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A BEURLING-CARLESON SET WHICH IS A UNIQUENESS SET FOR A GIVEN WEIGHTED SPACE OF ANALYTIC FUNCTIONS

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ABSTRACT. Let $p = \big(p(n)\big)_{n\geq 0}$ be a sequence of positive real numbers. We define B_p as the space of functions f which are analytic in the unit disc $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ and such that

$$\|f\|_p := \sum_{n=0}^{+\infty} |\hat{f}(n)|\, p(n) < +\infty,$$

where $\hat{f}(n)$ is the n^{th} Fourier coefficient of the restriction of f to the unit circle $\mathbb{T}$. Let E be a closed subset of $\mathbb{T}$. We say that E is a Beurling-Carleson set if

$$\int_0^{2\pi} \log^+ \frac{1}{d(e^{it},E)} \mathrm{d}t < +\infty,$$

where $d(e^{it},E)$ denotes the distance between e^{it} and E. In 1980, A. Atzmon asked whether there exists a sequence p of positive real numbers such that $\lim\limits_{n\to+\infty} \frac{p(n)}{n^k} = +\infty$ for all $k \geq 0$ and that has the following property: for every Beurling-Carleson set E, there exists a non-zero function in B_p that vanishes on E. In this note, we give a negative answer to this question.

1. Introduction

We denote by $\mathbb{T}$ the unit circle and by $\mathbb{D}$ the unit disc. We denote by $\mathcal{A}$ the space of functions which are analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. For $n \geq 0$, we denote by $\mathcal{A}^n$ the space of functions in $\mathcal{A}$ such that $f^{(k)} \in \mathcal{A}$ for all $0 \leq k \leq n$ (we have $\mathcal{A}^0 = \mathcal{A}$). We also set $\mathcal{A}^\infty = \bigcap\limits_{n\geq 0} \mathcal{A}^n$. Let $p = \big(p(n)\big)_{n\geq 0}$ be a sequence of positive real numbers. We define B_p as the space of functions f in $\mathcal{A}$ such that

$$\|f\|_p := \sum_{n=0}^{+\infty} |\hat{f}(n)|\, p(n) < +\infty,$$

where $\hat{f}(n)$ is the n^{th} Fourier coefficient of $f_{|\mathbb{T}}$. Let E be a closed subset of $\mathbb{T}$. We say that E is a Beurling-Carleson set if

$$\text{(C)} \qquad \int_0^{2\pi} \log^+ \frac{1}{d(e^{it},E)} \mathrm{d}t < +\infty,$$

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where $d(e^{it},E)$ denotes the distance between e^{it} and E. Let X be a subspace of $\mathcal{A}$. We say that E is ZX if there exists a non-identically zero function f in X that vanishes on E. If E is not ZX, we say that E is a set of uniqueness for X (see [2]). L. Carleson proved in [2] that E satisfies condition (C) if and only if E is $Z\mathcal{A}^n$, for a given $n>0$. Then, V. S. Korolevich proved in [5] (Theorem 2) that if E satisfies condition (C), E is $Z\mathcal{A}^\infty$. Almost simultaneously, B. A. Taylor and D. L. Williams published similar results in [6]. These developments culminated in a description of closed ideals of $\mathcal{A}^\infty$ (see [6]) and a much deeper result for $\mathcal{A}^n$ $(n\geq 1)$ due to B. I. Korenblyum in [4].

We denote by Ω the set of all the sequences p of positive real numbers such that for all $k>0$, $\lim\limits_{n\to+\infty}\frac{p(n)}{n^k}=+\infty$. We remark that $\mathcal{A}^\infty=\bigcup\limits_{p\in\Omega}B_p$. Indeed, if $f\in\mathcal{A}^\infty$, then the sequence p defined by $p(n)=\frac{1}{\max\left(\left|\widehat{f}(n)\right|,e^{-n}\right)(|n|+1)^2}$ $(n\in\mathbb{Z})$ belongs to Ω and $f\in B_p$. This proves that $\mathcal{A}^\infty\subset\bigcup\limits_{p\in\Omega}B_p$, and the other inclusion is clear. A. Atzmon wrote in [1] that it would be interesting to know whether there exists a sequence p in Ω such that every Beurling-Carleson set is ZB_p. We give a negative answer to this question. More precisely, given a sequence p in Ω, we exhibit a Beurling-Carleson set E such that the only function in B_p that vanishes on E is the zero function.

