THE REDUCED MINIMUM MODULUS OF DRAZIN INVERSES OF LINEAR OPERATORS ON HILBERT SPACES

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Abstract. In this article, we study the reduced minimum modulus of the Drazin inverse of an operator on a Hilbert space and give lower and upper bounds of the reduced minimum modulus of an operator and its Drazin inverse, respectively. Using these results, we obtain a characterization of the continuity of Drazin inverses of operators on a Hilbert space.

1. Introduction

Let \( H \) be a complex Hilbert space. Denote by \( \mathcal{B}(H) \) the Banach space of all bounded linear operators on \( H \). For an operator \( T \in \mathcal{B}(H) \), the symbols \( N(T) \) and \( R(T) \) will denote the null space and the range space of \( T \), respectively. The spectrum of \( T \) is denoted by \( \sigma(T) \). Let \( S_M = \{ x \in M : \|x\| = 1 \} \) if \( M \) is a subspace of \( H \) and let \( \overline{K} \) be the closure of a subset \( K \) of \( H \). For \( T \in \mathcal{B}(H) \), if there exists an operator \( T^D \in \mathcal{B}(H) \) satisfying the three operator equations

\[
TT^D = T^DT, \quad T^D TT^D = T^D, \quad T^{k+1}T^D = T^k,
\]

then \( T^D \) is called a Drazin inverse of \( T \). For \( T \in \mathcal{B}(H) \), if its Drazin inverse exists, then the Drazin inverse \( T^D \) of \( T \) is unique ([1]). The eigenprojection \( T^\pi = I - T^D T \) of \( T \) corresponding to the eigenvalue 0 is the uniquely determined idempotent operator with

\[
R(TT^D) = R(T^k) = R(T^D) = N(T^\pi) \quad \text{and} \quad N(TT^D) = R(T^\pi).
\]

Recall that \( \text{asc}(T) \) (\( \text{des}(T) \)), the ascent (descent) of \( T \in \mathcal{B}(H) \), is the smallest non-negative integer \( n \) such that \( N(T^n) = N(T^{n+1}) \) (\( R(T^n) = R(T^{n+1}) \)), that is, \( \text{asc}(T) := \min\{ k : N(T^k) = N(T^{k+1}) \} \) (\( \text{des}(T) := \min\{ k : R(T^k) = R(T^{k+1}) \} \)). If no such \( n \) exists, then \( \text{asc}(T) = \infty \) (\( \text{des}(T) = \infty \)). It is well known that \( \text{des}(T) = \text{asc}(T) \) if \( \text{asc}(T) \) and \( \text{des}(T) \) are finite ([9], [12]). An operator \( T \in \mathcal{B}(H) \) has the Drazin inverse \( T^D \) if and only if \( T \) has finite ascent and descent, which is equivalent with the fact that 0 is a finite order pole of the resolvent operator \( R_\Lambda(T) = (\Lambda - T)^{-1} \), say of order \( k \). In such a case, \( \text{ind}(T) = \text{asc}(T) = \text{des}(T) = k \), and 0 is not the accumulated point of \( \sigma(T) \) ([3]). If \( \dim H \), the dimension of \( H \), is finite,
then, for \( T \in \mathcal{B}(\mathcal{H}) \), it is clear that \( \text{asc}(T) \) and \( \text{des}(T) \) are finite. So all of the operators in \( \mathcal{B}(\mathcal{H}) \) are Drazin invertible when \( \dim \mathcal{H} \) is finite. But if \( \dim \mathcal{H} \) is infinite, then there exist operators which are not Drazin invertible. For example, let \( S \) be a unilateral shift in \( \mathcal{B}(\mathcal{H}) \). Since \( \mathcal{R}(S^{k+1}) \subseteq \mathcal{R}(S^k) \) and \( \mathcal{R}(S^{k+1}) \neq \mathcal{R}(S^k) \) for each positive integer \( n \), \( \text{des}(S) = \infty \). Thus \( S \) is not Drazin invertible. Some details about Drazin inverses of operators appear in [1], [2], [4], [6], [7], [10]-[12] and [14]-[15].

