

## NONABELIAN FREE SUBGROUPS IN HOMOMORPHIC IMAGES OF VALUED QUATERNION DIVISION ALGEBRAS

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(Communicated by Jonathan I. Hall)

ABSTRACT. Given a quaternion division algebra  $D$ , a noncentral element  $e \in D^\times$  is called *pure* if its square belongs to the center. A theorem of Rowen and Segev (2004) asserts that for any quaternion division algebra  $D$  of positive characteristic  $> 2$  and any pure element  $e \in D^\times$  the quotient  $D^\times/X(e)$  of  $D^\times$  by the normal subgroup  $X(e)$  generated by  $e$ , is abelian-by-nilpotent-by-abelian. In this note we construct a quaternion division algebra  $D$  of characteristic zero containing a pure element  $e \in D$  such that  $D^\times/X(e)$  contains a nonabelian free group. This demonstrates that the situation in characteristic zero is very different.

### 1. INTRODUCTION

Let  $D$  be a quaternion division algebra with center  $K$ . An element  $e \in D \setminus K$  is called *pure* if  $e^2 \in K$ . Given an element  $a \in D \setminus K$ , we denote by

$$X(a) \text{ and } Y(a)$$

the normal subgroups of  $D^\times$  generated by  $\langle a \rangle$  and  $SL_1(K(a))$  respectively, where as usual  $SL_1(K(a)) = K(a) \cap SL_1(D)$  and  $SL_1(D)$  is the subgroup of elements having reduced norm 1.

In [7] quotients of the form  $D^\times/X(e)$  were considered for pure elements  $e \in D^\times$ . These quotients arise in the analysis of the Whitehead group  $W(G, k)$  of an absolutely simple simply connected algebraic  $k$ -group  $G$  of type  ${}^3,6D_4$  having  $k$ -rank 1 (see Tits's Bourbaki talk [11] for the relevant terminology). It was shown that if  $D$  has positive characteristic  $> 2$ , then  $D^\times/X(e)$  is abelian-by-nilpotent-by-abelian for any pure element  $e \in D^\times$ , which, by the explicit description of  $W(G, k)$  given in [5], implies the solvability of  $W(G, k)$  for  $G$  as above over a field  $k$  of characteristic  $> 2$ . Even though the solvability of  $W(G, k)$  is expected to hold in any characteristic (see [6] for a general conjecture), the results of this note indicate

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Received by the editors March 3, 2005 and, in revised form, May 14, 2005.

2000 *Mathematics Subject Classification*. Primary 16K20, 16U60; Secondary 20G15, 12J20.

*Key words and phrases*. Quaternion division algebra, multiplicative group, valuation, residue algebra.

The first author was partially supported by BSF grant 2000-171, and by NSF grants DMS-0138315 and DMS-0502120.

The second author was partially supported by the Israel Science Foundation Center of Excellence.

The third author was partially supported by BSF grant 2000-171.

that a more delicate analysis will be needed to prove this in characteristic zero. More precisely, we will prove the following.

**Theorem 1.1.** *There exists a quaternion division algebra  $D$  of characteristic zero and pure quaternions  $e, f \in D$  such that the quotients  $D^\times/X(e)$  and  $D^\times/Y(f)$  contain nonabelian free groups. (In particular these quotients are nonsolvable.)*

In §3 we construct a quaternion division algebra  $D$  that supports a valuation  $v$  such that the residue algebra  $\bar{D}_v$  is non-commutative of characteristic two over the residue field  $\bar{K}_v$ . This algebra  $D$  has the properties asserted in Theorem 1.1 by the following more general result.

**Theorem 1.2.** *Let  $D$  be a quaternion division algebra of characteristic zero that supports a valuation  $v$  for which the residue algebra  $\bar{D}_v$  is non-commutative and has characteristic two. Then  $D$  contains pure quaternions  $e$  and  $f$  such that the quotients  $D^\times/X(e)$  and  $D^\times/Y(f)$  have homomorphic images containing  $\bar{D}_v^\times/\bar{K}_v^\times$ , and consequently  $D^\times/X(e)$  and  $D^\times/Y(f)$  contain nonabelian free groups.*

The proof of Theorem 1.2 proceeds as follows. Let  $D$  be a quaternion division algebra as in Theorem 1.2 and let  $e, f \in D$  be pure quaternions such that  $ef = -fe$ . We first observe that  $Y(e)$  commutes both with  $X(e)$  and  $Y(f)$  modulo the congruence subgroup  $1 + \mathfrak{m}_{D,v}$ , where  $\mathfrak{m}_{D,v}$  is the valuation ideal. (This follows from a stronger result that comes from [7]; see Lemma 2.1 for a short proof). We then apply the following lemma which is an easy consequence of the Cartan-Brauer-Hua Theorem (see, e.g., [3, Theorem 3.9.2, pg. 144]).

