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ON THE BOUNDARIES OF SELF-SIMILAR TILES IN $\mathbb{R}^1$

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ABSTRACT. The aim of this note is to study the construction of the boundary of a self-similar tile, which is generated by an iterated function system $\{\phi_i(x) = \frac{1}{N}(x + d_i)\}_{i=1}^N$. We will show that the boundary has complicated structure (no simple points) in general; however, it is a regular fractal set.

1. INTRODUCTION

Let N be a positive integer and let $\mathcal{D} = \{d_1, d_2, \ldots, d_q\} \subset \mathbb{R}$ be a set of real numbers. In this note we consider an *iterated function system* (IFS) $\{\phi_i(x)\}_{i=1}^q$ defined as

$$\phi_i(x) = \frac{1}{N}(x + d_i), \qquad i = 1, 2, \ldots, q.$$

It is well known that there exists a unique nonempty compact set T satisfying

$$T = \bigcup_{i=1}^{q} \phi_i(T)$$

(see, e.g., [F]). We call T a *self-similar set*. If T, written as $T(N, \mathcal{D})$, has nonempty interior and $q = N$, T is termed a *self-similar tile*. It was proved by Kenyon [K] and Lagarias and Wang [LW] that, if T is a self-similar tile, the set $\mathcal{D}$ can be rationalized, that is, there exist real numbers a and c such that $\mathcal{D} = c\mathcal{D}' + a$ and $\mathcal{D}' \subset \mathbb{Z}$. We will mainly study the geometric properties of self-similar tiles, so we can assume that $\mathcal{D}$ lies in $\mathbb{Z}$. Note that $T(N, \mathcal{D} - k) = T(N, \mathcal{D}) - k/(N-1)$. Without loss of generality we can assume that $d_1 = 0 < d_2 < \cdots < d_q$ throughout this paper.

Now we introduce the concept *product form* defined by Odlyzko [O] and Lagarias and Wang [LW2]. Denote $E + F := \{x + y : x \in E,\ y \in F\}$ for any two sets E and F. For the given N, $\mathcal{D}$ is said to have the product form if there is a residue system $\mathcal{E} \pmod N$ with $0 \in \mathcal{E}$ so that

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_k$$

with all sums distinct and

$$\mathcal{D} = N^{l_1}\mathcal{E}_1 + N^{l_2}\mathcal{E}_2 + \cdots + N^{l_k}\mathcal{E}_k,$$

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where $0 \in \mathcal{E}_i$ for $1 \le i \le k$, $\#\mathcal{E}_i$, the cardinality of $\mathcal{E}_i$, is larger than one and all l_i are integers with $0 \le l_1 \le l_2 \le l_3 \le \cdots \le l_k$. $\mathcal{D}$ is said to have the *strict product form* if $\mathcal{E} = \{0, 1, \ldots, N-1\}$. For example, for $N = 4$, $\mathcal{D}_1 = \{0,1\} + 4\{0,2\} = \{0,1,8,9\}$ is a strict product form and $\mathcal{D}_2 = \{0,5\} + 4\{0,2\} = \{0,5,8,13\}$ is a (not strict) product form.

Theorem 1.1 ([O], [LW2])**.** *If $\mathcal{D}$ has the product form, then the self-similar set $T(N,\mathcal{D})$ is a self-similar tile. Moreover, $T(N,\mathcal{D})$ is a union of finite closed intervals if and only if $\mathcal{D}$ has the strict product form.*

The sufficient condition ($\mathcal{D}$ is a product form) in Theorem 1.1 is far from being necessary. In general it is difficult to characterize the set $\mathcal{D}$ for an integer N so that $T(N,\mathcal{D})$ is a tile; only the case $N = p^n$ and $N = pr$, where p and r are prime, was solved by Lagarias and Wang [LW2], Lau and Rao [LR] respectively. Theorem 1.1 implies that the interior of a tile $T(N,\mathcal{D})$ contains an infinite number of disjoint open intervals if $\mathcal{D}$ is not a strict product form. What can we say about the construction of the tile in this case?

Some notions and initial ideas come from Xu [X]. Let $x \in \partial T$ be a point on the boundary of T; x is called a *simple point* if there exists $\epsilon > 0$ such that either $(x-\epsilon, x) \cap T = \emptyset$ and $(x, x+\epsilon) \cap T = (x, x+\epsilon)$ or $(x-\epsilon, x) \cap T = (x-\epsilon, x)$ and $(x, x+\epsilon) \cap T = \emptyset$.