2. A Beurling-Carleson set which is not ZB_p for $p\in\Omega$

Let E be a non-empty closed subset of $\mathbb{T}$ and let λ be a non-negative function on $[0,+\infty)$. We say that E is a λ-Beurling-Carleson set if

$$\int_0^{2\pi}\lambda\Big(\log^+\frac{1}{d(e^{it},E)}\Big)\mathrm{d}t<+\infty,$$

where $d(e^{it},E)$ is the distance from e^{it} to E. If $\lambda(x)=x$ $(x\geq 0)$, we recognize the Beurling-Carleson sets defined in the Introduction.

The main result of this paper follows from the proposition below. Its proof is based on an argument that was pointed out by L. Carleson in [2], and then by B. A. Taylor and D. L. Williams in [7].

Proposition 2.1. *Let λ be a non-negative function on $[0,+\infty)$ which is twice differentiable with $\lambda''>0$ on $[0,+\infty)$ and such that $\lim\limits_{x\to+\infty}\frac{\lambda(x)}{x}=+\infty$. We define a sequence p by $p(0)=1$ and $p(n)=e^{\lambda(\log n)}$ for $n\geq 1$. Let E be a closed subset of $\mathbb{T}$. We suppose that there exists a non-zero function F in B_p that vanishes with all its derivatives on E. Then E is a λ-Beurling-Carleson set.*

Proof. Let F be a non-zero function in B_p that vanishes with all its derivatives on E. Using the Taylor formula, we get, for all $n\geq 0$,

$$\big|F(z)\big|\leq\frac{d(z,E)^n}{n!}\big\|F^{(n)}\big\|_\infty\qquad(z\in\overline{\mathbb{D}}),$$

where $\|\ \|_\infty$ is the supremum norm over $\overline{\mathbb{D}}$. Now, an easy calculation shows that

$$\big\|F^{(n)}\big\|_\infty\quad\leq\quad\sup_{k\geq n}\frac{k!}{(k-n)!\,p(k)}\big\|F\big\|_p.$$

Without loss of generality, we can assume that $\|F\|_p = 1$, so that for all $n \geq 1$,

$$\begin{aligned} |F(z)| &\leq \frac{d(z,E)^n}{n!} \sup_{k\geq n} \frac{k!}{(k-n)!\,p(k)} \\ &\leq d(z,E)^n \sup_{k\geq n} \frac{k^n}{p(k)} \\ &\leq d(z,E)^n \exp\big\{ \sup_{t\geq 0} \big(nt - \lambda(t)\big)\big\} \qquad (z \in \overline{\mathbb{D}}). \end{aligned} \tag{2.1}$$

The conditions satisfied by λ imply that λ' increases strictly and that $\lim_{x\to+\infty} \lambda'(x) = +\infty$. So λ' has an inverse function λ'^{-1} defined on $[\lambda'(0), +\infty)$. It is easy to see that for all $n \geq \lambda'(0)$, we have

$$\sup_{t\geq 0} \big(nt - \lambda(t)\big) = n\lambda'^{-1}(n) - \lambda\big(\lambda'^{-1}(n)\big).$$

So we deduce from (2.1) that for all $n \geq \max\big(\lambda'(0), 1\big)$,

$$-\log\big|F(e^{i\theta})\big| \geq n \log \frac{1}{d(e^{i\theta},E)} - n\lambda'^{-1}(n) + \lambda\big(\lambda'^{-1}(n)\big). \tag{2.2}$$

Since $\lim_{x\to+\infty} \lambda'(x) = +\infty$ and $\lim_{x\to+\infty} \frac{\lambda(x)}{x} = +\infty$, there exists $\delta > 0$ such that for all $x \geq \delta$, we have $\lambda'(x) \geq 1 + \max\big(\lambda'(0), 1\big)$ and $\lambda(x) - x \geq \frac{1}{2}\lambda(x)$. Let $e^{i\theta}$ be such that $d(e^{i\theta}, E) \leq e^{-\delta}$. We set $n = \Big[\lambda'\Big(\log \frac{1}{d(e^{i\theta},E)}\Big)\Big]$, where, for a real number x, $[x]$ denotes the integer such that $[x] \leq x < [x]+1$. Observing that the function $x \mapsto -x\lambda'^{-1}(x) + \lambda\big(\lambda'^{-1}(x)\big)$ is non-increasing and $n \leq \lambda'\Big(\log \frac{1}{d(e^{i\theta},E)}\Big)$, we get