The reduced minimum modulus \( \gamma(T) \) of \( T \) (see [8]) is defined by

\[
\gamma(T) = \inf \{ \|Tx\| : \text{dist}(x,\mathcal{N}(T)) = 1 \}.
\]

Thus, from the definition of \( \gamma(T) \), we deduce that \( \|Tx\| \geq \gamma(T) \text{dist}(x,\mathcal{N}(T)) \) for any \( x \in \mathcal{H} \). It is well known that \( \mathcal{R}(T) \) is closed if and only if \( \gamma(T) > 0 \). In this case, \( \gamma(T) = \|T^+\|^{-1} \), where \( T^+ \) is the Moore-Penrose inverse of \( T \), is the unique solution of the four operator equations

\[
TXT = T, \quad XTX = X, \quad TX = (TX)^*, \quad XT = (XT)^*
\]

if there exists the Moore-Penrose inverse of \( T \).

Let \( \mathcal{M} \) and \( \mathcal{N} \) be two closed linear subspaces of a Hilbert space \( \mathcal{H} \). The gap between \( \mathcal{M} \) and \( \mathcal{N} \) (see [5], [8]) is defined by

\[
\text{gap}(\mathcal{M},\mathcal{N}) = \max\{\delta(\mathcal{M},\mathcal{N}),\delta(\mathcal{N},\mathcal{M})\},
\]

where \( \delta(\mathcal{M},\mathcal{N}) = \sup\{\text{dist}(x,\mathcal{N}) : x \in \mathcal{M}, \|x\| \leq 1\} \).

In this note, we study the reduced minimum modulus \( \gamma(T) \) of an operator \( T \in \mathcal{B}(\mathcal{H}) \) and the reduced minimum modulus \( \gamma(T^D) \) of its Drazin inverse \( T^D \), and give out their lower and upper bounds. Simultaneously, we obtain a characterization of the continuity of Drazin inverses.

2. Main results and proofs

As a motivation of this paper, we observe that there are many differences among the inverse, the Moore-Penrose inverse and the Drazin inverse of operators. In this section, we begin with some examples to show these differences. Let \( T \in \mathcal{B}(\mathcal{H}) \). It is known that \( T \) has the Moore-Penrose inverse \( T^+ \) if and only if \( \mathcal{R}(T) \) is closed (that is, \( \gamma(T) > 0 \)). But the operator \( T \) is not necessary to have the closed range if its Drazin inverse \( T^D \) exists. The following example can illustrate it.

Example 1. Let a linear operator \( T \) on the Hilbert space \( \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \) with \( \dim \mathcal{H} = \infty \) be given by the \( 3 \times 3 \) operator matrices,

\[
T = \begin{pmatrix}
A & 0 & 0 \\
0 & 0 & S \\
0 & 0 & 0
\end{pmatrix},
\]

where \( A \in \mathcal{B}(\mathcal{H}) \) is invertible and \( S \in \mathcal{B}(\mathcal{H}) \) is an arbitrary operator with \( \mathcal{R}(S) \) not being closed. As we know, there exist many operators in \( \mathcal{B}(\mathcal{H}) \) with non-closed ranges. It is easy to verify that \( T^D \) exists and

\[
T^D = \begin{pmatrix}
A^{-1} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

But \( \mathcal{R}(T) = \mathcal{H} \oplus \mathcal{R}(S) \oplus 0 \) is not closed.
If $T \in \mathcal{B}(\mathcal{H})$ is invertible and $T_n \to T$ in norm as $n \to \infty$, then $T_n^{-1}$ exists for large enough $n$ and $T_n^{-1} \to T^{-1}$ as $n \to \infty$. The analogue is not available for the Drazin inverse.

**Example 2.** Let $A \in \mathcal{B}(\mathcal{H})$ be an invertible operator and let $S$ be the unilateral shift operator defined by $Se_l = e_{l+1}$, where $\{e_l, l = 1, 2, 3, \ldots\}$ is an orthonormal basis for $\mathcal{H}$. Let $T_n$ and $T$ on the Hilbert space $\mathcal{H} \oplus \mathcal{H}$ be given by the $2 \times 2$ operator matrices

$$T_n = \begin{pmatrix} A & 0 \\ 0 & \frac{1}{n} S \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$  