Theorem 1.2. *If $T = T(N,\mathcal{D})$ is a self-similar tile but $\mathcal{D} \subset \mathbb{Z}$ is not a strict product form, then the boundary ∂T of T contains no simple points.*

Theorem 1.2 implies that ∂T is a nonempty compact set which has no isolated points. Then it contains infinite members. Moreover we have the following result:

Theorem 1.3. *If $T = T(N,\mathcal{D})$ is a self-similar tile but $\mathcal{D} \subset \mathbb{Z}$ is not a strict product form, then the boundary ∂T of T is a fractal, that is,* $0 < \dim_H \partial T < 1$.

We remark that $\dim_H \partial T < 1$ was already established in Strichartz and Wang [SW]. Also the above two theorems no longer hold in the nonuniform dilations setting. For example, let $f_1(x) = \frac{2}{3}x$, $f_2(x) = \frac{2}{9}x + \frac{4}{9}$ and $f_3(x) = \frac{1}{9}x + \frac{8}{9}$. Then it can be checked that the self-similar set is

$$T = \bigcup_{k=0}^{\infty} [1 - \frac{1}{3^{2k}}, 1 - \frac{1}{3^{2k+1}}] \cup \{1\}.$$

Then ∂T has simple points and $\dim_H \partial T = 0$. For the definitions of Hausdorff and box dimensions we refer to [F]. In general we have

Theorem 1.4. *Let $T = T(N,\mathcal{D})$ be a self-similar set with $\mathcal{D} \subset \mathbb{Z}$ and $\#\mathcal{D} = q \ge N$. Then ∂T is a regular set, i.e., $\dim_H \partial T = \dim_B \partial T$.*

2. Proof of Theorem 1.2

Throughout this section we shall assume that the digit set $\mathcal{D}$ for the self-affine tile $T(N,\mathcal{D})$ has the form $\mathcal{D} = \{0, d_2, \ldots, d_N\}$ with $d_j \in \mathbb{Z}$ and $0 < d_2 < \cdots < d_N$.

Let $\Sigma_q = \{1, 2, \ldots, q\}$, $\Sigma_q^n = \{(i_1, i_2, \ldots, i_n) : \text{all } i_s \in \Sigma_q\}$ and $\Sigma_q^* = \bigcup_{n=1}^{\infty} \Sigma_q^n$. For the IFS $\phi_i = \frac{1}{N}(x + d_i)$, $i = 1, 2, \ldots, q$, and $\sigma = (i_1, i_2, \ldots, i_n) \in \Sigma_q^*$, as usual we define $\phi_\sigma(x) = \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_n}(x)$, the composition of ϕ_{i_s} $(1 \le s \le n)$. We say that an IFS $\{\phi_i(x)\}_{i=1}^q$ satisfies the *open set condition* (OSC) if there exists a nonempty open set O (bounded) such that

$$\bigcup_{i=1}^{q} \phi_i(O) \subseteq O \quad \text{and} \quad \phi_i(O) \cap \phi_j(O) = \emptyset \quad \forall i \neq j.$$

We remark that an IFS which generates a self-similar tile satisfies the OSC. This property will be used in the following. We say that a sequence of open intervals $\{(a_n, b_n)\}_{n=1}^\infty$ is *monotonically decreasing to* a if $b_{n+1} < a_n$ for all $n \geq 1$ and $\lim_{n\to\infty} a_n = a$; Similarly a sequence of open intervals $\{(a_n, b_n)\}_{n=1}^\infty$ is *monotonically increasing to* b if $b_n < a_{n+1}$ for all $n \geq 1$ and $\lim_{n\to\infty} b_n = b$. Since T can be expressed explicitly as

$$T = \{\sum_{k=1}^{\infty} N^{-k} d_k : d_k \in \mathcal{D}\},$$

clearly $T \subset [0, d_q/(N-1)]$.

