\mathcal{B}(\mathcal{H})$ are Drazin invertible when $\dim \mathcal{H}$ is finite. But if $\dim \mathcal{H}$ is infinite, then there exist operators which are not Drazin invertible. For example, let $S$ be a unilateral shift in $\mathcal{B}(\mathcal{H})$. Since $\mathcal{R}(S^{k+1}) \subseteq \mathcal{R}(S^k)$ and $\mathcal{R}(S^{k+1}) \neq \mathcal{R}(S^k)$ for each positive integer $n$, $\text{des}(S) = \infty$. Thus $S$ is not Drazin invertible. Some details about Drazin inverses of operators appear in [1], [2], [4], [6], [7], [10]-[12] and [14]-[15].

The reduced minimum modulus $\gamma(T)$ of $T$ (see [8]) is defined by

$$
\gamma(T) = \inf \{ \|Tx\| : \text{dist}(x, \mathcal{N}(T)) = 1 \}.
$$

Thus, from the definition of $\gamma(T)$, we deduce that $\|Tx\| \geq \gamma(T)\text{dist}(x, \mathcal{N}(T))$ for any $x \in \mathcal{H}$. It is well known that $\mathcal{R}(T)$ is closed if and only if $\gamma(T) > 0$. In this case, $\gamma(T) = \|T^+\|^{-1}$, where $T^+$ is the Moore-Penrose inverse of $T$, is the unique solution of the four operator equations

\[
TXX = T, \quad XTX = X, \quad TX = (TX)^*, \quad XT = (XT)^*.
\]

if there exists the Moore-Penrose inverse of $T$.

Let $\mathcal{M}$ and $\mathcal{N}$ be two closed linear subspaces of a Hilbert space $\mathcal{H}$. The gap between $\mathcal{M}$ and $\mathcal{N}$ (see [5], [8]) is defined by

$$
gap(\mathcal{M}, \mathcal{N}) = \max \{ \delta(\mathcal{M}, \mathcal{N}), \delta(\mathcal{N}, \mathcal{M}) \},
$$

where $\delta(\mathcal{M}, \mathcal{N}) = \sup \{ \text{dist}(x, \mathcal{N}) : x \in \mathcal{M}, \|x\| \leq 1 \}$.

In this note, we study the reduced minimum modulus $\gamma(T)$ of an operator $T \in \mathcal{B}(\mathcal{H})$ and the reduced minimum modulus $\gamma(T^D)$ of its Drazin inverse $T^D$, and give out their lower and upper bounds. Simultaneously, we obtain a characterization of the continuity of Drazin inverses.

## 2. Main results and proofs

As a motivation of this paper, we observe that there are many differences among the inverse, the Moore-Penrose inverse and the Drazin inverse of operators. In this section, we begin with some examples to show these differences. Let $T \in \mathcal{B}(\mathcal{H})$. It is known that $T$ has the Moore-Penrose inverse $T^+$ if and only if $\mathcal{R}(T)$ is closed (that is, $\gamma(T) > 0$). But the operator $T$ is not necessary to have the closed range if its Drazin inverse $T^D$ exists. The following example can illustrate it.

**Example 1.** Let a linear operator $T$ on the Hilbert space $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ with $\dim \mathcal{H} = \infty$ be given by the $3 \times 3$ operator matrices,

$$
T = \begin{pmatrix}
A & 0 & 0 \\
0 & 0 & S \\
0 & 0 & 0
\end{pmatrix},
$$

where $A \in \mathcal{B}(\mathcal{H})$ is invertible and $S \in \mathcal{B}(\mathcal{H})$ is an arbitrary operator with $\mathcal{R}(S)$ not being closed. As we know, there exist many operators in $\mathcal{B}(\mathcal{H})$ with non-closed ranges. It is easy to verify that $T^D$ exists and

$$
T^D = \begin{pmatrix}
A^{-1} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
$$

But $\mathcal{R}(T) = \mathcal{H} \oplus \mathcal{R}(S) \oplus 0$ is not closed.
If \( T \in \mathcal{B}(\mathcal{H}) \) is invertible and \( T_n \to T \) in norm as \( n \to \infty \), then \( T_n^{-1} \) exists for large enough \( n \) and \( T_n^{-1} \to T^{-1} \) as \( n \to \infty \). The analogue is not available for the Drazin inverse.