Then $T_n \to T$ in norm as $n \to \infty$, $T^D$ exists and

$$T^D = \begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$  

But, from $\mathcal{R}(T_n) = \mathcal{R}(A) \oplus \mathcal{R}(S)$, $\mathcal{R}(T^D_n) = \mathcal{R}(A^k) \oplus \mathcal{R}(S^k) = \mathcal{R}(A) \oplus \mathcal{R}(S^k)$, $\mathcal{R}(S^{k+1}) \subseteq \mathcal{R}(S^k)$ and $\mathcal{R}(S^{k+1}) \neq \mathcal{R}(S^k)$, we know $\mathcal{R}(T^D_n) \subseteq \mathcal{R}(T^D_{n-1})$ and $\mathcal{R}(T^D_n) \neq \mathcal{R}(T^D_{n-1})$ for all $n = 1, 2, 3, \ldots$, so des$(S) = \infty$. Hence $T^D_n$ does not exist for all $n = 1, 2, 3, \ldots$.

Although $T^D_n$ and $T^D$ exist for all $n$, $T_n \to T$ as $n \to \infty$ does not imply that $T^D_n \to T^D$ as $n \to \infty$.

**Example 3.** Let us consider the operator matrices

$$T_n = \begin{pmatrix} \frac{1}{n} I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix}$$

on the Hilbert space $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$. Then $T_n$ and $T$ are not invertible and $T_n \to T$ in norm for large enough $n$, $T^D_n$ and $T^D$ exist and

$$T^D_n = \begin{pmatrix} nI & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T^D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

But $T^D_n$ does not converge to $T^D$ as $n \to \infty$.

Next, we will give some lemmas which will be useful later.

**Lemma 1.** For an operator $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{R}(T)$ is closed if and only if there exists $X \in \mathcal{B}(\mathcal{H})$ such that $\operatorname{TXT} = T$.

**Lemma 2.** Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse $T^D$ and $\operatorname{ind}(T) = k$. Then $\mathcal{R}(T^k)$ is closed and $\mathcal{R}(T^{D}) = \mathcal{R}(T^{k})$.

**Proof.** From $T^D T T^D = T^D$ and $T T^D = T^D T$, we have $T^D = T^k (T^D)^{k+1}$. So $\mathcal{R}(T^D) \subseteq \mathcal{R}(T^k)$. On the other hand, from $T T^D = T^D T$ and $T^{k+1} T^D = T^k$, we can get $T^k = T^D T^{k+1}$ and $\mathcal{R}(T^k) \subseteq \mathcal{R}(T^D)$. Therefore, $\mathcal{R}(T^D) = \mathcal{R}(T^k)$.

Since $T^D$ is the Drazin inverse of $T$, $T^D T T^D = T^D$. By Lemma 1, $\mathcal{R}(T^D)$ is closed. \hfill $\Box$

The following result gives a characterization of the spectral radius $r_\sigma(T^D)$ of the Drazin inverse $T^D$ of $T$.

**Lemma 3.** Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse $T^D$ and $\operatorname{ind}(T) = k$. Then $\operatorname{dist}(0, \sigma(T) \setminus \{0\}) = (r_\sigma(T^D))^{-1}$, where $r_\sigma(A)$ denotes the spectral radius of $A$. 

Proof. Since $T$ has the Drazin inverse $T^D$, then $\mathcal{R}(T_k) = \mathcal{R}(T^D)$ is closed. From $TT^k = T^{k+2}T^D = T^{k+2}T^D$, $\mathcal{R}(T_k)$ is an invariant subspace of $T$. So $T$ and $T^D$ have the operator matrices

\begin{equation}
T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \quad \text{and} \quad T^D = \begin{pmatrix} T_{11} & \sum_{i=0}^{k-1} T_{11}^{-1} T_{12} T_{22}^{k-1-i} \\ 0 & 0 \end{pmatrix} \tag{3}
\end{equation}

with respect to the space decomposition $\mathcal{H} = \mathcal{R}(T_k) \oplus \mathcal{R}(T_k)^\perp$, where $T_{11}$ is invertible and $T_{22}$ is $k$-nilpotent (see [6], theorem 2.5). So $\sigma(T) = \sigma(T_{11}) \cup \{0\}$ and $\sigma(T^D) = \sigma(T_{11}^{-1}) \cup \{0\}$, since $\sigma(T_{22}) = \{0\}$ is a singleton. Hence

\[ \sigma(T^D) = \{ \lambda^+ : \lambda \in \sigma(T) \}, \]

where

\[ \lambda^+ = \begin{cases} \lambda^{-1} & \text{if } \lambda \neq 0, \\ 0 & \text{if } \lambda = 0. \end{cases} \]

By the definition of $\text{dist}(x, \mathcal{M})$, it is easy to get

\[ \text{dist}(0, \sigma(T) \setminus \{0\}) = (\tau_{\sigma}(T^D))^{-1}. \]

For the reduced minimum modulus $\gamma(T^D)$ of $T^D$, we have the following estimation.

