

ON BOUNDED SOLUTIONS TO CONVOLUTION EQUATIONS

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ABSTRACT. Periodicity of bounded solutions for convolution equations on a separable abelian metric group G is established, and related Liouville type theorems are obtained. A non-constant Borel and bounded harmonic function is constructed for an arbitrary convolution semigroup on any infinite-dimensional separable Hilbert space, generalizing a classical result by Goodman (1973).

1. INTRODUCTION

Let μ be a probability measure defined on the σ -algebra $\mathcal{B}(G)$ of Borel subsets of a separable abelian metric group G , with the group operation “+”. This group might be non-locally compact. This paper is concerned with convolution equations of the type

$$(1.1) \quad f * \mu(x) := \int_G f(x - y)\mu(dy) = f(x), \quad x \in G,$$

where $f : G \rightarrow \mathbb{R}$ is a Borel and bounded function, written $f * \mu = f$ for short.

Our aim is to investigate bounded Borel solutions f to (1.1). These functions are also called bounded μ -harmonic functions; see [6]. Special attention will be paid to the case when G is a real separable Hilbert space.

Convolution equations arise naturally in several areas of pure and applied mathematics such as harmonic analysis (see [6], [4] and [15]), the theory of Markovian semigroups (see [9], [1] and [11]), and the renewal theory (see [10]).

Define the shift μ_h of the measure μ by the element $h \in G$, through the formula

$$\mu_h(A) = \mu(A - h), \quad A \in \mathcal{B}(G).$$

Consider the set $M_\mu \subset G$ of all elements $h \in G$ such that measures μ_h are absolutely continuous with respect to μ . The set M_μ has been introduced in [12] and investigated, for instance, in [24], [23] and [2]. Our main result (see Theorem 2.5) shows that each Borel and bounded solution f to (1.1) is periodic with periods in M_μ i.e.,

$$(1.2) \quad f(x + h) = f(x), \quad x \in G, \quad h \in M_\mu.$$

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The theorem holds in the more general setting of measurable abelian groups; see Remark 2.11. Moreover the set M_μ can be replaced by a larger set E_μ ; see (2.1).

The result seems particularly useful in infinite dimensions, when G is a separable Hilbert space H and there is no reference to any Haar measure. In Section 3.1 we consider an application to α -stable measures μ on H , $\alpha \in (0, 2]$.

A related equation,

$$(1.3) \quad \nu * \mu = \nu,$$

on a locally compact group G , with unknown non-negative measure ν , was the subject of a classical paper [3] by Choquet and Deny. Under suitable assumptions on ν (see Section 2.3), they proved that ν is periodic with periods in the subgroup generated by the support S_μ of μ (i.e., S_μ is the smallest closed set of G on which μ is concentrated). Within the class of locally compact groups, this result and related Liouville theorems have been extended in several directions; see [6], [15], [4], [21] and the references therein.

Even when G is locally compact, our result does not follow from [3]; see Remark 2.8. Moreover (1.2) does not hold if M_μ is replaced by S_μ . On the other hand, as a consequence of our main result, in Proposition 2.12 we obtain a version of the Choquet-Deny theorem concerning (1.3), which holds on metric groups (replacing the subgroup generated by S_μ with a smaller subgroup which contains M_μ). We also show in Theorem 2.9 that uniformly continuous and bounded solutions to (1.1) are periodic with periods in S_μ (see also Corollary 3.3.2 in [20]).

Equation (1.1) and related Liouville type theorems on a separable Hilbert space H are considered in Section 3. Let $(\mu_t)_{t \geq 0}$ be a convolution semigroup of probability measures on H . A function $f \in B_b(H)$ such that, for any $t \geq 0$,

$$(1.4) \quad f * \mu_t(x) = f(x), \quad x \in H,$$

is called a bounded harmonic function with respect to (μ_t) . In Theorem 3.3 we prove that if the space H is infinite dimensional, then one can always construct a discontinuous non-constant solution f to (1.4). This result generalises a well-known example of Goodman [14], concerning the case when (μ_t) are Gaussian measures. The Goodman result shows clearly that (1.2) cannot hold if M_μ is replaced by S_μ . This was one motivation for us to investigate (1.1).

Theorem 3.3 also implies that Markovian convolution semigroups associated to (μ_t) are never strong Feller in infinite dimensions; cf. [7].

2. CONVOLUTION EQUATIONS ON METRIC GROUPS

Let G be a separable abelian metric group with group operation indicated by $+$; see [16] and [23]. For $A, B \subset G$, we set

$$A + B = \{x + y, x \in A, y \in B\}, \quad -A = \{-x, x \in A\}.$$

Moreover $\text{Gr}(A)$ denotes the smallest subgroup containing A , and \overline{A} denotes the closure of A . The indicator function of a set $F \subset G$ will be indicated by 1_F .