Lemma 2.1. *Let $T(N, \mathcal{D})$ be a self-similar tile and let $a_1 = 0$. Suppose that $T \cap [0, b_k] = \bigcup_{i=1}^{k} [a_i, b_i]$ with $b_i < a_{i+1}$, $i = 1, 2, \ldots, k-1$, and $b_k \in \partial T$. Then b_k is a simple point.*

Proof. Suppose that b_k is not a simple point. Then there exists a sequence of open intervals $\{(\alpha_n, \beta_n)\}_{n=1}^\infty$ monotonically decreasing to b_k satisfying $(\alpha_n, \beta_n) \subseteq T$ and for each $n \geq 1$ there is an open interval $(\alpha'_n, \beta'_n) \subseteq (\beta_{n+1}, \alpha_n)$ such that $(\alpha'_n, \beta'_n) \cap T = \emptyset$. Note that $T = \bigcup_{i=1}^{N} \phi_i(T)$ and $\phi_1(b_k) = b_k/N < b_k$. Then there exists i, $1 \leq i \leq k$, such that $\phi_1(b_k) \in [a_i, b_i]$. We claim that $\phi_1(b_k) \neq b_i$. In fact, if $\phi_1(b_k) = b_i$, since the sequence $\{(\alpha_n, \beta_n)\}_{n=1}^\infty$ is monotonically decreasing to b_k, then $\phi_1((\alpha_n, \beta_n)) \subseteq (b_i, a_{i+1})$ for n large enough, which contradicts $\phi_1(T) \subseteq T$. Hence $\phi_1(b_k) \in [a_i, b_i)$.

Now we show that $\phi_1(b_k) < \phi_2(a_k)$. Suppose otherwise, that is, $\phi_1(b_k) \in \phi_2([a_k, b_k))$. Then for n large enough, we have $\phi_1((\alpha_n, \beta_n)) \subseteq \phi_2([a_k, b_k))$, which contradicts the OSC. Hence $\phi_1(b_k) < \phi_2(a_k)$. In this case there is an integer m such that $\phi_1((\alpha'_n, \beta'_n)) \subseteq [a_i, b_i)$ and $\phi_1(\beta'_n) < \phi_2(a_k)$ for all $n \geq m$. This implies that $\bigcup_{n=m}^{\infty} \phi_1((\alpha'_n, \beta'_n))$ can be covered by $\bigcup_{i=2}^{N} \bigcup_{j=1}^{k} \phi_i([a_j, b_j])$. Thus one of the intervals of the latter contains two adjacent intervals of the former, which contradicts the OSC again by the definitions of the sequences. So the result follows. □

Lemma 2.2. *Let $T(N, \mathcal{D})$ be a self-similar tile and let $a_1 = 0$. Suppose that $T \cap [0, a_{k+1}] = a_{k+1} \cup \bigcup_{i=1}^{k} [a_i, b_i]$ with $b_i < a_{i+1}$, $i = 1, 2, \ldots, k$. Then a_{k+1} is a simple point.*

Proof. Let $\eta = a_{k+1} - b_k > 0$ and choose n such that, for any $\sigma \in \Sigma_N^n$, the diameter of $\phi_\sigma(T)$ is less than η. Since $T = \bigcup_{\sigma \in \Sigma_N^n} \phi_\sigma(T)$, there exists $\sigma \in \Sigma_N^n$ such that $a_{k+1} \in \phi_\sigma(T)$. Observing $(b_k, a_{k+1}) \cap T = \emptyset$ and the choice of n, we have $a_{k+1} = \phi_\sigma(0)$, and thus a_{k+1} is the left end point of the closed interval $\phi_\sigma([0, b_1])$ which is contained in T. This implies that a_{k+1} is a simple point. □

Lemma 2.3. *Suppose that $T = T(N, \mathcal{D})$ is a self-similar tile but $\mathcal{D}$ is not of the strict product form. Then both* 0 *and $d_N/(N-1)$ are not simple points.*

Proof. Suppose that 0 is a simple point of ∂T. Since $\mathcal{D}$ is not of the strict product form, the interior T^o of T consists of countable disjoint open intervals without common end points. Let $b_1 = \max\{x : [0,x] \subseteq T\}$; then by Lemma 2.1 b_1 is a simple point. Denote by a_2 the smallest point in T which is larger than b_1. By Lemma 2.2 the point a_2 is simple too. With the same idea it is easy to see that there exists a point $a \in T$ and a monotonically increasing sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$ which converges to a such that $a_1 = 0$ and