**Theorem 4.** Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse $T^D \neq 0$ and $\text{ind}(T) = k$. Then

\[ \frac{1}{\|T\|} \leq \gamma(T^D) \leq \frac{\|T^D T\|^2}{\|T^D T^D\|^2}. \]

Proof. For any $x \in \mathcal{H}, y \in \mathcal{N}(T^D)$, since $TT^D = T^D T$, we have

\[ \|T^D T(x)\| = \|T^D T(x - y)\| \leq \|T^D T\| \|x - y\| \]

and

\[ \text{dist}(x, \mathcal{N}(T^D)) \leq \|x - (I - T^D T)x\| = \|T^D T x\|. \]

Thus

\[ \|T\| \|T^D x\| \geq \|T^D T x\| \geq \text{dist}(x, \mathcal{N}(T^D)) \geq \gamma(T^D) \frac{\|T^D T x\|}{\|T^D T\|}. \tag{4} \]

Combining the definition of $\gamma(T)$ with (4), we have $\gamma(T^D) \geq \|T\|^{-1}$ and

\[ \|T^D x\| \geq \gamma(T^D) \text{dist}(x, \mathcal{N}(T^D)) \geq \gamma(T^D) \frac{\|T^D T x\|}{\|T^D T\|}. \tag{5} \]

Replacing $x$ by $T z, \forall z \in \mathcal{H}$ in (5), we have $\|T^D T z\| \geq \gamma(T^D) \frac{\|T^D T^2 z\|}{\|T^D T\|^2}$, that is, $\gamma(T^D) \leq \frac{\|T^D T^2\|^2}{\|T^D T\|^2}$. \hfill \square

By the definition of the gap between two subspaces, we get the following results.

**Theorem 5.** Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse $T^D$ and $\text{ind}(T) = k$. If $\mathcal{N}(T) \neq \mathcal{N}(T^D)$, then

\[ \delta(\overline{\mathcal{R}(T)}, \mathcal{R}(T^D)) = 1, \delta(\mathcal{R}(T^D), \overline{\mathcal{R}(T)}) = 0 \tag{6} \]

and

\[ \delta(\mathcal{N}(T^D), \mathcal{N}(T)) = 1, \delta(\mathcal{N}(T), \mathcal{N}(T^D)) = 0. \tag{7} \]
Proof. Since \( T \) has the Drazin inverse \( T^D \) and \( \text{ind}(T) = k \), we have

\[
R(T^D) = R(T^k) \subseteq R(T) \subseteq R(T).
\]

Since \( T^D = (T^D)^2 T \) and \( T^k = T^{k+1} T^D \),

(8) \[
N(T) \subseteq N(T^D) \subseteq N(T^k).
\]

If \( N(T) \neq N(T^D) \), then \( \text{ind}(T) = k > 1 \) and \( R(T^D) \neq R(T) \). Since \( R(T^D) \subseteq R(T) \), by definition of the gap function \( \delta \) between two subspaces, we have

\[
\delta(R(T^D), R(T)) = 0.
\]

Since \( R(T^D) \neq R(T) \), there exists \( x \in \overline{R(T)} \cap R(T^D)^\perp \) with \( \|x\| = 1 \) such that

\[
\text{dist}(x, R(T^D))^2 = \|x\|^2 - \|P_{R(T^D)}x\|^2 = 1.
\]

Hence

\[
\delta(\overline{R(T)}, R(T^D)) = 1.
\]

By the similar argument, we can obtain formula (7). Hence the proof of this part is omitted. \( \square \)

Remark. (1) Theorem 3 gives us an example to show that, for two closed subspaces \( \mathcal{M} \) and \( \mathcal{N} \), \( \delta(\mathcal{M}, \mathcal{N}) \neq 0 \) is not symmetric, that is, in general \( \delta(\mathcal{M}, \mathcal{N}) \neq \delta(\mathcal{N}, \mathcal{M}) \).