When G is complete, we call it an abelian Polish group. When G is a real separable Hilbert space, we denote it by H . The inner product of H is then $\langle \cdot, \cdot \rangle$ and its norm is $|\cdot|$.

The probability measures on G we consider will always be Borel probability measures on the σ -algebra $\mathcal{B}(G)$. Let σ be a probability measure on G . The

support of σ is denoted by S_σ ; it is the smallest closed set in G which has measure 1 with respect to σ .

With $\tilde{\sigma}$ we indicate its reflection measure with respect to 0 (see [22, Chapter 1]), i.e. $\tilde{\sigma}(A) = \sigma(-A)$ for any $A \in \mathcal{B}(G)$. The probability measure σ is called symmetric if $\sigma = \tilde{\sigma}$, i.e. $\sigma(A) = \sigma(-A)$, $A \in \mathcal{B}(G)$.

For μ and ν probability measures on G , the convolution measure $\mu * \nu$ is defined by $\mu * \nu(A) := \int_G \mu(A - x)\nu(dx)$, $A \in \mathcal{B}(G)$; see for instance [16] or [23]. Note that the operation $*$ is commutative and associative.

We write $\mu^n = \mu * \dots * \mu$ (n -times), $n \in \mathbb{N}$ (\mathbb{N} denotes the set of all positive integers). We also set $\mu^0 = \delta_0$, where δ_0 is the Dirac measure concentrated in 0.

By $B_b(G)$ we denote the Banach space of all real, Borel and bounded functions $f : G \rightarrow \mathbb{R}$, endowed with the supremum norm $\|\cdot\|_\infty$. If $g \in B_b(G)$, we set

$$g * \mu(x) = \int_G g(x - y)\mu(dy), \quad x \in G.$$

2.1. Admissible shifts. Let T_a , $a \in G$, be the translation operator, i.e. $T_a(x) = x + a$, $x \in G$, and denote by $T_a \circ \mu$ the image of a probability measure μ under T_a , i.e. $(T_a \circ \mu)(A) = \mu(A - a)$, $A \in \mathcal{B}(G)$. We also set $(T_a \circ \mu) = \mu_a$.

According to [12, page 449], $a \in G$ is called an admissible shift for μ if $T_a \circ \mu$ is absolutely continuous with respect to μ . Let us denote by M_μ the set of all admissible shifts. Note that $0 \in M_\mu$. Moreover it is known that M_μ is always a semigroup of G ; see [12, page 450]. Since $M_{\tilde{\mu}} = -M_\mu$, where $\tilde{\mu}$ is the reflection measure of μ (see [24]), we have that M_μ is a subgroup of G if μ is symmetric.

If G is locally compact and μ is equivalent to the Haar measure of G , then it is easy to show that $M_\mu = G$.

We introduce the set

$$(2.1) \quad E_\mu = \bigcup_{n \geq 0} M_{\mu^n}, \quad \mu^n = \mu * \dots * \mu \text{ (n-times) } n \in \mathbb{N}.$$

We will need the following elementary result.

Proposition 2.1. *Let μ and ν be probability measures on G . Then*

$$(2.2) \quad M_\mu + M_\nu \subset M_{\mu * \nu}.$$

Moreover E_μ is a semigroup of G .

Proof. Take any $A \in \mathcal{B}(G)$ such that $\mu * \nu(A) = 0$ and $a \in M_\mu$, $b \in M_\nu$. It is enough to show that $(\mu * \nu)_{a+b}(A) = 0$. We have

$$(\mu * \nu)_{a+b}(A) = \int_G \int_G 1_A(x + y)p(x)q(y)\mu(dx)\nu(dy),$$

where p and q are the densities of μ_a and ν_b , respectively. For any $N \in \mathbb{N}$, set $p_N = N \wedge p$ and $q_N = N \wedge q$. Then

$$\int_G \int_G 1_A(x + y)p_N(x)q_N(y)\mu(dx)\nu(dy) \leq N^2 \int_G \int_G 1_A(x + y)\mu(dx)\nu(dy) = 0.$$

Taking $N \rightarrow \infty$, we get the first assertion. This implies that E_μ is an increasing union of semigroups. The second assertion follows easily. \square

The next example shows that the inclusion in (2.2) can be strict.

Example 2.2. There exists a probability measure μ on \mathbb{R}^n such that $M_\mu = 0$ and $M_{\mu^2} = \mathbb{R}^n = E_\mu$.