$$T \cap [0,\ a] = \bigcup_{n=1}^{\infty} [a_n, b_n] \cup \{a\}.$$

It is obvious that there exists an i such that $\phi_1(a) \in (a_i, b_i]$ and m_1 such that $\phi_1([a_n, b_n]) \subset (a_i, \phi_1(a))$ for $n \geq m_1$. Since $\phi_j(a) > \phi_1(a)$ for $j \geq 2$, there exists m_2 such that $\phi_j(a_n) > \phi_1(a)$ for $n \geq m_2$ and $2 \leq j \leq N$. Let $m = \max\{m_1, m_2\}$. Hence $\bigcup_{s=m}^{\infty} \phi_1((b_s, a_{s+1})) \subset (a_i, \phi_1(a))$ can be covered by $\bigcup_{j=2}^{N} \bigcup_{n=1}^{m} \phi_j([a_n, b_n])$. This is impossible by the proof of Lemma 2.1. Hence 0 is not a simple point. The proof of the result about the point $d_N/(N-1)$ is similar (symmetric). □

Proof of Theorem 1.2. Suppose that $a \in \partial T$ is a simple point. By the definition of a simple point, without loss of generality we assume that $(a - \epsilon, a) \cap T = \emptyset$ and $(a, a + \epsilon) \cap T = (a, a + \epsilon)$ for some $\epsilon > 0$. Similar to the proof of Lemma 2.2, we choose n so that the diameter of $\phi_\sigma(T)$ is less than ϵ for all $\sigma \in \Sigma_N^n$. Then there exists a $\sigma \in \Sigma_N^n$ satisfying $a = \phi_\sigma(0)$. Note that, for different $\sigma' \in \Sigma_N^n$, either $\max_{x \in T} \phi_{\sigma'}(x) \leq a - \epsilon$ or $\phi_{\sigma'}(0) \geq \phi_\sigma(0) + N^{-n}$ by OSC. This implies that $[a, a + N^{-n}) \cap T = [a, a + N^{-n}) \cap \phi_\sigma(T)$, which leads to $a = \phi_\sigma(0)$ is not a simple point by Lemma 2.3. The result follows from this contradiction. □

3. Proofs of Theorems 1.3 and 1.4

In this section we prove Theorems 1.3 and 1.4. Note that the IFS $\{\phi_i(x) = \frac{1}{N}(x + d_i)\}_{i=1}^{q}$ does not satisfy the OSC in general, which causes some difficulties in studying the dimensions. To overcome these difficulties several methods have been used. Here we follow the approach of [HLR] to obtain a graph-directed system with OSC such that one of the graph-directed sets is ∂T. A different method was used in [SW]. Since $T(N, \mathcal{D}) \subseteq [0, d_q/(N-1)]$, let b be the minimal integer which is larger than or equal to $d_q/(N-1)$. We construct an auxiliary tile $\Gamma := T(N, \mathcal{C}) = [0, b] \supseteq T[N, \mathcal{D}]$ where $\mathcal{C} = \{0, b, 2b, \ldots, (N-1)b\}$, and define $\psi_j(x) = \frac{1}{N}(x + (j-1)b)$, $j = 1, 2, \ldots, N$. Let $\Gamma_J = \psi_J(\Gamma)$ for all $J \in \Sigma_N^*$. The sequence $\{\Gamma_J : J \in \Sigma_N^k\}_{k \geq 1}$ forms a nested family of partitions of $[0, b]$. We can select a graph-directed system from these partitions: for $J \in \Sigma_N^k$, we give a label to Γ_J as

$$\Delta(J) = \{d_I - c_J : I \in \Sigma_q^k, \phi_I(\Gamma) \cap \psi_J(\Gamma) \neq \emptyset\},$$

where $c_J = c_{j_k} + N c_{j_{k-1}} + \cdots + N^{k-1} c_{j_1}$ if $J = (j_1, j_2, \ldots, j_k) \in \Sigma_N^k$ and the definition of d_I is similar. Let $\mathcal{S}_k = \{J \in \Sigma_N^k : \Delta(J) \neq \emptyset\}$, $\mathcal{S}_k^o = \{J \in \Sigma_N^k : \Delta(JJ') \neq \emptyset, \forall J' \in \Sigma_N^*\}$ and $\mathcal{S}_k' = \mathcal{S}_k - \mathcal{S}_k^o$. It is easy to see that the interval $\psi_J(\Gamma) \subseteq T$ if and only if $J \in \mathcal{S}_k^o$ and by the construction

$$T = \bigcap_{k=1}^{\infty} \Big(\bigcup_{J \in \mathcal{S}_k} \psi_J(\Gamma) \Big) \quad \text{and} \quad \partial T = \bigcap_{k=1}^{\infty} \Big(\bigcup_{J \in \mathcal{S}_k'} \psi_J(\Gamma) \Big). \tag{3.1}$$

(We remark that it may be necessary to add one or two end points $\{0, b\}$ to the right side of the second identity above according to the simplicities of 0 and b, which causes trivial changes in the following proofs and no influence at all on the results.)

