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A UNIQUENESS THEOREM FOR A FREE BOUNDARY PROBLEM

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Abstract. In this paper, we prove a uniqueness theorem for a free boundary problem which is given in the form of a variational inequality. This free boundary problem arises as the limit of an equation that serves as a basic model in population biology. Apart from the interest in the problem itself, the techniques used in this paper, which are based on the regularity theory of variational inequalities and of harmonic functions, are of independent interest, and may have other applications.

1. Introduction and main result

In this paper, we prove a uniqueness theorem for a free boundary problem. The free boundary problem arises as the limit of an equation which serves as a basic model in population biology. Apart from the interest in the problem itself, we feel that some of the techniques we use in this paper are of independent interest, and may have other applications.

As will be shown later, the free boundary problem we wish to study is in fact equivalent to a variational inequality which we now describe precisely. Suppose Ω is a bounded smooth domain in R^N and Ω_0 is a smooth subdomain of Ω such that $\overline{\Omega}_0 \subset \Omega$. Define

$$K = \{w \in H_0^1(\Omega) : \ w \leq 1 \text{ a.e. in } \Omega \setminus \Omega_0\}.$$

Our variational inequality has the form

$$\int_\Omega \nabla u \cdot \nabla (v-u) dx - \int_\Omega au(v-u)dx \geq 0, \ \forall v \in K. \tag{1.1}$$

Here a is a real parameter, and we are interested in the nonnegative solutions $u \in K$ of (1.1). Clearly $u \equiv 0$ is a nonnegative solution of (1.1) in K. We call this the trivial solution and want to understand the nontrivial nonnegative solutions.

Our main result of this paper is the following.

Theorem 1.1. *For $a \in (\lambda_1^\Omega, \lambda_1^{\Omega_0})$, the variational inequality* (1.1) *has a unique nontrivial nonnegative solution, and it has no such solution for $a \notin [\lambda_1^\Omega, \lambda_1^{\Omega_0})$; if $a = \lambda_1^\Omega$,* (1.1) *has infinitely many nontrivial nonnegative solutions.*

Here λ_1^Ω stands for the first eigenvalue of $-\Delta$ over Ω under Dirichlet boundary conditions; $\lambda_1^{\Omega_0}$ is defined analogously.

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Remark 1.2. If $a = \lambda_1^\Omega$, it is a consequence of our proof below that the positive solutions of (1.1) are given by $\alpha\phi_1$ with $\alpha > 0$ such that $\alpha\phi_1(x) \le 1$ on $\Omega \setminus \Omega_0$, where ϕ_1 is the principal positive eigenfunction corresponding to λ_1^Ω.

Let us now explain in some detail how the problem arises. In a number of recent papers, see for example, [DD1] and [Du], the problem

$$-\Delta u = au - b(x)u^p \text{ in } \Omega,\ u|_{\partial\Omega} = 0 \tag{1.2}$$

occurs naturally. Here $p > 1$ and $b(x)$ is a Hölder continuous, nonnegative function with $b^{-1}(0) = \overline{\Omega}_0$. Problem (1.2) is sometimes known as a degenerate logistic model in population biology, and has been used to understand the inhomogeneous spatial effects on various population models (see, e.g., [DD1], [Du]). It is by now well known that (1.2) has a unique positive solution u_p for any fixed $a \in (\lambda_1^\Omega, \lambda_1^{\Omega_0})$ and no positive solution otherwise. Moreover, if $p_i \to \infty$, it is shown in [DDM] that a subsequence of $\{u_{p_i}\}$ converges in $L^2(\Omega)$ to a nontrivial nonnegative solution of (1.1). Theorem 1.1 then implies that the entire sequence $\{u_p\}$ converges to the unique nontrivial nonnegative solution of (1.1) as $p \to \infty$. It will become clear in our proof later that any nontrivial nonnegative solution of (1.1) is in fact a member of $C^1(\overline{\Omega} \setminus \partial\Omega_0) \cap C(\overline{\Omega})$ and is strictly positive in Ω.

The proof of Theorem 1.1, which constitutes the main body of this paper, will be carried out in section 2. The existence part of Theorem 1.1 already follows from Theorem 1.6 of [DDM], but the proof for uniqueness turns out to be difficult. The free boundary nature of (1.1) will be clearly demonstrated in the process of our proof. As will become clear, our arguments are based on the regularity theory for variational inequalities and a result in [HW] for harmonic functions. In the very short section 3, we collect several remarks and briefly discuss some extensions of our main results.