Define $\mu = \sum_{k \geq 1} p_k \nu_k$, where ν_k are uniform distributions on the spheres centered in 0, of radiuses $k \in \mathbb{N}$, and $\sum_{k \geq 1} p_k = 1$, $p_k > 0$, $k \in \mathbb{N}$. Since the measures $\nu_k * \nu_l$ are absolutely continuous with respect to Lebesgue measure (see [10]) and the density of $\nu_k * \nu_k$ is positive on the ball $B(0, 2k)$, we have $\mu * \mu = \sum_{k,l=1}^\infty p_k p_l \nu_k * \nu_l$, and $\mu * \mu$ has a positive density on \mathbb{R}^n . The assertion follows.

We now compare the set E_μ with the subgroup generated by the support S_μ of μ .

Proposition 2.3. *Let μ be a probability measure on G . Then $E_\mu \subset Gr(S_\mu)$.*

Proof. Fix any $h \in M_\mu$. It is straightforward to check that $S_\mu + h \subset S_\mu$.

Now take $x \in S_\mu$, $x \neq 0$ (if $S_\mu = \{0\}$, then μ is the Dirac measure concentrated in 0 and $M_\mu = \{0\}$ as well). We know that $x + h \in S_\mu \subset Gr(S_\mu)$. Since also $-x \in Gr(S_\mu)$, it follows that $h \in Gr(S_\mu)$. We have proved that $M_\mu \subset Gr(S_\mu)$.

For any $n \in \mathbb{N}$, one has $M_{\mu^n} \subset Gr(S_{\mu^n}) = Gr(S_\mu)$. Hence the assertion holds. \square

In general the sets M_μ and S_μ are different.

Example 2.4. There exists a probability measure μ on \mathbb{R}^n such that M_μ and S_μ are disjoint sets.

Take $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$, and $u \in \mathbb{R}^n$ such that $|u| = 1$. Consider the line $L = \{x \in \mathbb{R}^n : x = \lambda u + x_0, \text{ for } \lambda \in \mathbb{R}\}$.

Let μ be a probability measure on \mathbb{R}^n , concentrated on L , having a positive density with respect to the one-dimensional Lebesgue measure. We have that $M_\mu = \{\lambda u\}_{\lambda \in \mathbb{R}}$ and $S_\mu = L$.

2.2. Characterization theorems.

Theorem 2.5. *Let G be a separable abelian metric group. Let μ be a probability measure on G and let $E_\mu \subset G$ be defined in (2.1). Let $f \in B_b(G)$ be a μ -harmonic function. Then one has*

$$(2.3) \quad f(x + a) = f(x), \quad x \in G, \quad a \in Gr(E_\mu).$$

If, in addition, f is continuous on G and $Gr(E_\mu)$ is dense in G , then f is constant.

The proof uses the following result; see [8, Theorem 9, page 292].

Theorem 2.6. *Let $(\Omega, \mathcal{F}, \nu)$ be a measure space (with ν a positive measure). Let K be a bounded subset of $L^1(\Omega, \nu)$. Assume that, for each decreasing sequence $(E_n) \subset \mathcal{F}$ with empty intersection, the limit $\lim_{n \rightarrow \infty} \int_{E_n} f(s) \nu(ds) = 0$ is uniform with respect to $f \in K$. Then, for any sequence $(f_n) \subset K$, there exists a subsequence (f_{n_k}) which converges weakly in $L^1(\Omega, \nu)$.*

Proof of Theorem 2.5. We first define, similar to [3], suitable auxiliary functions and then obtain the required characterization arguing by contradiction. Both steps are accomplished differently from [3]. In particular, instead of the Ascoli-Arzelà theorem, we use arguments based on L^1 -weak compactness.

Let us introduce \tilde{f} , $\tilde{f}(x) = f(-x)$, $x \in G$. The equation $f * \mu = f$ is equivalent to

$$\tilde{f} * \tilde{\mu} = \tilde{f},$$

where $\tilde{\mu}$ is the reflection measure of μ . Fix $a \in M_\mu$ and introduce the function

$$g(x) = \tilde{f}(x) - \tilde{f}(x + a).$$

It is clear that $g \in B_b(G)$ and $g * \tilde{\mu} = g$ on G . Let

$$2c = \sup_{x \in G} g(x)$$

and $(x_n) \subset G$ such that $g(x_n) \rightarrow 2c$ as $n \rightarrow \infty$. Consider the functions $g_n : G \rightarrow \mathbb{R}$,

$$g_n(x) = g(x + x_n), \quad x \in G.$$

Each $g_n \in B_b(G)$ and solves the convolution equation (2.3). Now we set $L^1 = L^1(G, \mu)$ and use L^1 -weak convergence ($L^\infty(G, \mu)$ is identified with the topological dual of L^1). The proof proceeds in some steps.

Step I. The sequence (g_n) is relatively L^1 -weak sequentially compact.