**Lemma 1.3.** *Let  $D$  be a finite-dimensional division algebra with center  $K$  that supports a valuation  $v$  so that the residue division algebra  $\bar{D}_v$  is not commutative. Let  $\mathcal{U}_v := \{x \in D^\times \mid v(x) = 0\}$  be the group of units and  $\mathfrak{m}_{D,v} := \{x \in D^\times \mid v(x) > 0\}$  be the valuation ideal. Let  $*$ :  $D^\times \rightarrow D^\times/K^\times(1 + \mathfrak{m}_{D,v})$  be the canonical homomorphism. Suppose  $N, M$  are normal subgroups of  $D^\times$  such that  $[N, M]^* = 1^*$ . Then*

- (1)  $\mathcal{U}_v^* = \bar{D}_v^\times/\bar{K}_v^\times$ , and either  $((MK^\times) \cap \mathcal{U}_v)^* = 1^*$  or  $((NK^\times) \cap \mathcal{U}_v)^* = 1^*$ ;
- (2) for  $H \in \{M, N\}$  such that  $((HK^\times) \cap \mathcal{U}_v)^* = 1^*$  we have

$$(HK^\times \mathcal{U}_v)/(HK^\times(1 + \mathfrak{m}_{D,v})) \cong \bar{D}_v^\times/\bar{K}_v^\times.$$

In §3 we extend a well-known construction of valuations on fields of rational functions (cf. [2, Section 10.1, Proposition 2]) to finite-dimensional division algebras of non-commutative rational functions. We then apply this construction to obtain the quaternion division algebra  $D$  satisfying the hypotheses of Theorem 1.2 (see Proposition 3.4 and Corollary 3.5). The center of the resulting algebra  $D$  has transcendence degree 2 over  $\mathbb{Q}$ , so in this context we would like to mention that for a finite-dimensional division algebra  $\mathcal{D}$  over a global field, any quotient of  $SL_1(\mathcal{D})$  by a noncentral subgroup is finite and solvable (see [6]), implying that all quotients of  $\mathcal{D}^\times$  by a noncentral subgroup are solvable.

In a preliminary version of this paper (see [9] for a report on this joint result) we proved a weaker version of Theorem 1.1 using ultra-products, however now we have a stronger result that does not require the use of ultra-products. Also, since our construction is *explicit* it is possible that the algebra  $D$  we construct could be used to demonstrate further properties of the Whitehead group.

2. THE PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2. We start with a lemma that comes from [7].

**Lemma 2.1.** *Let  $D$  be a quaternion division algebra of characteristic  $\neq 2$ , let  $S$  be the (normal) subgroup of  $D^\times$  generated by the set  $1 + 2SL_1(D)$ , and let  $\bullet: D^\times \rightarrow D^\bullet := D^\times/S$  be the canonical homomorphism. Then*

- (1) *for any pure  $e \in D$ , we have  $[Y(e), X(e)]^\bullet = 1^\bullet$ ;*
- (2) *for any pure  $e, f \in D$  such that  $ef = -fe$ , we have  $[Y(e), Y(f)]^\bullet = 1^\bullet$ .*

*Proof.* First observe that

(i)  $[x^\bullet, (x + s)^\bullet] = 1^\bullet$ , for all  $x \in D^\times$  and  $s \in S$  such that  $x + s \neq 0$ ,

because  $(x + s)^\bullet = (s(s^{-1}x + 1))^\bullet = (s^{-1}x + 1)^\bullet$ , and  $(s^{-1}x + 1)^\bullet$  commutes with  $(s^{-1}x)^\bullet = x^\bullet$ . Next, using the notation  $[a, b] = a^{-1}b^{-1}ab$ ,  $a^b = b^{-1}ab$ ,

$$[g, (1 + x)^{-1}] = (1 + x^g)(1 + x)^{-1} \quad \forall x, g \in D^\times \text{ with } x \neq -1,$$