2. Proof of Theorem 1.1

The proof of Theorem 1.1 will be broken into several steps.

Step 1. *An equivalent characterization of the variational inequality.*

In this step, we will find a PDE description of (1.1) which clearly reveals its free boundary nature.

By pages 137–138 in [KS], a solution $u \in K$ of (1.1) is Lipschitz in $\overline{\Omega}$, and is C^1 and $H^{2,q}$ ($\forall q > 1$) on $\Omega \setminus \overline{\Omega}_0$. (The theory in [F] can probably be adapted to show that u is $C^{1,\alpha}$ on $\overline{\Omega}$ except for a possible jump in the normal derivative across $\partial\Omega_0 \cap u^{-1}(1)$, but we do not use this here. Instead, we will follow an approach which has the advantage that it applies under weaker smoothness assumptions.) It follows that

$$-\Delta u = au \text{ in } \Omega_0 \cup \{u < 1\}. \tag{2.1}$$

Here and in what follows, we will use the notation $\{u < 1\} = \{x \in \Omega : u(x) < 1\}$. By bootstrapping, u is $C^{2,\alpha}$ on this set. Moreover, applying Theorem 6.9 in chapter II of [KS] to $(1 - u)$, we see that

$$-\Delta u = au - m \text{ in } \Omega, \tag{2.2}$$

where m is a nonnegative Radon measure.

We now set to identify m. By (2.1), obviously m is supported on $\{x \in \Omega \setminus \Omega_0 : u(x) = 1\}$. By Theorem 8.6 and pages 137–138 in [KS], $u \in H^{2,q}_{loc}(\overline{\Omega} \setminus \overline{\Omega}_0)$, $\forall q > 1$, and hence $m \in L^q_{loc}(\overline{\Omega} \setminus \overline{\Omega}_0)$. But $\Delta u = 0$ a.e. on the set $(\Omega \setminus \overline{\Omega}_0) \cap \{u = 1\}$ because u is identically 1 there. Hence we see from (2.2) that

$$m = au \text{ a.e. on } (\Omega \setminus \overline{\Omega}_0) \cap \{u = 1\}.$$

In view of (2.1), we have

$$-\Delta u = au\chi_{\{u<1\}} \text{ on } \Omega \setminus \overline{\Omega}_0.$$

To examine the behaviour of m on $\partial\Omega_0$, which is the most difficult part in this analysis, we first prove that normal derivatives $\partial^+_\nu u$ and $\partial^-_\nu u$ can be defined almost everywhere on $\partial\Omega_0$ (here and in what follows, *almost everywhere on* $\partial\Omega_0$ is understood in the sense of $(N-1)$-dimensional Hausdorff measure), where ν denotes the unit normal of $\partial\Omega_0$ pointing outward of Ω_0, $\partial^+_\nu u$ is called the outside normal derivative of u, and $\partial^-_\nu u$ is called the inside normal derivative of u.

We will give the detailed definition for the outside normal derivative only; the inside normal derivative is defined analogously. Let w be the unique solution of the problem

$$-\Delta w = g \text{ in } \Omega,\ w|_{\partial\Omega} = 0,$$

where $g = au$ on Ω_0 and $g = au\chi_{\{u<1\}}$ in $\Omega \setminus \overline{\Omega}_0$. By standard regularity theory, $w \in C^1(\overline{\Omega}) \cap H^{2,q}(\Omega)$, $\forall q > 1$. Since $u - w$ is harmonic in $\Omega \setminus \overline{\Omega}_0$ and is Lipschitz up to the boundary of this set, $\partial_{x_i}(u - w)$ is harmonic in this set and bounded on the closure of this set. By [HW], $\partial_{x_i}(u - w)$ has a nontangential limit a.e. on $\partial\Omega_0$, i.e., for a.e. $x \in \partial\Omega_0$, when $y \in \Omega \setminus \overline{\Omega}_0$ converges to x along a nontangential direction to $\partial\Omega_0$ at x, $\partial_{x_i}(u - w)(y)$ has a finite limit. It follows that $\partial_{x_i} u$ has the same property. Let $\nu(x)$ be a continuous extension of the unit normal vector of $\partial\Omega_0$ into a small neighbourhood of $\partial\Omega_0$. By the existence of the nontangential limit above, we see that $\partial_{\nu(y)}u(y)$, $y \in \Omega \setminus \overline{\Omega}_0$, has a limit as y converges to $x \in \partial\Omega_0$ in the direction parallel to $\nu(x)$, for a.e. $x \in \partial\Omega_0$. We denote this limit by $\partial^+_\nu u(x)$. It is easily seen that $\partial^+_\nu u$ is independent of the particular extension of $\nu(x)$ and hence is well defined a.e. on $\partial\Omega_0$. Similarly, $\partial^-_\nu u(x)$ is defined as the limit of $\partial_{\nu(y)}u(y)$ when $y \in \Omega_0$ converges to $x \in \partial\Omega_0$ in the direction parallel to $\nu(x)$.