**Example 2.** Let \( A \in \mathcal{B}(\mathcal{H}) \) be an invertible operator and let \( S \) be the unilateral shift operator defined by \( Se_i = e_{i+1} \), where \( \{e_i, i = 1, 2, 3, \ldots \} \) is an orthonormal basis for \( \mathcal{H} \). Let \( T_n \) and \( T \) on the Hilbert space \( \mathcal{H} \oplus \mathcal{H} \) be given by the \( 2 \times 2 \) operator matrices

\[
T_n = \begin{pmatrix} A & 0 \\ 0 & \frac{1}{n} S \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.
\]

Then \( T_n \to T \) in norm as \( n \to \infty \), \( T^D \) exists and

\[
T^D = \begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix}.
\]

But, from \( \mathcal{R}(T_n) = \mathcal{R}(A) \oplus \mathcal{R}(S) \), \( \mathcal{R}(T^k_n) = \mathcal{R}(A^k) \oplus \mathcal{R}(S^k) = \mathcal{R}(A) \oplus \mathcal{R}(S^k) \), \( \mathcal{R}(S^k) \subseteq \mathcal{R}(S^k) \) and \( \mathcal{R}(S^k+1) \neq \mathcal{R}(S^k) \), we know \( \mathcal{R}(T^k_n) \subseteq \mathcal{R}(T^k_n) \) and \( \mathcal{R}(T^k_n) \neq \mathcal{R}(T^k_n) \) for all \( n = 1, 2, 3, \ldots \), so \( \text{des}(S) = \infty \). Hence \( T_n^D \) does not exist for all \( n = 1, 2, 3, \ldots \).

Although \( T_n^D \) and \( T^D \) exist for all \( n \), \( T_n \to T \) as \( n \to \infty \) does not imply that \( T_n^D \to T^D \) as \( n \to \infty \).

**Example 3.** Let us consider the operator matrices

\[
T_n = \begin{pmatrix} \frac{1}{n} I & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

on the Hilbert space \( \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \). Then \( T_n \) and \( T \) are not invertible and \( T_n \to T \) in norm for large enough \( n \), \( T_n^D \) and \( T^D \) exist and

\[
T_n^D = \begin{pmatrix} nI & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T^D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

But \( T_n^D \) does not converge to \( T^D \) as \( n \to \infty \).

Next, we will give some lemmas which will be useful later.

**Lemma 1.** For an operator \( T \in \mathcal{B}(\mathcal{H}) \), \( \mathcal{R}(T) \) is closed if and only if there exists \( X \in \mathcal{B}(\mathcal{H}) \) such that \( T^TX = T \).

**Lemma 2.** Let \( T \in \mathcal{B}(\mathcal{H}) \) with the Drazin inverse \( T^D \) and \( \text{ind}(T) = k \). Then \( \mathcal{R}(T^k) \) is closed and \( \mathcal{R}(T^k) = \mathcal{R}(T^k) \).

**Proof.** From \( T^D TT^D = T^D \) and \( TT^D = T'T \), we have \( T^D = T^k(T^D)^k+1 \). So \( \mathcal{R}(T^k) \subseteq \mathcal{R}(T^k) \). On the other hand, from \( TT^D = T^D T \) and \( T^{k+1} T^D = T^k \), we can get \( T^k = T^D T^k+1 \) and \( \mathcal{R}(T^k) \subseteq \mathcal{R}(T^D) \). Therefore, \( \mathcal{R}(T^D) = \mathcal{R}(T^D) \).

Since \( T^D \) is the Drazin inverse of \( T \), \( T^D TT^D = T^D \). By Lemma 1, \( \mathcal{R}(T^D) \) is closed. \(\square\)

The following result gives a characterization of the spectral radius \( r_\sigma(T^D) \) of the Drazin inverse \( T^D \) of \( T \).