The crux of this construction is that $\{\Delta(J) : J \in \Sigma_N^*\}$ is a finite set. This allows us to construct a graph-directed system to reproduce T and ∂T in view of (3.1). Here we consider ∂T only. Let $\{\Delta(J_i)\}_{i=1}^m$ be all different words in $\mathcal{S}' = \bigcup_{k=1}^\infty \mathcal{S}'_k$. Then we define the vertices V as:

$$V = \{\Delta(0), \Delta(J_1), \dots, \Delta(J_m)\},$$

where $\Delta(0) = \{0\}$ is the "root" (define $\Delta(0J) = \Delta(J)$). The corresponding directed edges $E = \{E_{ij}\}_{i,j=0}^m$ on V are

$$E_{ij} = \{c_s \in \mathcal{C} : \Delta(J_i s) = \Delta(J_j), 1 \le s \le N\},$$

which come from the partition relationship

$$\Delta(J_i) \to \Delta(J_i 1)\Delta(J_i 2)\cdots\Delta(J_i N), \qquad i = 0, 1, \dots, m.$$

It is clear that, for any vertex $\Delta(J_j)$, there is a path from the root $\Delta(0)$ to it. If we let

$$\phi_{i,j}^e = N^{-1}(x+e), \qquad e \in E_{i,j}, \qquad i,j = 0, 1, \dots, m,$$

then according to [HLR, Proposition 3.3], there are nonempty compact subsets $\{F_0 = \partial T, F_1, \dots, F_m\}$ satisfying the following graph-directed relationship for ∂T:

$$F_i = \bigcup_{j=0}^m \bigcup_{e \in E_{i,j}} \phi_{i,j}^e(F_j) = \bigcup_{j=0}^m N^{-1}(F_j + E_{i,j}), \qquad i = 0, 1, \dots, m. \tag{3.2}$$

From (3.2) we can define an $(m+1) \times (m+1)$ matrix B with the (i,j)th entry given by

$$b_{ij} = \# E_{i,j}, \qquad i,j = 0, 1, \dots, m$$

[F, p. 48], where B is called the *adjacency* matrix of ∂T. The adjacency matrix is used to count the number of paths of the graph-directed sets in the iteration. Let e be the $(m+1)$-vector with all entries equal to 1 and let e_i be an $(m+1)$-vector with the ith entry 1 and zero otherwise. It is not difficult to prove that $\#\mathcal{S}'_n = e_0^t B^n e$ where $\mathcal{S}'_n$ is used in (3.1) [HLR, Proposition 4.1], which satisfies

$$\lim_{n\to\infty} (\#\mathcal{S}'_n)^{1/n} = \lambda_B,$$

where λ_B is the spectral radius of B. The Hausdorff dimension of ∂T can be calculated by the following theorem [HLR, Theorem 4.3].

Theorem 3.1. *Suppose the self-similar set $T(N, \mathcal{D})$ has nonempty interior and let B be the adjacency matrix of ∂T. Then*

$$\dim_H(\partial T) = \frac{\log \lambda_B}{\log N},$$

where λ_B is the spectral radius of B.

To prove Theorem 1.3 we need the following lemmas.

Lemma 3.2. *Suppose that the adjacency matrix B of ∂T is irreducible with spectral radius $\lambda_B = 1$. Then the cardinalities of all graph-directed sets are equal to one.*

Proof. By the Perron-Frobenius Theorem there exists a positive eigenvector v such that $v = Bv$. Since B is irreducible, for each j there exists an integer k such that the (j,j)-entry $b_{jj}^{(k)}$ of B^k is positive. Moreover, using $v = B^k v$, it is easy to get $b_{ji}^{(k)} = 0$ for $i \neq j$ and $b_{jj}^{(k)} = 1$. Hence we have, by iterating (3.2) k times,

$$N^k F_j = F_j + c_J$$

for some $J \in \Sigma_N^k$. Consequently $F_j = \{\sum_{n=1}^{\infty} N^{-kn} c_J = c_J/(N^k - 1)\}$ is a singleton. □