We are now ready to establish an integral representation for m by making use of the outside and inside normal derivatives of u on $\partial\Omega_0$. With $\nu(x)$ as above and for small $t > 0$, we let Ω_t be the perturbation of $\Omega \setminus \Omega_0$ with boundaries $\partial\Omega$ and $\{x + t\nu(x) : x \in \partial\Omega_0\}$. Roughly speaking, Ω_t is obtained by moving the boundary $\partial\Omega_0$ of $\Omega \setminus \Omega_0$ a distance t into this set, and hence Ω_t is contained in $\Omega \setminus \Omega_0$. Integration by parts gives, for any smooth $w \in H^1_0(\Omega)$,

$$\int_{\Omega_t} \nabla u \cdot \nabla w dx = \int_{\Omega_t} -\Delta u w dx - \int_{\partial\Omega_t} \partial_{\nu_t} u w dx, \tag{2.3}$$

where ν_t denotes the unit normal on $\{x + t\nu(x) : x \in \partial\Omega_0\}$ which varies continuously with t and agrees with ν at $t = 0$.

Recall that when restricted to $\Omega \setminus \overline{\Omega}_0$, $-\Delta u = au\chi_{\{u<1\}} \in L^\infty(\Omega \setminus \Omega_0)$. Moreover, as $t \to 0$, $\partial_{\nu_t} u$ converges to $\partial^+_\nu u$. Hence we can let $t \to 0$ in (2.3) to obtain

$$\int_{\Omega \setminus \Omega_0} \nabla u \cdot \nabla w dx = \int_{\Omega \setminus \Omega_0} au\chi_{\{u<1\}} w dx - \int_{\partial\Omega_0} \partial^+_\nu u w ds. \tag{2.4}$$

Using inside normal derivatives and $-\Delta u = au$ in Ω_0, we similarly obtain

$$\int_{\Omega_0} \nabla u \cdot \nabla w dx = \int_{\Omega_0} au w dx + \int_{\partial\Omega_0} \partial^-_\nu u w ds. \tag{2.5}$$

Clearly (2.4) and (2.5) also hold for all $w \in H_0^1(\Omega)$. Adding these two identities we obtain

$$(2.6) \qquad \int_\Omega [\nabla u \cdot \nabla w - auw]dx = \int_{\partial\Omega_0} (\partial_\nu^- u - \partial_\nu^+ u) w ds - \int_{\Omega\setminus\Omega_0} au\chi_{\{u=1\}} w dx, \ \ \forall w \in H_0^1(\Omega).$$

This gives the integral representation for the measure m as we wanted.

Finally, in this step, we use (2.6) to prove the following inequality:

$$(2.7) \qquad \partial_\nu^+ u \geq \partial_\nu^- u \text{ a.e. on } \partial\Omega_0.$$

Suppose $u \in K$ satisfies (1.1). Then it follows from (2.6) that for all $v \in K$,

$$(2.8) \qquad \int_{\partial\Omega_0} (\partial_\nu^- u - \partial_\nu^+ u)(v-u) ds \geq \int_{\Omega\setminus\Omega_0} au\chi_{\{u=1\}}(v-u)dx.$$

For any given $\phi \in C(\partial\Omega_0)$ satisfying $\phi(x) \leq 1 - u(x)$ we can find a sequence of smooth functions $v_n \in K$ such that

(i) $v_n - u \to 0$ uniformly on any compact $G \subset \Omega \setminus \overline{\Omega}_0$,
(ii) $\{\|v_n\|_{L^\infty(\Omega)}\}$ is bounded, and
(iii) $v_n - u \to \phi$ uniformly on $\partial\Omega_0$.