But \( \delta(\mathcal{M}, \mathcal{N}) = \delta(\mathcal{N}^\perp, \mathcal{M}^\perp) \) since \( \text{dist}(x, \mathcal{M}) = \sup_{y \in S_{\mathcal{M}^\perp}} \langle x, y \rangle \) (see \( \mathcal{S} \)). In fact,

\[
\delta(\mathcal{M}, \mathcal{N}) = \sup_{x \in S_{\mathcal{M}^\perp}} \text{dist}(x, \mathcal{N}) = \sup_{x \in S_{\mathcal{M}^\perp}} \text{dist}(x, \mathcal{M}) = \sup_{y \in S_{\mathcal{N}^\perp}} \text{dist}(y, \mathcal{M}^\perp) = \delta(\mathcal{N}^\perp, \mathcal{M}^\perp).
\]

(2) Let \( T \in \mathcal{B}(\mathcal{H}) \) with the Drazin inverse \( T^D \) and \( \text{ind}(T) = k \). If \( T \) and \( T^D \) have the operator matrices (3), respectively, and \( x_0 + x \in \mathcal{N}(T) \), where \( x_0 \in R(T^k) \) and \( x \in R(T^k)^\perp \), then

\[
\begin{pmatrix}
T_{11} & T_{12} \\
0 & T_{22}
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x
\end{pmatrix} = 0.
\]

So we have

\[
\mathcal{N}(T) = \left\{ \begin{pmatrix}
-T_{11}^{-1}T_{12}x \\
x
\end{pmatrix} : x \in \mathcal{N}(T_{22}) \right\}.
\]

Similarly, if \( y_0 + y \in \mathcal{N}(T^D) \), where \( y_0 \in R(T^k) \) and \( y \in R(T^k)^\perp \), then

\[
\begin{pmatrix}
T_{11}^{-1} & \sum_{i=0}^{k-1} T_{11}^{-i}T_{12}T_{22}^{-k-1-i} \\
0 & T_{22}^{-k-1-i}
\end{pmatrix}
\begin{pmatrix}
y_0 \\
y
\end{pmatrix} = 0.
\]

So we have

\[
\mathcal{N}(T^D) = \left\{ \begin{pmatrix}
-T_{11}^{-1}T_{12}^{-k}T_{22}^{-k-1-i}y \\
y
\end{pmatrix} : y \in R(T^k)^\perp \right\}.
\]

In general, \( \mathcal{N}(T_{22}) \subseteq \mathcal{N}(T^k)^\perp \); this shows that \( \mathcal{N}(T) \subseteq \mathcal{N}(T^D) \). Moreover, \( \mathcal{N}(T) = \mathcal{N}(T^D) \) if and only if \( \mathcal{N}(T_{22}) = \mathcal{N}(T^k)^\perp \). In this case, \( \text{ind}(T) = 1 \), that is, \( \mathcal{N}(T_{22}) = \mathcal{N}(T)^\perp \).

Theorem 6. Let \( T \in \mathcal{B}(\mathcal{H}) \) with the Drazin inverse \( T^D \) and \( \text{ind}(T) = k \). If \( \mathcal{N}(T) \neq \mathcal{N}(T^D) \), then

\[
\gamma(T) \leq \|T - T^D\|.
\]
Proof. If $\mathcal{R}(T)$ is not closed, then $\gamma(T) = 0$. It is clear that $\gamma(T) \leq \|T - T^D\|$.
Next, we assume that $\mathcal{R}(T)$ is closed.
For any $z \in \mathcal{N}(T)$ and $u \in \mathcal{R}(T)$ with $\|u\| = 1$, there exists $x \in \mathcal{H}$ such that $Tx = u$ and
$$\text{dist}(u, \mathcal{R}(T^D)) \leq \|Tx - T^D(x - z)\| = \|T(x - z) - T^D(x - z)\| \leq \|T - T^D\|\|x - z\|.$$ 
So $\delta(\mathcal{R}(T), \mathcal{R}(T^D)) \leq \|T - T^D\|\|x - z\|$. Since $\|Tx\| \geq \gamma(T)\text{dist}(x, \mathcal{N}(T))$ and $\mathcal{R}(T)$ is closed,
$$\text{dist}(x, \mathcal{N}(T)) \leq \frac{\|Tx\|}{\gamma(T)} = \frac{1}{\gamma(T)}.$$
That is, $1 = \delta(\mathcal{R}(T), \mathcal{R}(T^D)) \leq \|T - T^D\|\frac{1}{\gamma(T)}$ by Theorem 5. The proof is completed. \hfill \Box