Lemma 3.3. *Let $\mathcal{D}$ be not a strict product form. Then we can modify all the graph-directed sets in* (3.2) *such that each of them has no isolated points (may be empty) and keep $F_0 = \partial T$ invariant.*

Proof. From the relation

$$F_0 = \bigcup_{j=0}^{m} N^{-1}(F_j + E_{0,j}), \tag{3.3}$$

we claim that, if $E_{0,j} \subseteq \mathcal{C}$ is nonempty for $j \neq 0$, then F_j can be selected so that no isolated points in it can be cancelled without influence on (3.3). Suppose there is an isolated point $x_0 \in F_j$. If

$$x_0 + E_{0,j} \subseteq \bigcup_{k=0,\ k\neq j}^{m} (F_k + E_{0,k}),$$

then we can omit x_0 from F_j. If the above inclusion is not true, then $x_0 + e$, for some $e \in E_{0,j}$, is an isolated point of $NF_0 = N\partial T$, which is impossible by Theorem 1.2. If $\#F_j$ is finite, that is, all points of it are isolated, then (3.3) holds for each point in F_j, in the case F_j can be cancelled from the graph-directed relation (3.3) without loss of anything. So the claim follows obviously. Note that in the graph-directed system for the boundary of T the "root" is $\partial T = F_0$ and for each j there is a path from the root F_0 to F_j. Those relations imply that all graph-directed sets can be modified by finite steps with the same method. □

Proof of Theorem 1.3. It is well known that the Hausdorff dimension of the boundary of a self-similar tile is less than one (see e.g. [SW]). Now we make use of Theorem 3.1 to prove that the dimension is positive. Since the adjacency matrix B can be decomposed as

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{kk} \end{bmatrix},$$

where all A_{ii}, $i \geq 2$, are irreducible and A_{11} is either a zero or an irreducible matrix [S], by Lemma 3.3 we can assume that all cardinalities of graph-directed sets in (3.2) are not finite, and then by Lemma 3.2 the spectral radius of A_{kk} is larger than one, and so is λ_B. Hence the result follows by Theorem 3.1. □

Before proving Theorem 1.4, we recall the definition of box dimension for a nonempty bounded subset E of $\mathbb{R}^1$. Let $N_r(E)$ be the smallest number of sets of diameter r that can cover E. The lower and upper box dimension of E are defined as

$$\underline{\dim}_B E = \liminf_{r\to 0} \frac{\log N_r(E)}{-\log r} \quad \text{and} \quad \overline{\dim}_B E = \limsup_{r\to 0} \frac{\log N_r(E)}{-\log r}$$

respectively. If they are equal we refer to the common value as the box dimension of E.

Proof of Theorem 1.4. Denote $s = \dim_H \partial T$; then $\lambda_B = N^s$ by Theorem 3.1. For any $t > s$ we have

$$\Big(\sum_{\sigma\in\mathcal{S}'_n} |\psi_\sigma(\Gamma)|^t\Big)^{1/n} = \big(\#\mathcal{S}'_n N^{-nt} b^t\big)^{1/n}.$$

Thus

$$\lim_{n\to\infty}\Big(\sum_{\sigma\in\mathcal{S}'_n} |\psi_\sigma(\Gamma)|^t\Big)^{1/n} = \lambda_B N^{-t} = N^{s-t} < 1.$$

This limit implies that there exists an integer m such that for $n \ge m$ we have

$$\sum_{\sigma\in\mathcal{S}'_n} |\psi_\sigma(\Gamma)|^t < 1 \le b^t,$$

which is equivalent to

$$\#\mathcal{S}'_n < N^{nt}.$$

For any r, $0 < r < bN^{-m}$, there is an integer n larger than m satisfying

$$bN^{-n} \le r < bN^{-n+1}.$$

Note that $\{\psi_\sigma(\Gamma)\}_{\sigma\in\mathcal{S}'_n}$ is a bN^{-n}-cover of ∂T. Hence

$$N_r(\partial T) \le \#\mathcal{S}'_n.$$

By the definition of box dimension we have

$$\begin{aligned}\overline{\dim}_B(\partial T) &= \limsup_{r\to 0} \tfrac{N_r(\partial T)}{-\log r}\\ &\le \limsup_{n\to\infty} \tfrac{\log(\#\mathcal{S}'_n)}{\log bN^{n-1}},\\ &\le \limsup_{n\to\infty} \tfrac{\log N^{nt}}{\log bN^{n-1}} \le t.\end{aligned}$$

The result follows by letting t tend to s. □

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