Taking $v = v_n$ in (2.8) and letting $n \to \infty$, we obtain

$$\int_{\partial\Omega_0} (\partial_\nu^- u - \partial_\nu^+ u)\phi ds \geq 0.$$

Since $u \in K$, ϕ can be any nonpositive function, and it follows that $\partial_\nu^+ u - \partial_\nu^- u \geq 0$ a.e on $\partial\Omega_0$, as we wanted. Moreover, since ϕ can be chosen positive as well as negative in the part of $\partial\Omega_0$ where $u(x) < 1$, we must have $\partial_\nu^+ u - \partial_\nu^- u = 0$ a.e. on $\partial\Omega_0 \cap \{u < 1\}$.

Summarizing, we find that if $u \in K$ solves (1.1), then it is Lipschitz on $\overline{\Omega}$, $u = 0$ on $\partial\Omega$, $u \in H_{loc}^{2,q}((\overline{\Omega} \setminus \partial\Omega_0) \cup \{u < 1\})$, $\forall q > 1$,

$$-\Delta u = au \text{ in } \Omega_0, \ \ -\Delta u = au\chi_{\{u<1\}} \text{ in } \Omega \setminus \overline{\Omega}_0,$$

and

$$\partial_\nu^- u \leq \partial_\nu^+ u \text{ a.e. on } \partial\Omega_0,$$

with equality holding on $\partial\Omega_0 \cap \{u < 1\}$.

Conversely, if $u \in H_0^1(\Omega)$ satisfies the above listed properties, then it is easily seen that (1.1) holds for all smooth $v \in K$ and hence for all $v \in K$. Therefore, the above description of u is equivalent to the variational inequality (1.1).

Step 2. *Existence of a minimal nontrivial nonnegative solution.*

In this step we will show that if $a > \lambda_1^\Omega$ and if (1.1) has a nontrivial nonnegative solution in K, then it has a minimal such solution u_*, i.e., any nontrivial nonnegative solution u of (1.1) in K satisfies $u \geq u_*$ in Ω.

We will mainly follow the arguments in [DD2], but the situation here is more complicated. By Theorem 1.4 in chapter III of [KS] on strict monotone operators, for each $f \in L^2(\Omega)$, the variational inequality

$$\int_\Omega \nabla u \cdot \nabla(v-u)dx \geq \int_\Omega f(v-u)dx, \ \forall v \in K,$$

has a unique solution in K, which we denote by $u = L(f)$. As in [DD2], one easily checks that L has the following properties:

(i) $L : L^2(\Omega) \to L^2(\Omega)$ is completely continuous;
(ii) $L(f) \in H^{2,s}(\Omega) \cap C^{1,\mu}(\bar{\Omega})$, $\mu = 1 - (N/s)$, if $f \in L^s(\Omega)$ with $s > N$;
(iii) $L(f) \geq 0$ a.e. in Ω when $f \geq 0$ a.e. in Ω;
(iv) $L(f_1) \leq L(f_2)$ a.e. in Ω when $f_1 \leq f_2$ a.e. in Ω;
(v) $L(f) = (-\Delta)^{-1} f$ if $f \geq 0$ and $(-\Delta)^{-1} f \leq 1$ on $\Omega \setminus \Omega_0$, where $(-\Delta)^{-1}$ denotes the inverse of $-\Delta$ over Ω under Dirichlet boundary conditions.

For $a > 0$ let us define $L_a : L^2(\Omega) \to L^2(\Omega)$ by

$$L_a(f) = L(af).$$

Clearly L_a has the same properties (i)–(v) above except that (v) should be modified slightly in an obvious way. Moreover, $u \in K$ solves (1.1) if and only if it is a fixed point of L_a in K. Therefore, it suffices to show that L_a has a minimal nontrivial nonnegative fixed point in K.