$$\begin{aligned} -n\lambda'^{-1}(n) + \lambda\big(\lambda'^{-1}(n)\big) &\geq -\log\frac{1}{d(e^{i\theta},E)}\,\lambda'\Big(\log\frac{1}{d(e^{i\theta},E)}\Big) + \lambda\Big(\log\frac{1}{d(e^{i\theta},E)}\Big) \\ &\geq -(n+1)\log\frac{1}{d(e^{i\theta},E)} + \lambda\Big(\log\frac{1}{d(e^{i\theta},E)}\Big). \end{aligned}$$

So we deduce from (2.2) that if $d(e^{i\theta}, E) \leq e^{-\delta}$,

$$\begin{aligned} -\log\big|F(e^{i\theta})\big| &\geq \lambda\Big(\log\frac{1}{d(e^{i\theta},E)}\Big) - \log\frac{1}{d(e^{i\theta},E)} \\ &\geq \frac{1}{2}\lambda\Big(\log\frac{1}{d(e^{i\theta},E)}\Big). \end{aligned}$$

Since $z \mapsto -\log\big|F(z)\big|$ is integrable on $\mathbb{T}$, E is a λ-Beurling-Carleson set. □

We need the following three lemmas.

Lemma 2.2. *Let λ be a non-negative function on $[0,+\infty)$ and non-decreasing on $[A,+\infty)$ for some $A \geq 0$. Let E be a closed subset of $\mathbb{T}$. Let $\big(I_n\big)_{n\geq 0}$ be the sequence of the complementary arcs of E, and denote by $|I_n|$ the length of the arc I_n. If $\sum_{n=0}^{+\infty} |I_n|\lambda\Big(\log^+ \frac{1}{|I_n|}\Big) = +\infty$, then E is not a λ-Beurling-Carleson set.*

Proof. There exists $n_0 \geq 0$ such that for all $n \geq n_0$, $|I_n| \leq e^{-A}$. Let $n \geq n_0$; for all $e^{it} \in I_n$, we have $d(e^{it}, E) \leq |I_n|$. Also, since λ is non-decreasing on $[A,+\infty)$, we have

$$\lambda\Big(\log\frac{1}{d(e^{it},E)}\Big) \geq \lambda\Big(\log\frac{1}{|I_n|}\Big).$$

Hence, for all $n \geq n_0$,

$$\begin{aligned} \int_{I_n} \lambda\Big(\log^+ \frac{1}{d(e^{it},E)}\Big)\mathrm{d}t &= \int_{I_n} \lambda\Big(\log \frac{1}{d(e^{it},E)}\Big)\mathrm{d}t \\ &\geq |I_n|\lambda\Big(\log\frac{1}{|I_n|}\Big). \end{aligned} \tag{2.3}$$

Then, to conclude, it suffices to use (2.3) and observe that

$$\int_0^{2\pi} \lambda\Big(\log^+ \frac{1}{d(e^{it},E)}\Big)\mathrm{d}t \geq \sum_{n=n_0}^{+\infty} \int_{I_n} \lambda\Big(\log^+ \frac{1}{d(e^{it},E)}\Big)\mathrm{d}t$$

and that $\sum\limits_{n=n_0}^{+\infty} |I_n|\lambda\Big(\log\frac{1}{|I_n|}\Big) = +\infty$. □

Remark. Note that it is well known that E is a Beurling-Carleson set if and only if

$$|E| = 0 \quad \text{and} \quad \sum_{n=0}^{+\infty} |I_n| \log\frac{1}{|I_n|} < +\infty,$$

where $|E|$ is the Lebesgue measure of E.