We apply Theorem 2.6. To this purpose note that (g_n) is bounded in L^1 and moreover, for any decreasing sequence $(E_k) \subset \mathcal{B}(G)$, with empty intersection, one has

$$\sup_{n \geq 0} \left| \int_{E_k} g_n(y) \mu(dy) \right| \leq 2 \|f\|_\infty \mu(E_k),$$

which tends to 0 as $k \rightarrow \infty$. Hence, possibly passing to a subsequence, still denoted by (g_n) , we know that there exists $g_0 \in L^1$ such that, for any $h \in L^\infty(G, \mu)$,

$$\int_G g_n(y) h(y) \mu(dy) \rightarrow \int_G g_0(y) h(y) \mu(dy), \quad \text{as } n \rightarrow \infty.$$

Step II. The limit function $g_0 = 2c$, μ -a.s.

Note that, for $x \in G$,

$$(2.4) \quad g_n(x) = \int_G g_n(x - y) \tilde{\mu}(dy) = \int_G g_n(x + y) \mu(dy), \quad n \in \mathbb{N}.$$

Set $x = 0$ in (2.4). Using the L^1 -weak convergence, we get

$$(2.5) \quad 2c = \lim_{n \rightarrow \infty} g_n(0) = \int_G g_0(y) \mu(dy).$$

Now we prove that $g_0(x) \leq 2c$, μ -a.s. If this does not hold, then there exists $\epsilon > 0$ such that $B = \{x \in G : g_0(x) \geq 2c + \epsilon\}$ verifies $\mu(B) > 0$. But then, using that $g_n(x) \leq 2c$, $x \in G$, we find

$$2c\mu(B) \geq \int_B g_n(y) \mu(dy) = \int_G g_n(y) I_B(y) \mu(dy), \quad n \in \mathbb{N}.$$

Passing to the limit as $n \rightarrow \infty$, we infer a contradiction. By (2.5), we get the claim.

Step III. There exists a subsequence of (g_n) , still denoted by (g_n) , which converges pointwise to $2c$, μ -a.s.

It is enough to show that (g_n) converges to $2c$ in probability (with respect to μ). To this purpose, we write, using that $g_n \leq 2c$, for any $n \geq 1$,

$$\mu\left(x \in G : |g_n(x) - 2c| > \epsilon\right) \leq \frac{1}{\epsilon} \int_G (2c - g_n(y)) \mu(dy) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Step IV. For any $x \in M_\mu$,

$$(2.6) \quad \lim_{n \rightarrow \infty} g_n(x) = 2c.$$

By (2.4) we have, for any $x \in M_\mu$,

$$(2.7) \quad g_n(x) = \int_G g_n(y)(T_x \circ \mu)(dy) = \int_G g_n(y)F^x(y)\mu(dy), \quad x \in M_\mu, \quad n \in \mathbb{N},$$

where T_x is the translation operator and F^x denotes the density of $(T_x \circ \mu)$ with respect to μ . Now we write, for any $M > 0$,

$$\begin{aligned} |g_n(x) - 2c| &= \left| \int_G (g_n(y) - 2c)(T_x \circ \mu)(dy) \right| \\ &\leq \left| \int_{\{y: |F^x(y)| > M\}} (g_n(y) - 2c)(T_x \circ \mu)(dy) \right| \\ &\quad + \left| \int_{\{y: |F^x(y)| \leq M\}} (g_n(y) - 2c)F^x(y)\mu(dy) \right| \\ &\leq 2(\|f\|_\infty + |c|)(T_x \circ \mu)(\{|F^x(y)| > M\}) \\ &\quad + \left| \int_G (g_n(y) - 2c)h^x(y)\mu(dy) \right|, \end{aligned}$$

where $h^x(y) = F^x(y)I_{\{y: |F^x(y)| \leq M\}}(y)$. For any $\epsilon > 0$, we can choose $M > 0$ and $n_0 \in \mathbb{N}$ large enough, such that $n \geq n_0$ implies $|g_n(x) - 2c| \leq 2\epsilon$. The claim is proved.

Final Step. Recall that M_μ is a semigroup in G ; see [12, page 450]. This fact and (2.6) imply that

$$g_0(ka) = 2c, \quad k \in \mathbb{N}.$$

Now we complete the proof similarly to Choquet-Deny [3]. For any integer m , there exists \hat{n} such that

$$(2.8) \quad g_{\hat{n}}(ka) = \tilde{f}(x_{\hat{n}} + ka) - \tilde{f}(x_{\hat{n}} + (k-1)a) > c,$$

for $k = 1, \dots, m$. Summing (2.8) m -times, we get $\tilde{f}(x_{\hat{n}} + ma) - \tilde{f}(x_{\hat{n}}) > mc$.