and hence

$$\begin{aligned} \left[ x^g, \frac{1-x}{1+x} \right]^\bullet &= [x^g, 1 - 2(1+x)^{-1}]^\bullet = [x^g, (1+x^g)^{-1}(1+x^g - 2[g, (1+x)^{-1}])]^\bullet \\ &= [x^g, x^g + (1 - 2[g, (1+x)^{-1}])]^\bullet = 1^\bullet, \end{aligned}$$

where the last equality follows from (i) using the fact that  $1 - 2[g, (1+x)^{-1}] \in S$ . Replacing  $x$  by  $\alpha e$ , where  $\alpha \in K^\times$  is arbitrary and observing that any element  $\neq -1$  in  $SL_1(K(e))$  has the form  $\frac{1-\alpha e}{1+\alpha e}$  (for example, by Hilbert’s Theorem 90), we get (1).

Part (2) follows from (1) because

$$[e, 1 + \alpha f] = ((1 + \alpha f)^e)^{-1}(1 + \alpha f) = (1 - \alpha f)^{-1}(1 + \alpha f), \text{ for all } \alpha \in K^\times,$$

and as we mentioned above any element  $\neq -1$  of  $SL_1(K(f))$  has the form  $(1 - \alpha f)^{-1}(1 + \alpha f)$  for some  $\alpha \in K^\times$ . Hence  $SL_1(K(f)) \leq X(e)$ , and so (2) follows from (1). □

The next lemma is a consequence of the Cartan-Brauer-Hua theorem and will be applied in the proof of Lemma 1.3.

**Lemma 2.2.** *Let  $D$  be a finite-dimensional division algebra with center  $K$ . Let  $A$  and  $B$  be two normal subgroup of  $D^\times$  such that  $[A, B] \leq K^\times$ . Then either  $A \subseteq K^\times$  or  $B \subseteq K^\times$ .*

*Proof.* First observe that

(ii) if  $[A, B] = 1$ , then either  $A \subseteq K$  or  $B \subseteq K$ .

To prove (ii) note that the  $K$ -subalgebra  $K[A]$  generated by  $A$  is a division subalgebra normalized by  $D^\times$ , so by the Cartan-Brauer-Hua Theorem (see [3, Theorem 3.9.2, pg. 144] for an easy proof), if  $A$  is noncentral, then  $K[A] = D$ , and since  $B$  centralizes  $K[A]$  it follows that  $B$  is central.

Assume now that  $B$  is noncentral. Since  $[B, A, A] = [A, B, A] = 1$ , the three subgroup lemma ([1, (8.7)]) implies that  $[[A, A], B] = 1$ . Since  $B$  is noncentral, it follows from (ii) that  $[A, A] \leq K^\times$ . But this shows that  $A$  is nilpotent. Hence  $A \leq K^\times$  since by a theorem of Scott [8],  $D^\times$  contains no noncentral normal solvable subgroups. □

*Proof of Lemma 1.3.* (1) Let  $\mathcal{O}_{D,v} := \{x \in D^\times \mid v(x) \geq 0\} \cup \{0\}$  be the valuation ring of  $v$ . Of course the canonical homomorphism  $\mathcal{O}_{D,v} \rightarrow \bar{D}_v$  restricted to  $\mathcal{U}_v$  induces a surjective group homomorphism  $\mathcal{U}_v \rightarrow \bar{D}_v^\times$ , with kernel  $1 + \mathfrak{m}_{D,v}$ , so  $\mathcal{U}_v/(1 + \mathfrak{m}_{D,v}) = \bar{D}_v^\times$  and therefore  $\mathcal{U}_v^* \cong \bar{D}_v^\times/\bar{K}_v^\times$ .

Set  $A := (MK^\times) \cap \mathcal{U}_v$  and  $B := (NK^\times) \cap \mathcal{U}_v$ . Let  $\bar{\cdot} : \mathcal{U}_v \rightarrow \mathcal{U}_v/(1 + \mathfrak{m}_{D,v}) = \bar{D}_v^\times$  be the canonical homomorphism. Then  $[\bar{A}, \bar{B}] \leq \bar{K}_v$ . Since  $\bar{D}_v$  is not commutative Lemma 2.2 implies that either  $\bar{A} \leq \bar{K}_v$  or  $\bar{B} \leq \bar{K}_v$ , that is, either  $A^* = 1^*$  or  $B^* = 1^*$ .