Lemma 2.3. *Let F be a positive function on $[0,+\infty)$ such that $\lim\limits_{x\to+\infty} \frac{F(x)}{x} = +\infty$ and $\inf\limits_{x\geq 0} F(x) > 0$. Then, there exists a positive function λ on $[0,+\infty)$ such that*
(i) *λ is infinitely differentiable on $[0,+\infty)$ and $\lambda'' > 0$.*
(ii) $\lim\limits_{x\to+\infty} \frac{\lambda(x)}{x} = +\infty$.
(iii) *$\lambda \leq F$ on $[0,+\infty)$.*

Proof. For $x \geq 0$, we set $G(x) = \inf\limits_{t\geq x} F(t)$. G is a positive and non-decreasing function such that $G \leq F$ and $\lim\limits_{x\to+\infty} \frac{G(x)}{x} = +\infty$. So, without loss of generality, we will suppose that F is non-decreasing. Let m be the largest convex minorant of F. It is clear that m is a positive and non-decreasing function. Since $\lim\limits_{x\to+\infty} \frac{F(x)}{x} = +\infty$, for all $M > 0$, there exists $a > 0$ such that $F(x) \geq M(x-a)$ for all $x \geq 0$. By definition of m, we have $m(x) \geq M(x-a)$ for all $x \geq 0$. This proves that $\lim\limits_{x\to+\infty} \frac{m(x)}{x} = +\infty$. Now, let χ be a non-negative and infinitely differentiable function on $\mathbb{R}$, with support included in $[0,1]$ such that $\int_0^1 \chi(t)\mathrm{d}t = 1$. We extend m in a non-decreasing convex function on $\mathbb{R}$ by setting $m(x) = m(0)$ if $x \leq 0$. Also, we define a function μ on $\mathbb{R}$ by

$$\mu(x) = \int_0^1 m(x-t)\chi(t)\mathrm{d}t.$$

It is clear that μ is a positive and infinitely differentiable function on $\mathbb{R}$. Since m is non-decreasing, we have, for all $x \geq 0$,

$$\begin{aligned} \mu(x) &\leq m(x)\int_0^1 \chi(t)\mathrm{d}t = m(x), \\ \text{and} \quad \mu(x) &\geq m(x-1)\int_0^1 \chi(t)\mathrm{d}t = m(x-1), \end{aligned}$$

so that $\mu \leq m \leq F$ on $[0, +\infty)$ and $\lim_{x \to +\infty} \frac{\mu(x)}{x} = +\infty$. Furthermore, m is convex, so is μ, and in particular we have $\mu'' \geq 0$. Denote by $c = \inf_{x \geq 0} F(x)$, and set, for $x \geq 0$, $\lambda(x) = \frac{1}{2}\mu(x) + \frac{c}{2}e^{-x}$. It is clear that λ satisfies conditions (i)-(iii) of the lemma. □

Lemma 2.4. *Let λ be a non-negative convex function on $[0, +\infty)$ such that*

$$\lim_{x \to +\infty} \frac{\lambda(x)}{x} = +\infty.$$

Then there exists a perfect closed subset of $\mathbb{T}$ which is a Beurling-Carleson set but not a λ-Beurling-Carleson set.

Proof. Since $\lim_{x \to +\infty} \frac{\lambda(x)}{x} = +\infty$, we can find a sequence $\left(a'_k\right)_{k \geq 1}$ of real numbers in the interval $(0, e^{-1})$ such that for all $k \geq 1$,

$$a'_k \log \frac{1}{a'_k} \leq \frac{1}{k^2}$$

$$\text{and} \quad \lambda\Big(\log \frac{1}{a'_k}\Big) \geq k \log \frac{1}{a'_k}. \tag{2.4}$$

For all $k \geq 1$, let n_k be a non-negative integer such that

$$\frac{1}{k^2} \leq n_k\, a'_k \log \frac{1}{a'_k} \leq \frac{2}{k^2}. \tag{2.5}$$

We define $\left(a_n\right)_{n \geq 1}$ by

$$a_n = a'_k \quad \text{if} \quad 1 \leq n \leq n_1$$

$$\text{and} \quad a_n = a'_k \quad \text{if} \quad n_1 + \ldots + n_{k-1} + 1 \leq n \leq n_1 + \ldots + n_k \quad (k \geq 2).$$

Thanks to (2.4) and (2.5), the sequence $\left(a_n\right)_{n \geq 1}$ satisfies

$$\sum_{n=1}^{+\infty} a_n \log \frac{1}{a_n} < +\infty$$

$$\text{and} \quad \sum_{n=1}^{+\infty} a_n \lambda\Big(\log \frac{1}{a_n}\Big) = +\infty. \tag{2.6}$$