We claim that any nontrivial nonnegative solution of (1.1) satisfies $u \geq \mu\phi_1$ for some $\mu > 0$, where ϕ_1 denotes the normalized positive eigenfunction corresponding to λ_1^Ω:

$$-\Delta\phi_1 = \lambda_1^\Omega \phi_1 \text{ in } \Omega,\ \phi_1|_{\partial\Omega} = 0,\ \|\phi_1\|_{L^\infty(\Omega)} = 1.$$

To see this, we recall from Step 1 that u is continuous on $\overline{\Omega}$ and $-\Delta u = au$ on $\{u < 1\}$. Thus the weak Harnack inequality (see [GT]) can be used to conclude that $u(x) > 0$ in Ω. Since $u = 0$ on $\partial\Omega$, $-\Delta u = au$ near $\partial\Omega$, and hence the Hopf boundary lemma can be used to conclude that $\partial_n u < 0$ on $\partial\Omega$, where n is the outward unit normal on $\partial\Omega$. Our claim now follows immediately. Therefore, from now on, instead of saying a nontrivial nonnegative solution of (1.1), we are allowed to speak of a positive solution of (1.1).

Let us fix a positive solution v_0 of (1.1) in K, and choose $\epsilon_0 > 0$ such that $v_0 \geq \epsilon_0\phi_1$ in Ω. By property (v) for L above, if $\epsilon \in (0, \epsilon_0]$ is small enough, using $a > \lambda_1^\Omega$, we deduce

$$L_a(\epsilon\phi_1) = L(\epsilon a\phi_1) = (-\Delta)^{-1}(\epsilon a\phi_1) = \epsilon a(\lambda_1^a)^{-1}\phi_1 \geq \epsilon\phi_1.$$

Therefore,

$$\epsilon\phi_1 \leq L_a(\epsilon\phi_1) \leq L_a(v_0) = v_0.$$

If we denote $u_0 = \epsilon\phi_1, u_n = L_a^n(u_0)$, we easily deduce

$$u_0 \leq u_1 \leq ... \leq u_n \leq u_{n+1} \leq ... \leq v_0.$$

Since L_a is completely continuous, we easily see that $u_\epsilon^* := \lim_{n\to\infty} u_n$ exists and is a fixed point of L_a. We can now follow the argument in [DD2] to see that u_ϵ^* is independent of ϵ for small ϵ and is the minimal positive solution of (1.1).

We would like to point out that while the existence of a nontrivial nonnegative solution of (1.1) in K for each $a \in (\lambda_1^\Omega, \lambda_1^{\Omega_0})$ follows from [DDM] already, it can also be easily proved directly by using a variational or degree argument.

Step 3. *Uniqueness.*

In this step we will show that if $a > \lambda_1^\Omega$, and if (1.1) has two positive solutions u_1 and u_2 such that $u_1 \leq u_2$ in Ω, then $u_1 \equiv u_2$. By Step 2, and the existence result which follows from Theorem 1.6 of [DDM], this would prove that (1.1) has a unique positive solution for $a \in (\lambda_1^\Omega, \lambda_1^{\Omega_0})$.

Suppose now u_1 and u_2 are two positive solutions of (1.1) satisfying $0 < u_1 \leq u_2$. Using (2.6) with $(u,w) = (u_1,u_2)$ and $(u,w) = (u_2,u_1)$ and subtracting, we obtain

$$a\int_\Omega u_1u_2(\chi_{A_2} - \chi_{A_1})dx + \int_{\partial\Omega_0}[\gamma(u_2)u_1 - \gamma(u_1)u_2]ds = 0, \tag{2.9}$$

where for $i = 1, 2$,

$$A_i = (\Omega \setminus \Omega_0) \cap \{u_i = 1\},\ \gamma(u_i) = \partial_\nu^+ u_i - \partial_\nu^- u_i.$$

Since $0 < u_1 \leq u_2$ in Ω, and $u_1 \leq u_2 \leq 1$ on $\Omega \setminus \Omega_0$, we find that A_1 is contained in A_2 and hence the first term on the left hand side of (2.9) is nonnegative, and it is zero only if $A_2 \setminus A_1$ has measure zero. For the second term, we first observe that the integrand is identically zero a.e. on $\partial\Omega_0 \setminus C$, where

$$C = \partial\Omega_0 \cap (\{u_1 = 1\} \cup \{u_2 = 1\}).$$

Moreover, the set C can be decomposed as $C = C_1 \cup C_2$, where

$$C_1 = \{u_1 < 1\} \cap \{u_2 = 1\} \cap \partial\Omega_0,\ C_2 = \{u_1 = 1\} \cap \{u_2 = 1\} \cap \partial\Omega_0.$$