The following theorem gives a relation of the reduced minimum modulus of the operators $T$ and $T^D$.

**Theorem 7.** Let $T \in B(\mathcal{H})$ with the Drazin inverse $T^D$ and $\text{ind}(T) = k$. Then
$$\frac{1}{\|T^k + 1\|} (\gamma(T) - \|T^k - T\|) \leq \gamma(T^D) \leq \|(T^D)^{k+1}\|\gamma(T^k).$$

**Proof.** From $T^DT^D = T^D$ and $TT^D = T^DT$, we have $T^D = (T^D)^{k+1}T^k$. Equation (8) shows that $\mathcal{N}(T)^\bot \supseteq \mathcal{N}(T^D)^\bot \supseteq \mathcal{N}(T^k)^\bot$. Therefore, by the definition of $\gamma(T^D)$, we have
$$\gamma(T^D) = \inf \{\|T^Dx\| : \text{dist}(x, \mathcal{N}(T^D)) = 1\} \leq \inf \{\|T^Dx\| : x \in S_{\mathcal{N}(T^D)^\bot}\} \leq \inf \{\|T^Dx\| : x \in S_{\mathcal{N}(T^k)^\bot}\} \leq \inf \{\|T^Dx\| : x \in S_{\mathcal{N}(T^k)^\bot}\} \leq \|T^D\|\gamma(T^k).$$

On the other hand, observing that $T^k = T^k + T^D$, we get
$$\gamma(T^D) \geq \inf \{\|T^Dx\| : x \in S_{\mathcal{N}(T^D)^\bot}\} \geq \inf \{\|T^kx\| : x \in S_{\mathcal{N}(T^D)^\bot}\} \geq \inf \{\|T^kx\| : x \in S_{\mathcal{N}(T)^\bot}\} \geq \inf \{\|T^kx\| : x \in S_{\mathcal{N}(T)^\bot}\} \geq \|T^k\|\gamma(T) - \|T^k - T\| \geq \|T^k\|\gamma(T) - \|T^k - T\|.$$ 
Hence
$$\frac{1}{\|T^k + 1\|} (\gamma(T) - \|T^k - T\|) \leq \gamma(T^D) \leq \|(T^D)^{k+1}\|\gamma(T^k).$$ \hfill \Box

Combining the proceeding theorem with Theorem 4, we obtain the following result.

**Corollary 8.** Let $T \in B(\mathcal{H})$ with the Drazin inverse $T^D \neq 0$ and $\text{ind}(T) = k$. Then
$$\gamma(T^k) \geq \frac{1}{\|T\|\|(T^D)^{k+1}\|}. $$
Note that $\mathcal{N}(T^D) = \mathcal{N}(T^D TT^D) \supseteq \mathcal{N}(TT^D) \supseteq \mathcal{N}(T^D)$. So $\mathcal{N}(T^D) = \mathcal{N}(TT^D)$ and $\gamma(T^D) = \inf \{ \|TT^D x\| : x \in S_{\mathcal{N}(TT^D)^\perp} \} \leq \|T\| \inf \{\|T^D x\| : x \in S_{\mathcal{N}(T^D)^\perp} \} = \|T\| \gamma(T^D)$. Thus we have the following corollary.

**Corollary 9.** Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse $T^D$ and $\text{ind}(T) = k$. Then

$$\gamma(T^D) \|T\|^{-1} \leq \gamma(T^D) \leq \|T^D\|^{k+1} \|T^k\|^{-1}.$$ 

Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. If $P_\mathcal{M}$ and $P_\mathcal{N}$ are orthogonal projections onto $\mathcal{M}$ and $\mathcal{N}$, respectively, we have the following lemma for the gap between the two subspaces $\mathcal{M}$ and $\mathcal{N}$.