Now, since the function $x \mapsto x \log \frac{1}{x}$ is one-to-one and onto on $(0, e^{-1})$, there exists, for all $n \geq 1$, a non-negative real number $b_n \in (0, e^{-1})$ such that

$$a_n \log \frac{1}{a_n} = 2^{n-1} b_n \log \frac{1}{b_n}.$$

Hence, we have

$$\sum_{n=1}^{+\infty} 2^{n-1} b_n \log \frac{1}{b_n} < +\infty. \tag{2.7}$$

Furthermore, we have

$$2^{n-1} b_n \lambda\Big(\log \frac{1}{b_n}\Big) = a_n \log \frac{1}{a_n} \Big(\log \frac{1}{b_n}\Big)^{-1} \lambda\Big(\log \frac{1}{b_n}\Big). \tag{2.8}$$

Since λ is convex, the function $x \mapsto \frac{\lambda(x)-\lambda(0)}{x}$ is non-decreasing. So, as for all $n \geq 1$, $b_n \leq a_n$, we have

$$\begin{aligned} \Big(\log\frac{1}{b_n}\Big)^{-1}\lambda\Big(\log\frac{1}{b_n}\Big) &\geq \Big(\log\frac{1}{a_n}\Big)^{-1}\Big(\lambda\Big(\log\frac{1}{a_n}\Big)-\lambda(0)\Big)+\Big(\log\frac{1}{b_n}\Big)^{-1}\lambda(0) \\ &\geq \Big(\log\frac{1}{a_n}\Big)^{-1}\Big(\lambda\Big(\log\frac{1}{a_n}\Big)-\lambda(0)\Big). \end{aligned}$$

Therefore, we deduce from (2.8) that

$$\begin{aligned} 2^{n-1}b_n\lambda\Big(\log\frac{1}{b_n}\Big) &\geq a_n\lambda\Big(\log\frac{1}{a_n}\Big)-a_n\lambda(0) \\ &\geq a_n\lambda\Big(\log\frac{1}{a_n}\Big)-\lambda(0)a_n\log\frac{1}{a_n}. \end{aligned}$$

We deduce from this inequality and (2.6) that

$$\sum_{n=1}^{+\infty} 2^{n-1}b_n\lambda\Big(\log\frac{1}{b_n}\Big)=+\infty. \tag{2.9}$$

The condition (2.7) implies that the series $\sum_{n\geq 1} 2^{n-1}b_n$ converges. We can change a finite number of terms of the sequence $\big(b_n\big)_{n\geq 1}$ to have

$$\sum_{n=1}^{+\infty} 2^{n-1}b_n = 2\pi. \tag{2.10}$$

We will construct a perfect symmetric set with non-constant ratio of dissection as follows. We set $F_0=[0,2\pi]$. We cut off in the middle of F_0 an open interval of length b_1; there remain two closed intervals of length $\frac{1}{2}(2\pi-b_1)$. Then, we cut off, in the middle of each of these remaining intervals, an open interval of length b_2. So after the n^{th} step, it remains a closed subset F_n, which is a union of 2^n closed intervals of length $\frac{1}{2^n}(2\pi-\sum_{k=1}^{n}2^{k-1}b_k)$. We set $F=\bigcap_{n\geq 0}F_n$. F is a perfect symmetric set with non-constant ratio of dissection, and the complementary arcs of F consist in a union on $n\geq 1$ of 2^{n-1} open intervals of length b_n (see [3] for further details). We set

$$E=\Big\{e^{it}\in\mathbb{T} : t\in F\Big\}.$$

It follows from (2.10) that E is of Lebesgue measure zero and then from (2.7) that E is a Carleson set. Furthermore, as λ is convex and $\lim_{x\to+\infty}\lambda(x)=+\infty$, λ is non-decreasing on $[A,+\infty)$, for A large enough. So, according to Lemma 2.2 and condition (2.9), E is not a λ-Beurling-Carleson set. □

We can now give the main result of this paper.