Letting $m \rightarrow \infty$, we find that $c \leq 0$, since \tilde{f} is bounded. This means that $g(x) \leq 0$, $x \in G$, i.e.

$$\tilde{f}(x) \leq \tilde{f}(x+a), \quad x \in G.$$

Repeating the previous argument with $-f$ instead of f , one has $f(x) = f(x+a)$, $x \in G$. Thus (2.3) holds, for any $a \in M_\mu$. Now (1.1) implies that, for any $n \in \mathbb{N}$, $f * \mu^n(x) = f(x)$, $x \in G$. Hence (2.3) holds, for any $a \in E_\mu$. The assertion follows, remarking that the set of all periods of a given real function on G is a subgroup of G . The proof is complete. \square

Remark 2.7. If μ is symmetric, then $\text{Gr}(E_\mu) = E_\mu$, and so the formulation of Theorem 2.5 simplifies. Indeed if μ is symmetric, then M_μ is a subgroup of G . By Proposition 2.1 we know that E_μ is an increasing union of subgroups. Hence E_μ is a group.

Remark 2.8. Theorem 2.5 does not hold if we replace E_μ with the subgroup generated by the support S_μ of μ .

Let $G = \mathbb{R}^d$ and take $\mathbb{Q}^d = \{q_n\}$ to be the set of all points in \mathbb{R}^d having rational coordinates. Let $(p_n) \subset \mathbb{R}_+$ be such that $\sum_{n \geq 1} p_n = 1$. Define $\mu = \sum_{n \geq 1} p_n \delta_{q_n}$, where δ_{q_n} are Dirac measures concentrated in q_n .

Take $f = 1_{\mathbb{Q}^d}$ to be the indicator function of \mathbb{Q}^d . It is straightforward to check that $\mu * f(x) = f(x)$, $x \in \mathbb{R}^d$. Moreover the support $S_\mu = \mathbb{R}^d$ and $M_\mu = \mathbb{Q}^d$.

Note that if $h \notin \mathbb{Q}^d$, then $f(x+h) = f(x)$ only if $x \notin A_h$, where $A_h = -h + \mathbb{Q}^d$ has Lebesgue measure 0.

If we restrict our attention to bounded μ -harmonic functions which are also uniformly continuous on G , then we can prove periodicity with respect to $\text{Gr}(S_\mu)$. In the terminology of [4, page 396] the next result shows that any Polish abelian group has the Liouville property. We denote by $UC_b(G)$ the space of all uniformly continuous and bounded functions from G into \mathbb{R} .

Theorem 2.9. *Let G be a Polish abelian group. Let μ be a probability measure on G . Let $f \in UC_b(G)$ be a solution to $f * \mu = f$. Then,*

$$(2.9) \quad f(x+a) = f(x), \text{ for any } x \in G, a \in \text{Gr}(S_\mu).$$

In [19] we provide the complete proof. It uses the same arguments given in [3], but the next lemma is needed.

Lemma 2.10. *Let G be an abelian Polish group. There exists a subgroup $S_0 \subset G$, which is a countable union of compact sets and has the property that $\mu(S_0) = 1$.*

Proof. First choose compact sets G_n such that $0 \in G_n$, $G_n \subset G_{n+1}$ and $\mu(G \setminus G_n) < 1/n$.

Define new compacts $F_n = (-G_n) \cup G_n$, $n \geq 1$, and finally set

$$K_1 = F_1, K_2 = F_2 + F_2, \dots, K_n = F_n + \dots + F_n \text{ (} n\text{-times)}, \dots$$

It is easy to check that $S_0 = \bigcup_{n \geq 1} K_n$ has all the required properties. \square

Remark 2.11. Theorem 2.5 holds more generally, with the same proof, if G is a measurable abelian group; see [23, page 63]. A measurable space (G, \mathcal{A}) which is also a group (with additive notation) is said to be a measurable group if the group operations $(x, y) \mapsto x + y$ and $x \mapsto -x$ are both measurable (on $G \times G$ one considers the product σ -algebra $\mathcal{A} \times \mathcal{A}$). In measurable groups the convolution of finite measures on \mathcal{A} is naturally defined. Note that a separable metric group and a locally compact group, with \mathcal{A} being the Borel σ -algebra, are both examples of measurable groups.

2.3. Connections with the Choquet-Deny theorem. It is interesting to compare our result with the remarkable theorem due to Choquet and Deny, mentioned in the Introduction, valid in *locally compact groups* G ; see [3]. Their theorem is concerned with the equation

$$(2.10) \quad \nu * \mu = \nu,$$

where μ is a given probability measure on G and unknown ν is a σ -finite Borel measure on G such that, for any compact set $K \subset G$, the Borel non-negative function

$$(2.11) \quad G \rightarrow \mathbb{R}_+, \quad x \mapsto \nu(x - K) = 1_K * \nu(x) \text{ is finite and bounded on } G.$$

It turns out that ν is periodic with periods in the subgroup generated by the support S_μ of μ , i.e.

$$(2.12) \quad \nu(A+h) = \nu(A), \quad h \in \text{Gr}(S_\mu), A \in \mathcal{B}(G).$$

This result can be applied to the study of equation (1.1). Let us interpret function f as a density of a measure ν with respect to the Haar measure \mathcal{L} of G : $f = \frac{d\nu}{d\mathcal{L}}$.