**Lemma 3 (E).** Let \( T \in \mathcal{B}(\mathcal{H}) \) with the Drazin inverse \( T^D \) and \( \text{ind}(T) = k \). Then

\[
\text{dist}(0, \sigma(T) \setminus \{0\}) = (r_\sigma(T^D))^{-1}, \text{ where } r_\sigma(A) \text{ denotes the spectral radius of } A.
\]
Proof. Since $T$ has the Drazin inverse $T^D$, then $\mathcal{R}(T^k) = \mathcal{R}(T^D)$ is closed. From $TT^k = T^{k+2}T^D = T^{k+2}T^D$, $\mathcal{R}(T^k)$ is an invariant subspace of $T$. So $T$ and $T^D$ have the operator matrices

$$
T = \begin{pmatrix}
T_{11} & T_{12} \\
0 & T_{22}
\end{pmatrix}
$$

and

$$
T^D = \begin{pmatrix}
T_{11}^{-1} & \sum_{i=0}^{k-1} T_{11}^{-i-k}T_{12}T_{22}^{k-1-i} \\
0 & 0
\end{pmatrix}
$$

with respect to the space decomposition $\mathcal{H} = \mathcal{R}(T^k) \oplus \mathcal{R}(T^k)^\perp$, where $T_{11}$ is invertible and $T_{22}$ is $k$-nilpotent (see [6], theorem 2.5). So $\sigma(T) = \sigma(T_{11}) \cup \{0\}$ and $\sigma(T^D) = \sigma(T_{11}^{-1}) \cup \{0\}$, since $\sigma(T_{22}) = \{0\}$ is a singleton. Hence

$$
\sigma(T^D) = \{\lambda^+ : \lambda \in \sigma(T)\},
$$

where

$$
\lambda^+ = \begin{cases}
\lambda^{-1} & \text{if } \lambda \neq 0, \\
0 & \text{if } \lambda = 0.
\end{cases}
$$

By the definition of $\text{dist}(x, \mathcal{M})$, it is easy to get

$$
\text{dist}(0, \sigma(T) \setminus \{0\}) = \left(\tau_{\sigma}(T^D)\right)^{-1}.
$$

For the reduced minimum modulus $\gamma(T^D)$ of $T^D$, we have the following estimation.

**Theorem 4.** Let $T \in B(\mathcal{H})$ with the Drazin inverse $T^D \neq 0$ and $\text{ind}(T) = k$. Then

$$
\frac{1}{\|T\|} \leq \gamma(T^D) \leq \frac{\|T^DT\|^2}{\|T^DT^2\|}.
$$

Proof. For any $x \in \mathcal{H}, y \in \mathcal{N}(T^D)$, since $TT^D = T^DT$, we have

$$
\|T^DT(x)\| = \|T^DT(x - y)\| \leq \|T^DT\| \|x - y\|
$$

and

$$
\text{dist}(x, \mathcal{N}(T^D)) \leq \|x - (I - T^DT)x\| = \|T^DTx\|.
$$

Thus

$$
\|T\| \|T^Dx\| \geq \|T^DTx\| \geq \text{dist}(x, \mathcal{N}(T^D)) \geq \frac{\|T^DTx\|}{\|T^DT\|}.
$$

Combining the definition of $\gamma(T)$ with (4), we have $\gamma(T^D) \geq \|T\|^{-1}$ and

$$
\|T^Dx\| \geq \gamma(T^D) \text{dist}(x, \mathcal{N}(T^D)) \geq \gamma(T^D) \frac{\|T^DTx\|}{\|T^DT\|}.
$$

Replacing $x$ by $Tz, \forall z \in \mathcal{H}$ in (5), we have $\|T^DTz\| \geq \gamma(T^D) \frac{\|T^DT^2z\|}{\|T^DT\|}$, that is,

$$
\gamma(T^D) \leq \frac{\|T^DT^2\|^2}{\|T^DT\|^2}.
$$

By the definition of the gap between two subspaces, we get the following results.

**Theorem 5.** Let $T \in B(\mathcal{H})$ with the Drazin inverse $T^D$ and $\text{ind}(T) = k$. If $\mathcal{N}(T) \neq \mathcal{N}(T^D)$, then

$$
\delta(\overline{\mathcal{R}(T)} , \mathcal{R}(T^D)) = 1, \delta(\mathcal{R}(T^D), \overline{\mathcal{R}(T)}) = 0
$$

and

$$
\delta(\mathcal{N}(T^D), \mathcal{N}(T)) = 1, \delta(\mathcal{N}(T), \mathcal{N}(T^D)) = 0.
$$
Proof. Since $T$ has the Drazin inverse $T^D$ and $\text{ind}(T) = k$, we have

$$\mathcal{R}(T^D) = \mathcal{R}(T^k) \subseteq \mathcal{R}(T) \subseteq \overline{\mathcal{R}(T)}.$$ 