Theorem 2.5. *Let p be in Ω. Then there exists a Beurling-Carleson set which is not ZB_p.*

Proof. Without loss of generality, we assume that $p(n)>1$ for all $n\geq 0$. We extend p in a continuous function on $[0,+\infty)$, linear on each interval $[n,n+1]\,(n\geq 0)$, and we put $F(x)=\log\big(p(e^x)\big)$. We have $\inf_{x\geq 0}F(x)>0$, and since $p\in\Omega$, F satisfies $\lim_{x\to+\infty}\frac{F(x)}{x}=+\infty$. So according to Lemma 2.3, there exists a non-negative function

λ on $[0,+\infty)$ satisfying conditions (i)-(iii) of Lemma 2.3. So we apply Lemma 2.4 that gives a perfect closed subset E of $\mathbb{T}$, which is a Beurling-Carleson set but not a λ-Beurling-Carleson set. As $e^{\lambda(\log n)} \le p(n)$ for all $n \ge 1$, we have

$$B_p \subset B_{p_\lambda},$$

where $p_\lambda(0) = 1$ and $p_\lambda(n) = e^{\lambda(\log n)}$ if $n \ge 1$. Now, let F be a function in B_p that vanishes on E. Then $F \in B_{p_\lambda}$, and since E is a perfect set, it vanishes with all its derivatives on E. As E is not a λ-Beurling-Carleson set, it follows from Proposition 2.1 that F is the zero function. This means exactly that E is not ZB_p. □

Let $c > 0$; we set

$$A_c(\mathbb{T}) = \Big\{ f \in \mathcal{C}(\mathbb{T}) : \sum_{n=-\infty}^{\infty} \big|\widehat{f}(n)\big| e^{c|n|^{\frac{1}{2}}} < +\infty \Big\}.$$

Let E be a closed subset of $\mathbb{T}$. Let p be a sequence of positive real number and $c > 0$. We denote by $A_c(E), B_p(E)$ and $\mathcal{A}^\infty(E)$ the sets of restrictions to E of functions respectively in $A_c(\mathbb{T}), B_p$ and $\mathcal{A}^\infty$. A. Atzmon showed in [1] (Theorem 2) that if E is a Beurling-Carleson set, then there exists a constant $c > 0$ such that $A_c(E) \subset \mathcal{A}^\infty(E)$. It follows from Corollary 2.6 that this result cannot be refined in the following sense: for all $p \in \Omega$, there exists a Beurling-Carleson set E such that for all $c > 0$, we have

$$A_c(E) \not\subset B_p(E).$$

Corollary 2.6. *Let $p \in \Omega$; there exists a Beurling-Carleson set E such that*

$$z^{-1}_{|_E} \notin B_p(E). \tag{2.11}$$

Proof. Let $p \in \Omega$. We claim that we can find a sequence q in Ω such that for all $n \ge 0$, $q(n) \le p(n)$ and

$$\sup_{n\ge 0} \frac{q(n+1)}{q(n)} \le 2. \tag{2.12}$$

We can construct such a sequence as follows. Let $\tilde{p}$ be a sequence defined by $\tilde{p}(n) = \inf_{m \ge n} p(m)$. It is easily seen that $\tilde{p}$ is a non-decreasing sequence in Ω such that $\tilde{p} \le p$. Then we define a sequence q by

$$\begin{aligned} q(0) &= \tilde{p}(0), \\ q(n) &= \min\big(2q(n-1), \tilde{p}(n)\big) \quad (n \ge 1). \end{aligned}$$

It is clear that for all $n \ge 0$, $q(n) \le \tilde{p}(n) \le p(n)$, and that q satisfies (2.12). Then, an induction on n shows that

$$q(n) = \min\big\{2^k \tilde{p}(n-k) : 0 \le k \le n\big\}.$$

So, for all $n \ge 0$, there exists an integer k_n, $0 \le k_n \le n$, such that

$$q(n) = 2^{k_n} \tilde{p}(n-k_n).$$

Distinguishing the case $k_n \le \left[\frac{n}{2}\right]$ from the case $k_n > \left[\frac{n}{2}\right]$, we obtain

$$q(n) \ge \min\Big(\tilde{p}\Big(\Big[\frac{n}{2}\Big]\Big), 2^{[\frac{n}{2}]}\tilde{p}(0)\Big).$$

Then we deduce from this inequality that $q \in \Omega$. According to Theorem 2.5, there exists a Beurling-Carleson set E which is not ZB_q. As $B_p \subset B_q$, it suffices to prove that $z_{|_E}^{-1} \notin B_q(E)$ to show (2.11). Suppose that $z_{|_E}^{-1} \in B_q(E)$. Then, there exists $F \in B_q$ such that $\frac{1}{z} = F(z)$ for all $z \in E$. Consequently, $z \mapsto 1 - zF(z)$ is a non-zero function that vanishes on E, and thanks to (2.12), it belongs to B_q. This is in contradiction with the fact that E is not ZB_q, and concludes the proof. $\square$

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