(2) Suppose  $A^* = 1^*$ , that is,  $(MK^\times) \cap \mathcal{U}_v \leq K^\times(1 + \mathfrak{m}_{D,v})$ . Then

$$\begin{aligned} (MK^\times \mathcal{U}_v)/(MK^\times(1 + \mathfrak{m}_{D,v})) &\cong \mathcal{U}_v/\mathcal{U}_v \cap (MK^\times(1 + \mathfrak{m}_{D,v})) \\ &\cong \mathcal{U}_v/(\mathcal{U}_v \cap MK^\times)(1 + \mathfrak{m}_{D,v}) = \mathcal{U}_v^*. \end{aligned}$$

□

We can now complete the proof of Theorem 1.2.

*Proof of Theorem 1.2.* Let  $D$  be as in Theorem 1.2, and adopt the notation of Lemma 1.3 and its proof. Let  $e, f \in D$  be pure elements such that  $ef = -fe$ . We claim that

$$(iii) \quad [Y(e), X(e)]^* = 1^* = [Y(e), Y(f)]^*.$$

Recall that since  $D$  is a quaternion division algebra,  $SL_1(D) = [D^\times, D^\times]$ . Furthermore, for any valuation  $v$  of a finite-dimensional division algebra, the value group  $\Gamma_v$  is commutative. Thus  $SL_1(D) = [D^\times, D^\times] \leq \mathcal{U}_v$ , since  $\mathcal{U}_v$  is the kernel of the valuation  $v: D^\times \rightarrow \Gamma_v$ . Also the fact that the characteristic of  $\bar{D}_v$  is two implies that  $2 \in \mathfrak{m}_{D,v}$ , so  $2SL_1(D) \subseteq 2\mathcal{U}_v \subseteq \mathfrak{m}_{D,v}$  and therefore

$$S := \langle 1 + 2SL_1(D) \rangle \leq 1 + \mathfrak{m}_{D,v}.$$

Now (iii) follows from Lemma 2.1.

Let  $b, c \in \mathcal{U}_v$  such that  $[\bar{b}, \bar{c}]$  is a noncentral element in  $\bar{D}_v$  ( $\bar{b}, \bar{c}$  exist since  $\bar{D}_v$  is not commutative and hence not solvable; see, e.g., [8]). Let  $a = [b, c]$  ( $= b^{-1}c^{-1}bc$ ) and let  $e \in K(a)$  be a pure element. By the choice of  $a$  we have  $a^* \neq 1^*$  and  $a \in SL_1(K(e)) \leq \mathcal{U}_v$ . Thus  $(Y(e) \cap \mathcal{U}_v)^* \neq 1^*$ . Now (iii) together with Lemma 1.3 imply that  $((X(e)K^\times) \cap \mathcal{U}_v)^* = 1^* = ((Y(f)K^\times) \cap \mathcal{U}_v)^*$ . Now let  $H \in \{X(e), Y(f)\}$ . Then, by Lemma 1.3(2),

$$(HK^\times \mathcal{U}_v)/(HK^\times(1 + \mathfrak{m}_{D,v})) \cong \bar{D}_v^\times/\bar{K}_v^\times,$$

so  $D^\times/HK^\times(1 + \mathfrak{m}_{D,v})$  contains a subgroup isomorphic to  $\bar{D}_v^\times/\bar{K}_v^\times$ . This completes the first part of Theorem 1.2.

For the second part note first that since  $\bar{D}_v^\times$  is non-commutative, it contains nonabelian free groups (see, e.g., [4, Theorem 2.1]); of course, the proof of this fact uses the celebrated Tits' alternative [10]. It follows that  $\bar{D}_v^\times/\bar{K}_v^\times$  contain a nonabelian free group. Hence  $D^\times/X(e)$  and  $D^\times/Y(f)$  have homomorphic images containing nonabelian free groups. But then  $D^\times/X(e)$  and  $D^\times/Y(e)$  also contain nonabelian free groups because if we pick one preimage for each free generator of a free subgroup in a homomorphic image, the resulting elements will generate a free subgroup. □

*Remarks 2.3.* Note that the proof of Theorem 1.2 actually shows that if  $D$  is a quaternion division algebra that supports a valuation  $v$  such that the residue algebra  $\bar{D}_v$  is non-commutative of characteristic two, and if  $e, f \in D$  are pure elements such

that  $ef = -fe$ , then one of  $D^\times/Y(e)$  or  $D^\times/Y(f)$  contains a nonabelian free group (and in fact one of these groups has a homomorphic image that contains a subgroup isomorphic to  $\bar{D}_v^\times/\bar{K}_v^\times$ ).