Since $T^D = (T^D)^2T$ and $T^k = T^{k+1}T^D$,

$$(8) \quad \mathcal{N}(T) \subseteq \mathcal{N}(T^D) \subseteq \mathcal{N}(T^k).$$

If $\mathcal{N}(T) \neq \mathcal{N}(T^D)$, then $\text{ind}(T) = k > 1$ and $\mathcal{R}(T^D) \neq \mathcal{R}(T)$. Since $\mathcal{R}(T^D) \subseteq \overline{\mathcal{R}(T)}$, by definition of the gap function $\delta$ between two subspaces, we have

$$\delta(\mathcal{R}(T^D), \mathcal{R}(T)) = 0.$$ 

Since $\mathcal{R}(T^D) \neq \mathcal{R}(T)$, there exists $x \in \overline{\mathcal{R}(T)} \cap \mathcal{R}(T^D) \perp$ with $\|x\| = 1$ such that

$$\text{dist}(x, \mathcal{R}(T^D))^2 = \|x\|^2 - \|P_{\mathcal{R}(T^D)}x\|^2 = 1.$$ 

Hence

$$\delta(\overline{\mathcal{R}(T)}, \mathcal{R}(T^D)) = 1.$$ 

By the similar argument, we can obtain formula (7). Hence the proof of this part is omitted. 

Remark. (1) Theorem 3 gives us an example to show that, for two closed subspaces $\mathcal{M}$ and $\mathcal{N}$, $\delta(\mathcal{M}, \mathcal{N})$ is not symmetric, that is, in general $\delta(\mathcal{M}, \mathcal{N}) \neq \delta(\mathcal{N}, \mathcal{M})$. But $\delta(\mathcal{M}, \mathcal{N}) = \delta(\mathcal{N}^\perp, \mathcal{M}^\perp)$ since $\text{dist}(x, \mathcal{M}) = \sup_{y \in S_{\mathcal{M}^\perp}} |\langle x, y \rangle|$ (see [8]). In fact,

$$\delta(\mathcal{M}, \mathcal{N}) = \sup_{x \in S_{\mathcal{M}}} \text{dist}(x, \mathcal{N}) = \sup_{x \in S_{\mathcal{M}}} \sup_{y \in S_{\mathcal{N}^\perp}} |\langle x, y \rangle| = \sup_{y \in S_{\mathcal{N}^\perp}} \sup_{x \in S_{\mathcal{M}}} |\langle x, y \rangle| = \sup_{y \in S_{\mathcal{N}^\perp}} \text{dist}(y, \mathcal{M}^\perp) = \delta(\mathcal{N}^\perp, \mathcal{M}^\perp).$$

(2) Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse $T^D$ and $\text{ind}(T) = k$. If $T$ and $T^D$ have the operator matrices (3), respectively, and $x_0 + x \in \mathcal{N}(T)$, where $x_0 \in \mathcal{R}(T^k)$ and $x \in \mathcal{R}(T^k) \perp$, then

$$\left( \begin{array}{cc} T_{11} & T_{12} \\ 0 & T_{22} \end{array} \right) \left( \begin{array}{c} x_0 \\ x \end{array} \right) = 0.$$ 

So we have

$$\mathcal{N}(T) = \{ \left( \begin{array}{c} -T_{11}^{-1}T_{12}x \\ x \end{array} \right) : x \in \mathcal{N}(T_{22}) \}.$$ 

Similarly, if $y_0 + y \in \mathcal{N}(T^D)$, where $y_0 \in \mathcal{R}(T^k)$ and $y \in \mathcal{R}(T^k) \perp$, then

$$\left( \begin{array}{cc} T_{11}^{-1} & 0 \\ \sum_{i=0}^{k-1} T_{11}^{i-k}T_{12}T_{22}^{k-1-i} & 0 \end{array} \right) \left( \begin{array}{c} y_0 \\ y \end{array} \right) = 0.$$ 

So we have

$$\mathcal{N}(T^D) = \{ \left( \begin{array}{c} -\sum_{i=0}^{k-1} T_{11}^{i-k}T_{12}T_{22}^{k-1-i}y \\ y \end{array} \right) : y \in \mathcal{R}(T^k) \perp \}.$$ 

In general, $\mathcal{N}(T_{22}) \subseteq \mathcal{N}(T^k) \perp$; this shows that $\mathcal{N}(T) \subseteq \mathcal{N}(T^D)$. Moreover, $\mathcal{N}(T) = \mathcal{N}(T^D)$ if and only if $\mathcal{N}(T_{22}) = \mathcal{N}(T^k) \perp$. In this case, $\text{ind}(T) = 1$, that is, $\mathcal{N}(T_{22}) = \mathcal{N}(T) \perp$.