Clearly,

$$\gamma(u_2)u_1 - \gamma(u_1)u_2 = \gamma(u_2)u_1 \geq 0 \text{ a.e. on } C_1,$$

with equality holding only if $\gamma(u_2) = 0$ a.e. on C_1. On C_2,

$$\gamma(u_2)u_1 - \gamma(u_1)u_2 = \gamma(u_2) - \gamma(u_1).$$

Since $u_1 \leq u_2$ outside C_2 and $u_1 = u_2$ on C_2, it can be shown that $\gamma(u_2) \geq \gamma(u_1)$ a.e. on C_2. We will give a proof for this fact at the end of this step since it does not follow directly from our definitions of the inside and outside normal derivatives on $\partial\Omega_0$. Assuming this fact for the moment, we have

$$\gamma(u_2)u_1 - \gamma(u_1)u_2 \geq 0 \text{ a.e. on } C_2.$$

It follows that the second term on the left-hand side of (2.9) is also nonnegative, and it is zero only if $\gamma(u_2) = 0$ a.e. on C_1 and $\gamma(u_2) = \gamma(u_1)$ a.e. on C_2. In other words, the second term is always nonnegative and is zero only if $\gamma(u_2) = \gamma(u_1)$ a.e. on $\partial\Omega_0$. But by (2.9) both the first and the second terms have to be zero. Hence $A_2 \setminus A_1$ has measure zero and $\gamma(u_2) = \gamma(u_1)$ a.e. on $\partial\Omega_0$. Thus we can apply (2.6) to u_1 and u_2 and then subtract to obtain

$$\int_\Omega \nabla(u_2 - u_1) \cdot \nabla w dx = \int_\Omega a(1 - \chi_{A_2})(u_2 - u_1)w dx,\ \forall w \in H_0^1(\Omega).$$

That is, $u_2 - u_1$ is a nonnegative solution of the problem

$$-\Delta v = a(1 - \chi_{A_2})v \text{ in } \Omega,\ v|_{\partial\Omega} = 0.$$

Hence $u_2 - u_1$ is C^1 up to the boundary of Ω, and by the weak Harnack inequality, either $u_1 \equiv u_2$ or $u_1 < u_2$ in Ω. If the second alternative occurs, since u_1 is Lipschitz continuous, we necessarily have $u_1 < 1$ in a small neighbourhood of $\Omega \setminus \Omega_0$ in Ω, and hence, by the variational inequality (1.1) (see also (2.1)), u_1 is a weak positive solution of $-\Delta v = av,\ v|_{\partial\Omega} = 0$. This is only possible if $a = \lambda_1^\Omega$, a contradiction to our assumption that $a > \lambda_1^\Omega$. Thus we must have $u_1 \equiv u_2$, and our proof of Step 3 is complete provided we can show that $\gamma(u_2) \geq \gamma(u_1)$ a.e. on C_2.

To prove this last inequality, we suppose that $x_0 \in C_2$ is such that all the inside and outside normal derivatives appearing in $\gamma(u_1)$ and $\gamma(u_2)$ are defined at $x = x_0$. Denote $w = u_2 - u_1$ and $\nu_0 = \nu(x_0)$. We find that $w(x_0 + t\nu_0)$ is C^1 and nonnegative for all small $t \neq 0$, and $w(x_0) = 0$. By the mean value theorem, we can find $t_n^+ \downarrow 0$ and $t_n^- \uparrow 0$ such that $\frac{d}{dt}w(x_0 + t_n^+\nu_0) \geq 0$ and $\frac{d}{dt}w(x_0 + t_n^-\nu_0) \leq 0$. Therefore,

$$\gamma(u_2)(x_0) - \gamma(u_1)(x_0) = \lim_{n\to\infty}[\frac{d}{dt}w(x_0 + t_n^+\nu_0) - \frac{d}{dt}w(x_0 + t_n^-\nu_0)] \geq 0.$$

This implies that $\gamma(u_2) \geq \gamma(u_1)$ a.e. on C_2.