This should be compared with the fact that for any quaternion division algebra  $D$  and any pure element  $e \in D$ , the quotient of  $D^\times$  by the normal subgroup generated by  $K(e)^\times$  is an elementary abelian 2-group (i.e. all nonidentity elements have order 2). This last fact is rather easy to prove; cf. [7].

### 3. CONSTRUCTING VALUATIONS ON DIVISION ALGEBRAS OF NON-COMMUTATIVE RATIONAL FUNCTIONS

In this section we construct a quaternion division algebra whose center is an extension of  $\mathbb{Q}$  of transcendence degree 2 that supports a valuation  $v$  such that the residue algebra  $\bar{D}_v$  is non-commutative of characteristic two. In view of Theorem 1.2, this will complete the proof of Theorem 1.1. More generally, we give an explicit construction of valuations on finite-dimensional division algebras of non-commutative rational functions which enables us to control the structure of the residue algebra.

Let  $F$  be a field and let  $v: F^\times \rightarrow \Gamma$  be a non-archimedean valuation on  $F$ . Thus  $v: F^\times \rightarrow \Gamma$  is a nontrivial homomorphism from  $F^\times$  to a totally ordered commutative group  $\Gamma$  (written additively) satisfying  $v(a + b) \geq \min\{v(a), v(b)\}$ , for all  $a, b \in F^\times$ , with  $a + b \neq 0$ . Recall that if  $v(a) < v(b)$ , then it follows that  $v(a + b) = v(a)$ .

In this section we first generalize the well-known construction of a valuation on the field of rational function  $F(x)$ , extending  $v$  (cf. [2, Section 10.1, Proposition 2]), to a construction of a valuation on the division algebra of fractions  $F(x, \sigma)$ , where  $\sigma \in \text{Aut}(F)$  is an automorphism of finite order satisfying  $v(\sigma(a)) = v(a)$ , for all  $a \in F^\times$ . This construction has probably been well known to experts for some time; see for example [3].

Thus let  $\sigma \in \text{Aut}(F)$  and let  $R = F[x, \sigma]$  be the associated ring of skew polynomials in  $x$ . We recall that  $F[x, \sigma]$  consists of formal expressions  $a_0 + a_1x + \dots + a_mx^m$  with  $a_i \in F$  which are added in the obvious way and are multiplied according to the rule: if

$$a(x) = a_0 + a_1x + \dots + a_mx^m \quad \text{and} \quad b(x) = b_0 + b_1x + \dots + b_nx^n,$$

then

$$a(x)b(x) = c_0 + c_1x + \dots + c_{m+n}x^{m+n}, \quad \text{where } c_k = \sum_{i+j=k} a_i b_j^{\sigma^i}.$$

Now suppose that  $v(\sigma(a)) = v(a)$  for all  $a \in F$ . Define a function  $w: R \setminus \{0\} \rightarrow \Gamma$  as follows: given a nonzero  $a(x) = a_0 + a_1x + \dots + a_mx^m$ , we let

$$w(a(x)) = \min_{a_i \neq 0} v(a_i).$$

**Lemma 3.1.** *w is a valuation of R. In other words, for nonzero  $a(x), b(x) \in R$  we have*

- (1)  $w(a(x)b(x)) = w(a(x)) + w(b(x))$ , and
- (2)  $w(a(x) + b(x)) \geq \min\{w(a(x)), w(b(x))\}$  if  $b(x) \neq -a(x)$ .

*Proof.* The proof of (1) is identical to the usual proof of Gauss' Lemma. Namely, suppose  $w(a(x)) = \alpha$ ,  $w(b(x)) = \beta$ , and set

$$i_0 = \min\{i \mid a_i \neq 0, v(a_i) = \alpha\} \text{ and } j_0 = \min\{j \mid b_j \neq 0, v(b_j) = \beta\}.$$