**Lemma 10** ([5], [8]). Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. If $P_\mathcal{M}$ and $P_\mathcal{N}$ are orthogonal projections onto $\mathcal{M}$ and $\mathcal{N}$, respectively, then

$$\text{gap}(\mathcal{M}, \mathcal{N}) = \max \{ \|P_\mathcal{M}(I - P_\mathcal{N})\|, \|P_\mathcal{N}(I - P_\mathcal{M})\| \} = \text{gap}(\mathcal{M}^\perp, \mathcal{N}^\perp),$$

(9) $\| (I - P_\mathcal{N}) P_\mathcal{M} \| \leq \| I - P_\mathcal{N} \| \|P_\mathcal{M}\| \text{gap}(\mathcal{R}(P_\mathcal{M}), \mathcal{R}(P_\mathcal{N}))$

and

(10) $\| P_\mathcal{N}(I - P_\mathcal{M}) \| \leq \| P_\mathcal{N} \| \|I - P_\mathcal{M}\| \text{gap}(\mathcal{N}(P_\mathcal{M}), \mathcal{N}(P_\mathcal{N})).$

**Proof.** (1) See corollary 7 in [5].

(2) Let $y \in \mathcal{N}$. Then, for any $x \in \mathcal{H}$, $(I - P_\mathcal{N})P_\mathcal{M}x = (I - P_\mathcal{N})(P_\mathcal{M}x - y)$, which implies

$$\|(I - P_\mathcal{N})P_\mathcal{M}x\| \leq \|(I - P_\mathcal{N})\| \text{dist}(P_\mathcal{M}x, \mathcal{N}) \leq \|(I - P_\mathcal{N})\| \text{gap}(\mathcal{R}(P_\mathcal{M}), \mathcal{R}(P_\mathcal{N})) \|P_\mathcal{M}x\| \leq \|(I - P_\mathcal{N})\| \|P_\mathcal{M}\| \text{gap}(\mathcal{R}(P_\mathcal{M}), \mathcal{R}(P_\mathcal{N})) \|x\|.$$ 

Hence formula (9) holds.

Similarly, noting that $\mathcal{N}(P_\mathcal{M}) = \mathcal{R}(I - P_\mathcal{M}), \mathcal{N}(P_\mathcal{N}) = \mathcal{R}(I - P_\mathcal{N})$, we can show

$$\| P_\mathcal{N}(I - P_\mathcal{M}) \| \leq \| P_\mathcal{N} \| \|I - P_\mathcal{M}\| \text{gap}(\mathcal{N}(P_\mathcal{M}), \mathcal{N}(P_\mathcal{N})).$$

**Remark.** From the above proof, the inequalities (9) and (10) are also true when $P_\mathcal{M}$ and $P_\mathcal{N}$ are projections on $\mathcal{M}$ and $\mathcal{N}$, respectively, but not orthogonal projections.

To discuss the continuity of Drazin inverse in the rest of this paper, we need a lemma (see [5], I, Theorem 6.35).

**Lemma 11** ([8]). Let $P', Q'$ be two idempotents on $\mathcal{H}$ and let $\mathcal{M} = \mathcal{R}(P'), \mathcal{N} = \mathcal{R}(Q')$. Let $P, Q$ be the orthogonal projections on $\mathcal{M}, \mathcal{N}$, respectively. Then

$$\|P - Q\| \leq \|P' - Q'\|.$$ 

In the corollary 1 of [11], Y. Wei and G. Chen give sufficient and necessary conditions for the continuity of Moore-Penrose inverses in a Hilbert space. With the application of the characterization of gap between two subspaces, we give a similar theorem for Drazin inverses.

**Theorem 12.** Let $T \in \mathcal{B}(\mathcal{H})$ with the Drazin inverse $T^D$, $T_n \to T$ as $n \to \infty$, and let $T_n \in \mathcal{B}(\mathcal{H})$ with Drazin inverse $T_n^D$ as $n$ large enough. Then the following statements are equivalent:

1. $T_n^D \to T^D$ as $n \to \infty$. 