Then (2.12) implies that, for any $h \in S_\mu$, $f(x+h) = f(x)$, \mathcal{L} -a.s., where the set of \mathcal{L} -measure 0 depends, in general, on h ; see Remark 2.8.

The following corollary of Theorem 2.5 can be regarded as a version of [3, Theorem 1] in the non-locally compact case.

Proposition 2.12. *Let μ be a given probability measure on a Polish abelian group G . Let ν be a σ -finite Borel measure on G satisfying the condition (2.11) for any compact set $K \subset G$. If $\nu * \mu = \nu$, then ν is periodic with periods in E_μ ; see (2.12).*

Proof. First take a compact set $K \subset G$ and consider the indicator function of K , i.e. 1_K . We have $(1_K * \nu) * \mu = 1_K * \nu$. Indeed, by the Fubini theorem,

$$1_K * (\nu * \mu)(x) = \int_{G^2} 1_K(x-y-z)\nu(dy)\mu(dz) = (1_K * \nu) * \mu(x), \quad x \in G.$$

By Theorem 2.5 we get that $1_K * \nu(x+h) = 1_K * \nu(x)$, $x \in G$, $h \in E_\mu$. Hence, by taking $x = 0$, we get $\nu(h-K) = \nu(K)$, and so

$$(2.13) \quad \nu(h+K) = \nu(K), \quad \text{for any compact set } K \subset G, h \in E_\mu.$$

By the inner regularity of ν , for any Borel set $A \subset G$ with $\nu(A) < \infty$, there exists an increasing sequence of compact sets (K_n) , such that $K_n \subset A$, $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} \nu(K_n) = \nu(A)$. It follows that $\nu(h+A) = \nu(A)$, for any $h \in E_\mu$. The proof is complete. \square

We do not know if the previous result holds when G is a Polish abelian group and E_μ is replaced by the subgroup generated by the support of μ .

Remark 2.13. Proposition 2.12 holds more generally, with the same proof, if G is a Hausdorff topological abelian group. In this case we assume that the measures μ and ν are both *Radon measures* (a non-negative Borel σ -finite measure γ on a Hausdorff topological space X is called Radon, if for each Borel set $B \subset X$ with $\gamma(B) < \infty$, for any $\epsilon > 0$ there exists a compact set $K \subset B$ such that $\gamma(B \setminus K) < \epsilon$; see for instance [23]).

3. CONVOLUTION EQUATIONS ON HILBERT SPACES

3.1. The case of stable measures. Let $G = H$ be a real separable Hilbert space, Q a non-negative trace class operator on H and $\alpha \in (0, 2]$. A probability measure μ on H is said to be (α, Q) -stable, $\alpha \in (0, 2]$, centered at $x \in H$, if its characteristic function is

$$(3.1) \quad \hat{\mu}(h) := \int_H e^{i\langle h, y \rangle} \mu(dy) = \exp(i\langle x, h \rangle) \exp\left(-\left(\frac{\langle Qh, h \rangle}{2}\right)^{\alpha/2}\right), \quad h \in H;$$

see [16], [23], [22]. Such measures will be denoted by $N_\alpha(x, Q)$. Measures $N_2(x, Q)$ are Gaussian. In this case we also write $N(x, Q)$.

The following result extends Theorem 4.3.4 in [7].

Proposition 3.1. *Let $\mu = N_\alpha(x, Q)$, $\alpha \in (0, 2]$. If $f \in B_b(H)$ solves equation (1.1), then*

$$(3.2) \quad f(y + Q^{1/2}a) = f(y), \quad y \in H, \quad a \in H.$$

If, in addition, f is continuous and Q positive definite, then f is constant on H .

Proof. First we show the result for $\alpha = 2$.

Lemma 3.2. *Let $\mu = N(x, Q)$ and $\nu = N(y, S)$ be Gaussian measures on H . Then,*

$$(3.3) \quad M_{\mu*\nu} = M_\mu + M_\nu = Q^{1/2}H + S^{1/2}H.$$

In particular $E_\mu = M_\mu = Q^{1/2}H$.

Proof. It is well known that $M_\mu = Q^{1/2}H$; see [7]. Moreover $\mu*\nu = N(x+y, Q+S)$.