Then the coefficient  $c_{i_0+j_0}$  of  $x^{i_0+j_0}$  in  $c(x) = a(x)b(x)$  is

$$c_{i_0+j_0} = \sum_{i+j=i_0+j_0} a_i b_j^{\sigma^i}.$$

If  $i+j = i_0+j_0$ ,  $(i, j) \neq (i_0, j_0)$  and  $a_i, b_j \neq 0$ , we have either  $i < i_0$  or  $j < j_0$  and then respectively either  $v(a_i) > \alpha$  (and  $v(b_j) \geq \beta$ ) or  $v(b_j) > \beta$  (and  $v(a_i) \geq \alpha$ ). In all cases,

$$v(a_i b_j^{\sigma^i}) = v(a_i) + v(b_j) > \alpha + \beta.$$

It follows that

$$v(c_{i_0+j_0}) = v(a_{i_0} b_{j_0}^{\sigma^{i_0}}) = \alpha + \beta.$$

On the other hand, for any  $k$  we have

$$v(c_k) = v\left(\sum_{i+j=k} a_i b_j^{\sigma^i}\right) \geq \min_{a_i \neq 0 \neq b_j} v(a_i b_j^{\sigma^i}) \geq \alpha + \beta,$$

and (1) follows. Property (2) is obvious.  $\square$

From now on, we will assume that  $\sigma$  has finite order  $d$ . Then the center of  $R$  is  $R_0 = F^\sigma[x^d]$ , where  $F^\sigma$  is the fixed subfield. Let  $S = R_0 \setminus \{0\}$ . Then  $S$  is a central multiplicative subset of  $R$ , so the localization  $D := R_S$  exists, and every element of  $D$  has a presentation of the form  $as^{-1}$ , where  $a \in R$  and  $s \in S$ . Note that the localization  $K := (R_0)_S$  is simply the field of fractions of  $R_0$ , and  $D$  is a finite-dimensional algebra over  $K$  without zero divisors, hence a division algebra. In fact,  $\dim_K D = d^2$ , as the elements  $a_i x^j$ ,  $i, j = 1, \dots, d$ , where  $a_1, \dots, a_d$  is a basis of  $F$  over  $F^\sigma$ , form a basis of  $D$  over  $K$ . We will denote  $D$  by  $F(x, \sigma)$ . If  $as^{-1} = bt^{-1}$ , then  $at = bs$ , and using Lemma 3.1(1) we immediately obtain that

$$w(a) - w(s) = w(b) - w(t),$$

so the equation

$$\tilde{w}(as^{-1}) = w(a) - w(s)$$

yields a well-defined function on  $D^\times$ .

**Lemma 3.2.**  $\tilde{w}$  is a valuation on  $D$ .

*Proof.* The property that  $\tilde{w}(\tilde{a}\tilde{b}) = \tilde{w}(\tilde{a}) + \tilde{w}(\tilde{b})$  for all nonzero  $\tilde{a}, \tilde{b} \in D$  immediately follows from the definition. Now, suppose we have  $\tilde{a} = as^{-1}, \tilde{b} = bt^{-1} \in D^\times$  such that  $\tilde{b} \neq -\tilde{a}$  (i.e.  $bs \neq -at$ ). By taking a common denominator we may assume that  $t = s$ . Suppose in addition that  $\tilde{w}(\tilde{a}) \leq \tilde{w}(\tilde{b})$ . Then

$$\tilde{w}(\tilde{a} + \tilde{b}) = w(a + b) - w(s) \geq w(a) - w(s) = \tilde{w}(\tilde{a}),$$

and the property  $\tilde{w}(\tilde{a} + \tilde{b}) \geq \min\{\tilde{w}(\tilde{a}), \tilde{w}(\tilde{b})\}$  follows.  $\square$

Let  $\mathcal{O}_{F,v} = \{a \in F^\times \mid v(a) \geq 0\} \cup \{0\}$  and  $\mathfrak{m}_{F,v} = \{a \in F^\times \mid v(a) > 0\} \cup \{0\}$  be the valuation ring and the valuation ideal of  $v$ . Note that

$$\sigma \text{ induces an automorphism } \bar{\sigma} \text{ on } \bar{F}_v,$$

because  $\sigma$  preserves  $v$ .

**Lemma 3.3.** *Let  $S_0 = \{s \in S \mid w(s) = 0\}$ . Then the valuation ring and the valuation ideal of  $\tilde{w}$  are as follows:*

$$(iv) \quad \mathcal{O}_{D, \tilde{w}} = \mathcal{O}_{F, v}[x, \sigma]S_0^{-1} \quad \text{and} \quad \mathfrak{m}_{D, \tilde{w}} = \mathfrak{m}_{F, v}[x, \sigma]S_0^{-1}.$$

Furthermore, the residue algebra  $\bar{D}_{\tilde{w}} = \mathcal{O}_{D, \tilde{w}}/\mathfrak{m}_{D, \tilde{w}}$  is isomorphic to  $\bar{F}_v(x, \bar{\sigma})$ .

