

NIKOLSKII-TYPE INEQUALITIES FOR SHIFT INVARIANT FUNCTION SPACES

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(Communicated by David Preiss)

ABSTRACT. Let V_n be a vectorspace of complex-valued functions defined on \mathbb{R} of dimension $n + 1$ over \mathbb{C} . We say that V_n is shift invariant (on \mathbb{R}) if $f \in V_n$ implies that $f_a \in V_n$ for every $a \in \mathbb{R}$, where $f_a(x) := f(x - a)$ on \mathbb{R} . In this note we prove the following.

Theorem. Let $V_n \subset C[a, b]$ be a shift invariant vectorspace of complex-valued functions defined on \mathbb{R} of dimension $n + 1$ over \mathbb{C} . Let $p \in (0, 2]$. Then

$$\|f\|_{L_\infty[a+\delta, b-\delta]} \leq 2^{2/p^2} \left(\frac{n+1}{\delta}\right)^{1/p} \|f\|_{L_p[a, b]}$$

for every $f \in V_n$ and $\delta \in (0, \frac{1}{2}(b - a))$.

1. INTRODUCTION

The well-known results of Nikolskii assert that the essentially sharp inequality

$$\|h_n\|_{L_q[-1, 1]} \leq c(p, q)n^{2/p-2/q}\|h_n\|_{L_p[-1, 1]}$$

holds for all algebraic polynomials h_n of degree at most n with complex coefficients and for all $0 < p < q \leq \infty$, while the essentially sharp inequality

$$\|t_n\|_{L_q[-\pi, \pi]} \leq c(p, q)n^{1/p-1/q}\|t_n\|_{L_p[-\pi, \pi]}$$

holds for all trigonometric polynomials t_n of degree at most n with complex coefficients and for all $0 < p < q \leq \infty$. The subject started with two famous papers, [5] and [6]. There are quite a few related papers in the literature. A recent one, for example, is [3].

Let V_n be a vectorspace of complex-valued functions defined on \mathbb{R} of dimension $n + 1$ over \mathbb{C} . We say that V_n is shift invariant (on \mathbb{R}) if $f \in V_n$ implies that $f_a \in V_n$ for every $a \in \mathbb{R}$, where $f_a(x) := f(x - a)$ on \mathbb{R} . Let $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ be a set of distinct COMPLEX numbers. The collection of all linear combinations of $e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}$ over \mathbb{C} will be denoted by

$$E(\Lambda_n) := \text{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}\}.$$

Received by the editors May 17, 2005.

2000 *Mathematics Subject Classification.* Primary 41A17.

Key words and phrases. Nikolskii-type inequalities, shift invariant function spaces, exponential sums.

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Elements of $E(\Lambda_n)$ are called exponential sums of $n + 1$ terms. Examples of shift invariant spaces of dimension $n + 1$ include $E(\Lambda_n)$. In a recent paper [4] the following essentially sharp Nikolskii-type inequality is proved.

Theorem A. *Suppose $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ is a set of distinct real numbers, $a, b \in \mathbb{R}$, $a < b$, and $0 < p \leq q \leq \infty$. There are constants $c_1 = c_1(p, q, a, b) > 0$ and $c_2 = c_2(p, q, a, b) > 0$ depending only on p, q, a , and b such that*

$$c_1 \left(n^2 + \sum_{j=0}^n |\lambda_j| \right)^{\frac{1}{p} - \frac{1}{q}} \leq \sup_{0 \neq P \in E(\Lambda_n)} \frac{\|P\|_{L_q[a,b]}}{\|P\|_{L_p[a,b]}} \leq c_2 \left(n^2 + \sum_{j=0}^n |\lambda_j| \right)^{\frac{1}{p} - \frac{1}{q}}.$$

Using the L_∞ norm on a fixed subinterval $[a + \delta, b - \delta] \subset [a, b]$ in the numerator in the above theorem, we proved the following essentially sharp result in [2].

Theorem B. *If $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ is a set of distinct real numbers, then the inequality*

$$\|f\|_{L_\infty[a+\delta, b-\delta]} \leq e 8^{1/p} \left(\frac{n+1}{\delta} \right)^{1/p} \|f\|_{L_p[a,b]}$$

holds for every $f \in E(\Lambda_n)$, $p > 0$, and $\delta \in (0, \frac{1}{2}(b-a))$.

Having real exponents λ_j in the above theorems is essential in the proof using some Descartes system methods. In this note we prove an analogous result for complex exponents λ_j , in which case Descartes system methods cannot help us in the proof.

2. NEW RESULT

Theorem. *Let $V_n \subset C[a, b]$ be a shift invariant vectorspace of complex-valued functions defined on \mathbb{R} of dimension $n + 1$ over \mathbb{C} . Let $p \in (0, 2]$. Then*

$$\|f\|_{L_\infty[a+\delta, b-\delta]} \leq 2^{2/p^2} \left(\frac{n+1}{\delta} \right)^{1/p} \|f\|_{L_p[a,b]}$$

for every $f \in V_n$ and $\delta \in (0, \frac{1}{2}(b-a))$.