Define the linear operator $T : H \times H \rightarrow H$, $T(x, y) = Q^{1/2}x + S^{1/2}y$ (where as usual $\langle(x, y), (x', y')\rangle := \langle x, x'\rangle + \langle y, y'\rangle$). We easily check that

$$(3.4) \quad |(Q + S)^{1/2}h|^2 = |T^*h|^2, \quad h \in H,$$

where T^* denotes the adjoint of T . By a classical duality argument, we have that $(Q + S)^{1/2}H = Q^{1/2}H + S^{1/2}H$. The proof is complete. \square

Continuing the proof of the proposition note that by Lemma 3.2, $Q^{1/2}H = E_{N_2}$. Thus Theorem 2.5 gives the first claim. The second one follows from the density of $Q^{1/2}H$ in H when Q is non-degenerate.

Let us now consider $\alpha \in (0, 2)$ and set $N_\alpha = N_\alpha(x, Q)$. We show that $Q^{1/2}H \subset M_{N_\alpha}$, $\alpha \in (0, 2]$. For this we use subordination. Let ν_α be an α -stable distribution ν^α on $[0, +\infty)$, with the Laplace transform given by

$$\int_0^\infty e^{-\lambda s} \nu^\alpha(ds) = e^{-(\lambda)^\alpha/2}, \quad \lambda > 0.$$

It is easy to check that

$$(3.5) \quad N_\alpha(B) := \int_0^\infty N_2(x, sQ)(B) \nu^\alpha(ds), \quad B \in \mathcal{B}(H).$$

Take $A \in \mathcal{B}(H)$ such that $N_\alpha(A) = 0$; then, $N_2(x, sQ)(A) = 0$, for any $s \in S_{\nu^\alpha} = \mathbb{R}_+$. Let $g = Q^{1/2}h$, for some $h \in H$. By the absolute continuity of the Gaussian measures,

$$(T_g \circ N_\alpha)(A) = N_\alpha(A + g) = \int_0^\infty N_2(x - g, sQ)(A) \nu^\alpha(ds) = 0.$$

Hence, $Q^{1/2}(H) \subset M_{N_\alpha}$. By Theorem 2.5 we get the claim. \square

For information about the set of all admissible shifts for general α -stable measures, we refer to [2] and [24].

3.2. Liouville type theorems on Hilbert spaces. Let μ_t , $t \geq 0$, be a convolution semigroup of measures on a real separable Hilbert space H . This means that $\mu_t * \mu_s = \mu_{t+s}$, $t, s \geq 0$, μ_0 is the Dirac measure concentrated in 0 and μ_t is weakly continuous at $t = 0$. Let P_t be the Markovian convolution semigroup determined by μ_t , $t \geq 0$,

$$(3.6) \quad P_t f(x) = \int_H f(x - y) \mu_t(dy) = f * \mu_t(x), \quad x \in H, t \geq 0, f \in B_b(H).$$

See [16], [23], [13] and [22] for more information on convolution semigroups and Lévy processes. A function $h \in B_b(H)$ is said to be a *bounded harmonic function* for P_t , briefly a BHF for P_t (see [1], [9], [17] and [18]), if

$$(3.7) \quad P_t h = h, \quad t \geq 0.$$

In particular, when P_t is a compound convolution semigroup, i.e. $P_t f(x) = e^{-\lambda t} \sum_{k \geq 0} \frac{(t\lambda)^k}{k!} (f * (\nu)^k)(x)$, where $\lambda > 0$ and ν is a given probability measure on H , one has that h is a BHF for P_t if and only if h is a bounded ν -harmonic function.

We now present a Liouville type theorem about BHFs for convolution semigroups. The first part is a consequence of Theorem 2.5. The second one is a generalization of a surprising result obtained by Goodman [14]. It states that in infinite dimensions there exist non-constant BHFs for the heat semigroup (see also [7], Section 4.3.1).

Theorem 3.3. 1) Let P_t be the Markovian semigroup (3.6) on a separable Hilbert space H and let

$$(3.8) \quad \Gamma = Gr\left(\bigcup_{t \geq 0} M_{\mu_t}\right).$$

Then each BHF h for P_t is periodic with periods in Γ . If h is continuous and $\overline{\Gamma} = H$, then h is constant.

2) For arbitrary semigroups (3.6) there exists a non-constant BHF ϕ if H is infinite dimensional.

Proof. 1) It follows from Theorem 2.5. We only note that, by Proposition 2.1, one has $Gr\left(\bigcup_{t \geq 0} E_{\mu_t}\right) = \Gamma$.

2) We use a probabilistic representation of convolution semigroups; see [13].