*Proof.* Clearly,  $S = (F^\sigma)^\times S_0$ , from which it follows that any element in  $\tilde{a} \in D^\times$  has a presentation of the form  $\tilde{a} = as^{-1}$  with  $s \in S_0$ . Then  $\tilde{w}(\tilde{a}) = w(a)$ , and the descriptions in (iv) easily follow. Reducing the coefficients of polynomials in  $\mathcal{O}_{F, v}[x]$  modulo  $\mathfrak{m}_{F, v}$  defines a surjective ring homomorphism  $\varphi: \mathcal{O}_{F, v}[x, \sigma] \rightarrow \bar{F}_v[x, \bar{\sigma}]$  with  $\ker \varphi = \mathfrak{m}_{F, v}[x, \sigma]$ . Then  $\varphi$  uniquely extends to a homomorphism of localizations

$$\tilde{\varphi}: \mathcal{O}_{F, v}[x, \sigma]S_0^{-1} \rightarrow \bar{F}_v[x, \bar{\sigma}](\varphi(S_0))^{-1}$$

with  $\ker \tilde{\varphi} = \mathfrak{m}_{F, v}[x, \sigma]S_0^{-1}$ . It follows that  $\bar{D}_{\tilde{w}} \simeq \bar{F}_v[x, \bar{\sigma}](\varphi(S_0))^{-1}$ . As the left-hand side is a division ring, so is the right-hand side, from which it follows that it in fact coincides with  $\bar{F}_v(x, \bar{\sigma})$  (although in general  $\varphi(S_0)$  may be smaller than the set of nonzero elements of the center of  $\bar{F}_v[x, \bar{\sigma}]$ ).  $\square$

Now let  $k$  be a field and let  $u$  be a non-archimedean valuation on  $k$ . We use the above construction to construct a quaternion division algebra  $D$  equipped with a valuation  $\tilde{w}$  extending  $u$  such that  $\bar{D}_{\tilde{w}}$  is not commutative and has characteristic 2.

Consider the field of rational functions  $F := k(y)$  in the variable  $y$ . Extend  $u$  to a valuation of  $F$  as above, by taking in the construction above  $k$  in place of  $F$ ,  $u$  in place of  $v$ , and the identity map of  $k$  in place of  $\sigma$ . We thus obtain a valuation  $v$  on  $F$  extending  $u$ , and

$$\bar{F}_v = \bar{k}_u(y).$$

Now let  $\sigma \in \text{Aut}(F)$  be the unique automorphism (of order 2) taking  $y \rightarrow \frac{1}{y}$  and fixing  $k$  pointwise. Using the definition of  $v$  one easily checks that  $v(\sigma(r(y))) = v(r(y))$ , for any rational function  $r(y) \in k(y)$ . Let  $D = F(x, \sigma)$  be as above. Since the order of  $\sigma$  is 2,  $D$  is a quaternion division algebra. Also,  $\bar{\sigma}$  is the unique map on  $\bar{F}_v$  fixing  $\bar{k}_u$  pointwise and taking  $y$  to  $\frac{1}{y}$ . Thus  $\bar{\sigma}$  is nontrivial, and since  $\bar{D}_{\tilde{w}} \cong \bar{F}_v(x, \bar{\sigma})$ , it follows that  $\bar{D}_{\tilde{w}}$  is not commutative. Thus, we have shown

**Proposition 3.4.** *Let  $k$  be a field that supports a non-archimedean valuation  $u$ . Then there exists a quaternion division algebra  $D$  whose center has transcendence degree 2 over  $k$ , and a valuation  $v$  on  $D$  extending  $u$ , such that the residue division algebra  $\bar{D}_v$  is not commutative (and has characteristic equal to that of the residue field  $\bar{k}_u$ ).*

Taking in Proposition 3.4  $k = \mathbb{Q}$  and  $u$  the 2-adic valuation, we get the following corollary which in conjunction with Theorem 1.2 completes the proof of Theorem 1.1.

**Corollary 3.5.** *There exists a quaternion division algebra  $D$  of characteristic zero and a valuation  $v$  on  $D$  such that the residue division algebra  $\bar{D}_v$  is not commutative and has characteristic two.*

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