Theorem 6. Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse $T^D$ and $\text{ind}(T) = k$. If $\mathcal{N}(T) \neq \mathcal{N}(T^D)$, then

$$\gamma(T) \leq \|T - T^D\|.$$
Proof. If \( R(T) \) is not closed, then \( \gamma(T) = 0 \). It is clear that \( \gamma(T) \leq \|T - T^D\| \).

Next, we assume that \( R(T) \) is closed.

For any \( z \in N(T) \) and \( u \in R(T) \) with \( \|u\| = 1 \), there exists \( x \in H \) such that \( Tx = u \) and
\[
\text{dist}(u, R(T^D)) \leq \|Tx - T^D(x - z)\| = \|T(x - z) - T^D(x - z)\| \leq \|T - T^D\|\|x - z\|.
\]

So \( \delta(R(T), R(T^D)) \leq \|T - T^D\|\|x - z\| \). Since \( \|Tx\| \geq \gamma(T)\text{dist}(x, N(T)) \) and \( R(T) \) is closed,
\[
\text{dist}(x, N(T)) \leq \frac{\|Tx\|}{\gamma(T)} = \frac{1}{\gamma(T)}.
\]

That is, \( 1 = \delta(R(T), R(T^D)) \leq \|T - T^D\|\frac{1}{\gamma(T)} \) by Theorem 5. The proof is completed. \( \square \)

The following theorem gives a relation of the reduced minimum modulus of the operators \( T \) and \( T^D \).

**Theorem 7.** Let \( T \in B(H) \) with the Drazin inverse \( T^D \) and ind\((T) = k \). Then
\[
\frac{1}{\|T^{k+1}\|}(\gamma(T) - \|T^k - T\|) \leq \gamma(T^D) \leq \|(T^D)^{k+1}\|\gamma(T^k).
\]

Proof. From \( T^D TT^D = T^D \) and \( TT^D = T^D T \), we have \( T^D = (T^D)^{k+1}T^k \). Equation (8) shows that \( N(T) \perp \supseteq N(T^D) \perp \supseteq N(T^k) \perp \). Therefore, by the definition of \( \gamma(T^D) \), we have
\[
\gamma(T^D) = \inf\{\|T^D x\| : \text{dist}(x, N(T^D)) = 1\} = \inf\{\|T^D x\| : x \in S_{N(T^D)}\} \leq \inf\{\|T^D x\| : x \in S_{N(T^D)\perp}\} = \inf\{\|(T^D)^{k+1}T^k x\| : x \in S_{N(T^k)\perp}\} \leq \|(T^D)^{k+1}\inf\{\|T^k x\| : x \in S_{N(T^k)\perp}\} \leq \|(T^D)^{k+1}\|\gamma(T^k).
\]

On the other hand, observing that \( T^k = T^{k+1}T^D \), we get
\[
\gamma(T^D) = \inf\{\|T^D x\| : x \in S_{N(T^D)\perp}\} \geq \frac{1}{\|T^{k+1}\|}\inf\{\|T^k x\| : x \in S_{N(T^k)\perp}\} \geq \frac{1}{\|T^{k+1}\|}\inf\{\|T x\| : x \in S_{N(T)\perp}\} \geq \frac{1}{\|T^{k+1}\|}(\gamma(T) - \|T^k - T\|).
\]

Hence
\[
\frac{1}{\|T^{k+1}\|}(\gamma(T) - \|T^k - T\|) \leq \gamma(T^D) \leq \|(T^D)^{k+1}\|\gamma(T^k).
\]

\( \square \)

Combining the proceeding theorem with Theorem 4, we obtain the following result.

**Corollary 8.** Let \( T \in B(H) \) with the Drazin inverse \( T^D \neq 0 \) and ind\((T) = k \). Then
\[
\gamma(T^k) \geq \frac{1}{\|T\|\|(T^D)^{k+1}\|}.
\]
Note that \( N(T^D) = N(T^DTT^D) \supseteq N(TT^D) \supseteq N(T^D) \). So \( N(T^D) = N(TT^D) \) and \( \gamma(TT^D) = \inf\{\|TT^Dx\| : x \in S_{N(TT^D)^\perp}\} \leq \|T\| \inf\{\|T^Dx\| : x \in S_{N(T^D)^\perp}\} = \|T\| \gamma(T^D) \). Thus we have the following corollary.