There exists a Lévy process (Z_t) on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, with values in H , such that the law of each Z_t is μ_t , $t \geq 0$. The process (Z_t) can be represented as

$$(3.9) \quad Z_t = at + \eta_t + \xi_t, \quad t \geq 0,$$

where $a \in \mathbb{R}^n$, (η_t) is a square integrable martingale and (ξ_t) is a compound Poisson process. Moreover the processes (η_t) and (ξ_t) are independent, and so in particular

$$(3.10) \quad P_t f(x) = \mathbb{E}f(x - \xi_t - \eta_t - at) = f * \nu_t * r_t(x),$$

where ν_t is the law of $at + \eta_t$ and r_t the law of ξ_t . Thus it is enough to construct a non-constant function ϕ such that

$$(3.11) \quad \phi * \nu_t = \phi \quad \text{and} \quad \phi * r_t = \phi, \quad t \geq 0.$$

Let us first consider $\eta_t + at$ with law ν_t . Remark that there exists a non-negative trace class operator $Q : H \rightarrow H$, such that the following holds:

$$\langle Qh, k \rangle = \frac{1}{t} \mathbb{E}(\langle \eta_t, h \rangle \langle \eta_t, k \rangle), \quad t > 0, \quad h, k \in H.$$

Let us choose an orthonormal basis (e_k) in H , such that $Qe_k = \lambda_k e_k$. We have that $\sum_{k \geq 1} \lambda_k < \infty$. Let (α_k) be a sequence of positive numbers, diverging to $+\infty$, such that

$$(3.12) \quad \sum_{k \geq 1} \lambda_k \alpha_k + \sum_{k \geq 1} a_k^2 \alpha_k < \infty, \quad a_k = \langle a, e_k \rangle, \quad k \in \mathbb{N},$$

and define the linear subspace K ,

$$(3.13) \quad K = \{x \in H : g(x) < \infty\}, \quad g(x) = \sum_{k \geq 1} x_k^2 \alpha_k, \quad x \in H, \quad x_k = \langle x, e_k \rangle.$$

Since (α_k) diverges, one has that K is *strictly* contained in H . Moreover $a \in K$ by construction. It turns out that the law of $\eta_t + at$ is concentrated on K , for any $t \geq 0$. Indeed one has

$$\mathbb{E}g(\eta_t + at) = \sum_{k \geq 1} \alpha_k \langle \eta_t, e_k \rangle^2 + t \sum_{k \geq 1} a_k^2 \alpha_k < \infty,$$

and so $\eta_t + at$ is almost surely in K , for any $t \geq 0$. Note that

$$(I_K) * \nu_t(x) = \int_H I_K(x - y) \nu_t(dy) = \nu_t(x - K) = I_K(x), \quad x \in H, t \geq 0.$$

Indeed if $x \notin K$, then $x - k \notin K$, for any $k \in K$. This gives that I_K is a non-constant BHF for the convolution semigroup determined by the process $(\eta_t + at)$.

Let us consider the remainder compound Poisson process (ξ_t) ; see (3.9). Denote by $\lambda > 0$ its intensity, by ν its Lévy measure and by S_t the associated convolution semigroup. One has

$$(3.14) \quad S_t f(x) = \mathbb{E}f(x - \xi_t) = e^{-\lambda t} \sum_{k \geq 0} \frac{(t\lambda)^k}{k!} (f * (\nu)^k)(x),$$

where $\mathbb{E}(e^{i\langle \xi_t, h \rangle}) = e^{-t\psi(h)}$, $h \in H$, $t \geq 0$, and $\psi(h) = \int_H (e^{i\langle x, h \rangle} - 1) \nu(dx)$. As already noted, $h \in B_b(H)$ is a BHF for S_t if and only if $h * \nu = h$.

Let us first construct a non-constant bounded ν -harmonic function h . Let us introduce $\lambda'_k = \mathbb{E}(e^{-|U|} \langle U, e_k \rangle^2)$, where $U : \Omega \rightarrow H$ is a random variable with law ν . It is clear that

$$\sum_{k \geq 1} \lambda'_k = \mathbb{E}(e^{-|U|} |U|^2) < \infty.$$

Take a diverging sequence of positive real numbers (α'_k) , such that $\sum_{k \geq 1} \lambda'_k \alpha'_k < \infty$ and define the linear subspace K' , $K' = \{x \in H : \sum_{k \geq 1} x_k^2 \alpha'_k < \infty\}$. One has that h is a non-constant BHF for S_t .

To finish the proof, define $\tilde{\alpha}_k = \min(\alpha_k, \alpha'_k)$, and introduce the new subspace $\tilde{K} = \{x \in H, \sum_{k \geq 1} x_k^2 \tilde{\alpha}_k < \infty\}$. Repeating the previous arguments, we find that $\phi = I_{\tilde{K}}$ is non-constant and verifies (3.11). This completes the proof. \square

We recall that a Markovian semigroup P_t , acting on $B_b(H)$, is called *strong Feller* if $P_t f$ is continuous on H , for any $f \in B_b(H)$, $t > 0$; see for instance [7].

Corollary 3.4. *Markovian convolution semigroups P_t , given by (3.6) on an infinite dimensional Hilbert space H , are never strong Feller.*

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