Step 4. *Nonexistence and multiplicity.*

We first show that (1.1) has no positive solution when $a \geq \lambda_1^{\Omega_0}$ or $a < \lambda_1^{\Omega}$. Indeed, if it has a positive solution u_0, then u_0 satisfies $-\Delta u_0 = au_0$ in Ω_0 and is positive and continuous on $\overline{\Omega}_0$. By [BNV], this implies that $a < \lambda_1^{\Omega_0}$. Moreover, if we take $v = 0$ in the variational inequality for u_0, we obtain $\int_\Omega |\nabla u_0|^2 dx \leq a\int_\Omega u_0^2 dx$, which is possible only if $a \geq \lambda_1^{\Omega}$. Finally, if $a = \lambda_1^{\Omega}$, then clearly $u = \alpha\phi_1$ is a positive solution of (1.1) for $0 < \alpha \leq (\sup_{\Omega\setminus\Omega_0}\phi_1)^{-1}$, and these are all the possible positive solutions.

The proof of Theorem 1.1 is now complete.

3. Some remarks

(i) Our assumptions on the smoothness of $\partial\Omega$ and $\partial\Omega_0$ are more than necessary. It can be seen from our proof in section 2 and some care that Theorem 1.1 remains true if Ω is a Lipschitz domain and $\partial\Omega_0$ is C^1 except for a set of capacity zero, and furthermore, the regularity of Ω_0 ensures that the solution of

$$-\Delta v = av \text{ in } \Omega_0,\ v|_{\partial\Omega_0} = 1$$

is Lipschitz on $\overline{\Omega}_0$.

For example, in space dimension 2, we can make use of the theory in Grisvard [G] to study the cases where $\partial\Omega_0$ has corners.

(ii) By the method of sub- and super solutions, it is easy to prove that the unique positive solution u_a of (1.1) is increasing in a for $a \in (\lambda_1^{\Omega}, \lambda_1^{\Omega_0})$. Moreover, a simple compactness argument shows that u_a is continuous in a in the norm of $C(\overline{\Omega})$.

(iii) If $a > \lambda_1^{\Omega}$ and u is a positive solution of (1.1), then necessarily $u = 1$ somewhere in $\Omega \setminus \Omega_0$, for otherwise u would be a positive solution of $-\Delta u = au,\ u|_{\partial\Omega} = 0$, which forces $a = \lambda_1^{\Omega}$.

(iv) It follows easily from (iii) above that $\lim_{a\downarrow\lambda_1^{\Omega}} u_a = \alpha_0\phi_1$, where $\alpha_0^{-1} = \sup_{\Omega\setminus\Omega_0}\phi_1$.

(v) In contrast to (iv), because (1.1) has no positive solution when $a = \lambda_1^{\Omega_0}$, u_a must become unbounded in Ω as $a \uparrow \lambda_1^{\Omega_0}$. In particular, it is not difficult to prove that $u_a/\|u_a\|_{L^2(\Omega)}$ converges in $L^2(\Omega)$ to a nonnegative function ϕ, where ϕ is identically zero on $\overline{\Omega} \setminus \Omega_0$, and $\phi|_{\Omega_0}$ is the positive eigenfunction corresponding to $\lambda_1^{\Omega_0}$ with $L^2(\Omega_0)$ norm 1.

(vi) If $(\lambda_1^{\Omega_0} - a)$ is positive and small, there must be a jump in the derivative of u_a somewhere on $\partial\Omega_0$, for otherwise u_a would be a weak solution of $-\Delta u = b_a(x)u$ in Ω with some uniformly bounded function $b_a(x)$ whose bound is independent of a. By Harnack's inequality, this would imply a contradiction to (v) above.

(vii) Our results here and those in [DDM] remain valid if Ω_0 is not connected but has finitely many components $D_k,\ k = 1, ..., m$, each satisfying the smoothness conditions as in (i) above. In this case, $\lambda_1^{\Omega_0}$ is understood as the minimum of $\lambda_1^{D_k}$,

$k = 1, ..., m$. By remark (i) above, this applies even if some of the D_i's touch each other at isolated points.

(viii) Our methods can be adapted to cover some cases where Ω_0 touches the boundary of Ω. What helps in doing this is that u is always continuous on $\partial\Omega$ and hence the free boundary is never close to $\partial\Omega$. But we leave the details to the interested reader.

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