**Corollary 9.** Let \( T \in \mathcal{B}(\mathcal{H}) \) with the Drazin inverse \( T^D \) and \( \text{ind}(T) = k \). Then

\[
\frac{\gamma(TT^D)}{\|T\|} \leq \gamma(T^D) \leq (T^D)^{k+1}\|T^k\|.
\]

Let \( \mathcal{M} \) and \( \mathcal{N} \) be two closed subspaces of \( \mathcal{H} \). If \( P_M \) and \( P_N \) are orthogonal projections onto \( \mathcal{M} \) and \( \mathcal{N} \), respectively, we have the following lemma for the gap between the two subspaces \( \mathcal{M} \) and \( \mathcal{N} \).

**Lemma 10** ([3], [5]). Let \( \mathcal{M} \) and \( \mathcal{N} \) be two closed subspaces of \( \mathcal{H} \). If \( P_M \) and \( P_N \) are orthogonal projections onto \( \mathcal{M} \) and \( \mathcal{N} \), respectively, then

\[
gap(\mathcal{M}, \mathcal{N}) = \max\{\|P_M(I - P_N)\|, \|P_N(I - P_M)\|\}
\]

\[
= \gap(\mathcal{M}^\perp, \mathcal{N}^\perp),
\]

(9)

\[
\|I - P_N\|P_M\| \leq \|I - P_N\||P_M\|\gap(\mathcal{R}(P_M), \mathcal{R}(P_N))
\]

and

(10)

\[
\|P_N(I - P_M)\| \leq \|P_N\||I - P_M\|\gap(\mathcal{N}(P_M), \mathcal{N}(P_N)).
\]

**Proof.** (1) See corollary 7 in [5].

(2) Let \( y \in \mathcal{N} \). Then, for any \( x \in \mathcal{H}, (I - P_N)P_Mx = (I - P_N)(P_Mx - y) \), which implies

\[
\|(I - P_N)P_Mx\| \leq \|(I - P_N)||\text{dist}(P_Mx, \mathcal{N})\|
\]

\[
\leq \|(I - P_N)||\gap(\mathcal{R}(P_M), \mathcal{R}(P_N))||P_Mx\|
\]

\[
\leq \|(I - P_N)||P_M||\gap(\mathcal{R}(P_M), \mathcal{R}(P_N))\||x||.
\]

Hence formula (9) holds.

Similarly, noting that \( \mathcal{N}(P_M) = \mathcal{R}(I - P_M), \mathcal{N}(P_N) = \mathcal{R}(I - P_N) \), we can show

\[
\|P_N(I - P_M)\| \leq \|P_N\||I - P_M\|\gap(\mathcal{N}(P_M), \mathcal{N}(P_N)).
\]

\[\square\]

**Remark.** From the above proof, the inequalities (9) and (10) are also true when \( P_M \) and \( P_N \) are projections on \( \mathcal{M} \) and \( \mathcal{N} \), respectively, but not orthogonal projections.

To discuss the continuity of Drazin inverse in the rest of this paper, we need a lemma (see [5], I, Theorem 6.35).

**Lemma 11** ([5]). Let \( P', Q' \) be two idempotents on \( \mathcal{H} \) and let \( \mathcal{M} = \mathcal{R}(P'), \mathcal{N} = \mathcal{R}(Q') \). Let \( P, Q \) be the orthogonal projections on \( \mathcal{M}, \mathcal{N} \), respectively. Then

\[
\|P - Q\| \leq \|P' - Q'\|.
\]

In the corollary 1 of [11], Y. Wei and G. Chen give sufficient and necessary conditions for the continuity of Moore-Penrose inverses in a Hilbert space. With the application of the characterization of gap between two subspaces, we give a similar theorem for Drazin inverses.