Problem. Is it possible to extend a version of the theorem for ALL $p > 0$?

Proof. Since V_n is shift invariant, it is sufficient to prove only that

$$|f(0)| \leq 2^{2/p^2 - 1/p} (n+1)^{1/p} \|f\|_{L_p[-2,2]}$$

for every $f \in V_n$. Take an orthonormal basis $(L_k)_{k=0}^n$ on $[-\frac{1}{2}, \frac{1}{2}]$ so that

$$(1) \quad L_k \in V_n, \quad k = 0, 1, \dots, n,$$

and

$$(2) \quad \int_{-1/2}^{1/2} L_j(x) \overline{L_k(x)} dx = \delta_{j,k}, \quad 0 \leq j \leq k \leq n,$$

where $\delta_{j,k}$ is the Kronecker symbol. On writing $f \in V_n$ as a linear combination of L_0, L_1, \dots, L_n , and using the Cauchy-Schwarz inequality and the orthonormality of $(L_k)_{k=0}^n$ on $[-\frac{1}{2}, \frac{1}{2}]$, we obtain in a standard fashion that

$$\max_{0 \neq f \in V_n} \frac{|f(t_0)|}{\|f\|_{L_2[-1/2, 1/2]}} = \left(\sum_{k=0}^n |L_k(t_0)|^2 \right)^{1/2}, \quad t_0 \in \mathbb{R}.$$

Since

$$\int_{-1/2}^{1/2} \sum_{k=0}^n |L_k(x)|^2 dx = n + 1,$$

there exists a $t_0 \in [-\frac{1}{2}, \frac{1}{2}]$ such that

$$\max_{0 \neq f \in V_n} \frac{|f(t_0)|}{\|f\|_{L_2[-1/2, 1/2]}} = \left(\sum_{k=0}^n |L_k(t_0)|^2 \right)^{1/2} \leq \sqrt{n + 1}.$$

Observe that if $f \in V_n$, then g defined by $g(t) := f(t - t_0)$ is also in V_n , so

$$(3) \quad \max_{0 \neq f \in V_n} \frac{|f(0)|}{\|f\|_{L_2[-1, 1]}} \leq \sqrt{n + 1}.$$

We introduce

$$\tilde{V}_n := \{g : g(t) = f(\lambda t), \quad f \in V_n, \lambda \in [-2, 2]\}.$$

It follows from (3) that

$$\max_{0 \neq f \in \tilde{V}_n} \frac{|f(0)|}{\|f\|_{L_2[-1, 1]}} \leq \sqrt{n + 1}.$$

Let

$$C := \max_{0 \neq f \in \tilde{V}_n} \frac{|f(0)|}{\|f\|_{L_p[-2, 2]}}.$$

Let $0 \neq f \in \tilde{V}_n$. We define $g \in \tilde{V}_n$ by $g(t) = f(t/2 + y)$. Then

$$\frac{|f(y)|}{\|f\|_{L_p[-2, 2]}} \leq \frac{|f(y)|}{\|f\|_{L_p[y-1, y+1]}} \leq \frac{|g(0)|}{\|g\|_{L_p[-2, 2]}} 2^{1/p} \leq 2^{1/p} C, \quad y \in [-1, 1].$$

Hence

$$\max_{0 \neq f \in \tilde{V}_n} \frac{|f(y)|}{\|f\|_{L_p[-2, 2]}} \leq 2^{1/p} C, \quad y \in [-1, 1].$$

Therefore, for every $f \in \tilde{V}_n$,

$$\begin{aligned} |f(0)| &\leq \sqrt{n + 1} \|f\|_{L_2[-1, 1]} \\ &\leq \sqrt{n + 1} \left(\|f\|_{L_p[-1, 1]}^p \|f\|_{L_\infty[-1, 1]}^{2-p} \right)^{1/2} \\ &\leq \sqrt{n + 1} \left(\|f\|_{L_p[-1, 1]}^p \left(2^{1/p} C \right)^{2-p} \|f\|_{L_p[-2, 2]}^{2-p} \right)^{1/2} \\ &\leq \sqrt{n + 1} \left(2^{1/p} C \right)^{1-p/2} \|f\|_{L_p[-2, 2]} \\ &\leq 2^{1/p-1/2} \sqrt{n + 1} C^{1-p/2} \|f\|_{L_p[-2, 2]}. \end{aligned}$$

Hence

$$C = \max_{0 \neq f \in \tilde{V}_n} \frac{|f(0)|}{\|f\|_{L_p[-2, 2]}} \leq 2^{1/p-1/2} \sqrt{n + 1} C^{1-p/2},$$

and we conclude that

$$C \leq 2^{2/p^2-1/p} (n + 1)^{1/p}.$$

So

$$|f(0)| \leq 2^{2/p^2-1/p} (n + 1)^{1/p} \|f\|_{L_p[-2, 2]}$$

for every $f \in \tilde{V}_n$, and the result follows. □

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