(2) \( \text{gap}(\mathcal{R}(T_n^\pi), \mathcal{R}(T^\pi)) \to 0 \) and \( \text{gap}(\mathcal{N}(T_n^\pi), \mathcal{N}(T^\pi)) \to 0 \) as \( n \) large enough, where \( T_n^D = I - T_n^D T_n \) and \( T^\pi = I - T^D T \).

Proof. (1) \( \Rightarrow (2) \) Let \( O(T_n^\pi) \) and \( O(T^\pi) \) be the orthogonal projections of \( \mathcal{H} \) onto \( \mathcal{R}(T_n^\pi) \) and \( \mathcal{R}(T^\pi) \), respectively. By Lemma 11, we have

\[
\| O(T_n^\pi) - O(T^\pi) \| \leq \| T_n^\pi - T^\pi \|.
\]

Since \( T, T_n, T_n^D \in \mathcal{B}(\mathcal{H}), T_n \to T \) and \( T_n^D \to T^D \) as \( n \to \infty \), we have

\[
\| T_n^\pi - T^\pi \| = \| T_n^D T_n - T^D T \| \leq \| T_n \| \| T_n^D - T^D \| + \| T^D \| \| T_n - T \| \to 0
\]
as \( n \to \infty \). Hence \( O(T_n^\pi) \to O(T^\pi) \) as \( n \to \infty \). From Lemma 10, we get

\[
\text{gap}(\mathcal{R}(T_n^\pi), \mathcal{R}(T^\pi)) = \max \{\| O(T_n^\pi) (I - O(T^\pi)) \|, \| O(T^\pi) (I - O(T_n^\pi)) \|\} \to 0
\]
as \( n \to \infty \). Since \( O(T_n^\pi) \to O(T^\pi) \) as \( n \to \infty \),

\[
\text{gap}(\mathcal{N}(T_n^\pi), \mathcal{N}(T^\pi)) = \text{gap}(\mathcal{R}(T_n^\pi)^\perp, \mathcal{R}(T^\pi)^\perp) = \text{gap}(\mathcal{R}(T_n^\pi^*), \mathcal{R}(T^\pi^*)) \to 0
\]
as \( n \to \infty \).

(2) \( \Rightarrow (1) \) Since

\[
\| (I - T^\pi) T_n^\pi \| \leq \| I - T^\pi \| \| T_n^\pi \| \| \text{gap}(\mathcal{R}(T_n^\pi), \mathcal{R}(T^\pi)) \| \to 0
\]
as \( n \to \infty \) and

\[
\| T^\pi (I - T_n^\pi) \| \leq \| T^\pi \| \| I - T_n^\pi \| \| \text{gap}(\mathcal{N}(T_n^\pi), \mathcal{N}(T^\pi)) \| \to 0
\]
as \( n \to \infty \), we have

\[
\| T_n^D T_n - T^D T \| = \| T_n^\pi - T^\pi \| \leq \| T_n^\pi - T^\pi T_n^\pi \| + \| T^\pi T_n^\pi - T^\pi \| \to 0
\]
as \( n \to \infty \). Therefore,

\[
T_n^D - T^D = T_n^D T_n T - T_n^D (I - T_n^D T_n) T^D - T_n^D T_n T^D = T_n^D (T - T_n^- T_n^D) + T_n^D (I - T_n^D T_n) T^D - (I - T_n^D T_n^D) T^D
\]

\[
= T_n^D (T - T_n^- T_n^D) + T_n^D (T_n^D T_n - T^D) - (T^D T - T_n^D T_n^D) T^D
\]

\[
\to 0, \text{ as } n \to \infty
\]

Thus \( T_n^D \to T^D \) in \( \mathcal{B}(\mathcal{H}) \). \( \square \)

Remark. (1) Example 2 shows that, in general, it does not imply that \( T_n \) is Drazin invertible as \( n \) large enough if \( T_n \to T \) as \( n \to \infty \) and \( T \) is Drazin invertible. So in (1) of Theorem 12 the condition that \( T_n \) has the Drazin inverse \( T_n^D \) for \( n \) large enough is necessary.

(2) Theorem 12 shows there is a close relation between the continuity of the Drazin inverse and the gaps of the ranges and the gaps of the null-spaces of the eigenprojections \( T_n^\pi \) and \( T^\pi \) of \( T_n \) and \( T \) corresponding to the eigenvalue 0, respectively.

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