**Theorem 12.** Let \( T \in \mathcal{B}(\mathcal{H}) \) with the Drazin inverse \( T^D \), \( T_n \to T \) as \( n \to \infty \), and let \( T_n \in \mathcal{B}(\mathcal{H}) \) with Drazin inverse \( T_n^D \) as \( n \) large enough. Then the following statements are equivalent:

1. \( T_n^D \to T^D \) as \( n \to \infty \).
(2) \( \text{gap}(\mathcal{R}(T^n), \mathcal{R}(T^\pi)) \to 0 \) and \( \text{gap}(\mathcal{N}(T^n), \mathcal{N}(T^\pi)) \to 0 \) as \( n \) large enough, where \( T_n^D = I - T_n^D T_n \) and \( T^\pi = I - T^D T \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( O(T^n) \) and \( O(T^\pi) \) be the orthogonal projections of \( \mathcal{H} \) onto \( \mathcal{R}(T^n) \) and \( \mathcal{R}(T^\pi) \), respectively. By Lemma 11, we have

\[
\| O(T^n) - O(T^\pi) \| \leq \| T_n^\pi - T^\pi \|.
\]

Since \( T, T_n, T_n^D \in \mathcal{B}(\mathcal{H}), T_n \to T \) and \( T_n^D \to T^D \) as \( n \to \infty \), we have

\[
\| T_n^\pi - T^\pi \| = \| T_n^D T_n - T^D T \| \leq \| T_n \| \| T_n^D - T^D \| + \| T^D \| \| T_n - T \| \to 0
\]
as \( n \to \infty \). Hence \( O(T_n^\pi) \to O(T^\pi) \) as \( n \to \infty \). From Lemma 10, we get

\[
\text{gap}(\mathcal{R}(T_n^\pi), \mathcal{R}(T^\pi)) = \max\{\| O(T_n^\pi)(I - O(T^\pi)) \|, \| O(T^\pi)(I - O(T_n^\pi)) \|\} \to 0
\]
as \( n \to \infty \). Since \( O(T_n^\pi) \to O(T^\pi) \) as \( n \to \infty \),

\[
\text{gap}(\mathcal{N}(T_n^\pi), \mathcal{N}(T^\pi)) = \text{gap}(\mathcal{R}(T_n^\pi), \mathcal{R}(T^\pi)) = \text{gap}(\mathcal{R}(T_n^{\pi^*}), \mathcal{R}(T^{\pi^*})) \to 0
\]
as \( n \to \infty \).

(2) \( \Rightarrow \) (1) Since

\[
\| (I - T^\pi) T_n^\pi \| \leq \| I - T^\pi \| \| T_n^\pi \| \text{gap}(\mathcal{R}(T_n^\pi), \mathcal{R}(T^\pi)) \to 0
\]
as \( n \to \infty \) and

\[
\| T_n^\pi (I - T_n^\pi) \| \leq \| T_n^\pi \| \| I - T_n^\pi \| \text{gap}(\mathcal{N}(T_n^\pi), \mathcal{N}(T^\pi)) \to 0
\]
as \( n \to \infty \), we have

\[
\| T_n^D T_n - T^D T \| = \| T_n^\pi - T^\pi \| \leq \| T_n^\pi - T^\pi T_n^\pi \| + \| T^\pi T_n^\pi - T^\pi \| \to 0
\]
as \( n \to \infty \). Therefore,

\[
T_n^D - T^D = T_n^D T^D + T_n^D (I - T^D T) - (I - T_n^D T_n) T^D - T_n^D T_n T^D
\]

\[
= T_n^D (T - T_n) T^D + T_n^D (I - T^D T) - (I - T_n^D T_n) T^D
\]

\[
= T_n^D (T - T_n) T^D + T_n^D (T_n^D T_n - T^D T) - (T^D T - T_n^D T_n) T^D
\]

\[
\to 0, \text{ as } n \to \infty.
\]

Thus \( T_n^D \to T^D \) in \( \mathcal{B}(\mathcal{H}) \).

**Remark.** (1) Example 2 shows that, in general, it does not imply that \( T_n \) is Drazin invertible as \( n \) large enough if \( T_n \to T \) as \( n \to \infty \) and \( T \) is Drazin invertible. So in (1) of Theorem 12 the condition that \( T_n \) has the Drazin inverse \( T_n^D \) for \( n \) large enough is necessary.

(2) Theorem 12 shows there is a close relation between the continuity of the Drazin inverse and the gaps of the ranges and the gaps of the null-spaces of the eigenprojections \( T_n^\pi \) and \( T^\pi \) of \( T_n \) and \( T \) corresponding to the eigenvalue 0, respectively.

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References


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