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THE WEAK DIRICHLET PROBLEM FOR BAIRE FUNCTIONS

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ABSTRACT. Let X be a simplex and K a compact subset of the set of all extreme points of X. We show that any bounded function f of Baire class α on K can be extended to a function h of affine class α on X. Moreover, h can be chosen in such a way that $h(X) \subset \overline{\operatorname{co}} f(K)$.

1. INTRODUCTION

If X is a simplex in a locally convex space (for the notions not explained here we refer reader to the next section) and K is a compact subset of the set $\operatorname{ext} X$ of all extreme points of X, it is well known that any continuous function on K can be extended to an affine continuous function on X (see [1, Theorem II.3.12] or [2, Corollary 7.7]). The problem of finding an affine extension for a function defined on a compact subset of $\operatorname{ext} X$ is sometimes called the *weak Dirichlet problem.* In our paper we investigate this question for Baire functions.

We remark that the general question of finding an affine extension for a function defined on a compact subset of $\operatorname{ext} X$ is of some interest. Our Corollary 3.5 serves as an example of a purely topological application of "affine extension theorems".

Let $\mathcal{A}(X)$ stand for the space of all affine continuous functions on X. We set $\mathcal{B}_0^{bb}(\mathcal{A}(X)) := \mathcal{A}(X)$ and having $\mathcal{B}_\beta^{bb}(\mathcal{A}(X))$, $\beta < \alpha$, already defined for an ordinal number $\alpha \in (0, \omega_1)$, we set $\mathcal{B}_\alpha^{bb}(\mathcal{A}(X))$ to be the space of all pointwise limits of bounded sequences of functions from $\bigcup_{\beta<\alpha} \mathcal{B}_\beta^{bb}(\mathcal{A}(X))$. We say that a function $h \in \mathcal{B}_\alpha^{bb}(\mathcal{A}(X))$ is of *affine class* α (see [3]). With this notation we can formulate the main result of the paper.

Theorem 1.1. *Let X be a simplex and let $K \subset \operatorname{ext} X$ be compact. Then for any bounded function f on K of Baire class α there exists a function $h \in \mathcal{B}_\alpha^{bb}(\mathcal{A}(X))$ such that $h = f$ on K and $h(X) \subset \overline{\operatorname{co}} f(K)$.*

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We remark that the proof of Theorem 1.1 is easy if K is metrizable since we are able to use almost straightforwardly Lazar's selection theorem [5, Theorem 3.1]. We sketch the proof of this observation in Remark 3.4. However, for nonmetrizable sets we have to apply the above-mentioned Lazar's selection theorem more carefully as we explain below.

2. PRELIMINARIES

All topological spaces will be considered as Hausdorff. If X is a compact space, we denote by $\mathcal{C}(X)$ the space of all continuous functions on X. Let $\mathcal{M}^1(X)$ denote the set of all probability Radon measures on X and let ε_x stand for the Dirac measure at $x \in X$.

If $\mathcal{F}$ is a family of functions on a set X, we write $(\mathcal{F})_1$ for the family of all pointwise limits of sequences of functions from $\mathcal{F}$. We set $\mathcal{B}_0(\mathcal{F}) := \mathcal{F}$ and inductively define $\mathcal{B}_\alpha(\mathcal{F}) := (\bigcup_{\beta<\alpha} \mathcal{B}_\beta(\mathcal{F}))_1$ for each $\alpha \in (0, \omega_1)$. The family consisting of all pointwise limits of bounded sequences of functions from $\mathcal{F}$ is denoted as $(\mathcal{F})_1^{bb}$. Similarly as above we define $\mathcal{B}_\alpha^{bb}(\mathcal{F})$ for $\alpha \in [0, \omega_1)$.

If X is a topological space and $\alpha \in (0, \omega_1)$, we write $\mathcal{B}_\alpha(X)$ for the space of all *Baire functions of class* α on X, i.e., $\mathcal{B}_\alpha(X) = \mathcal{B}_\alpha(\mathcal{C}(X))$. It is easy to see that any bounded function of Baire class α belongs to $\mathcal{B}_\alpha^{bb}(\mathcal{C}(X))$.

Let X be a compact convex subset of a locally convex space. Given a point $x \in X$, we say that a measure $\mu \in \mathcal{M}^1(X)$ *represents* x if $\mu(h) = h(x)$ for every $h \in \mathcal{A}(X)$. The set $\operatorname{ext} X$ of all extreme points consists precisely of those points $x \in X$ for which there exists no representing measure except ε_x (see [1, Corollary I.2.4] or [2, Theorem 6.3]).

A convex set $F \subset X$ is called a *face* if each open segment $I \subset X$ intersecting F is contained in F.

The convex cone of all continuous convex functions on X determines a partial ordering $\preceq$ (called the *Choquet ordering*) on the space $\mathcal{M}^+(X)$ of all positive Radon measures on X: $\mu \preceq \nu$ if and only if $\mu(f) \leq \nu(f)$ for each convex continuous function f on X. According to the famous Choquet–Bishop–de-Leeuw theorem (see [1, Theorem I.4.8] or [2, Theorem 6.8]), for every $x \in X$ there exists a maximal (with respect to the Choquet ordering) measure μ representing x. If this maximal measure is uniquely determined for every $x \in X$, we say that X is a *simplex* (respectively *Choquet simplex*, see [1, Chapter II., §3] or [2, Section 2.7]). In this case we denote the unique maximal measure for x as δ_x.

If f is a bounded Borel function on a Borel subset B of simplex X, we define the function

$$H^f(x) := \delta_x(f) , \quad x \in X ,$$

where f is defined by 0 on $X \setminus B$.

If F is a subset of a locally convex space, we denote by $\overline{\operatorname{co}} F$ the closed convex hull of F. If X is a set, the restriction of a function $f : X \to \mathbb{R}$ to a set $F \subset X$ is denoted by $f \restriction_F$.

3. PROOF OF THEOREM 1.1

Unless otherwise stated, X is assumed to be a simplex in some locally convex space.

Lemma 3.1. *Let $\mathcal{H} = \{f_n : n \in \mathbb{N}\}$ be a bounded family of affine continuous functions on a closed face $F \subset X$. Then there exist a family $\mathcal{A} = \{h_n : n \in \mathbb{N}\}$ of affine continuous functions on X and a mapping $r : X \to F$ such that $r(x) = x$ on F and $h_n(x) = f_n(r(x))$ for all $n \in \mathbb{N}$.*

Proof. Let $\mathcal{H}$ be as in the premise. Without loss of generality we may assume that $f_n(F) \subset [0,1]$ for all $n \in \mathbb{N}$. We consider a mapping $\varphi : F \to [0,1]^{\mathbb{N}}$ (the space $[0,1]^{\mathbb{N}}$ is considered with the pointwise topology) defined as

$$x \mapsto \{f_n(x)\}_{n\in\mathbb{N}}\ , \quad x \in F\ .$$

Then φ is continuous and affine on F. We define a multivalued map $\Phi : X \to [0,1]^{\mathbb{N}}$ as

$$\Phi(x) := \begin{cases} \varphi(x) & \text{if } x \in F\ , \\ \varphi(F) & \text{if } x \in X \setminus F\ . \end{cases}$$

Then Φ has nonempty convex compact values and Φ is affine, i.e., for every $x, y \in X$ and $\alpha \in [0,1]$,

$$\alpha\Phi(x) + (1-\alpha)\Phi(y) \subset \Phi(\alpha x + (1-\alpha)y)\ .$$

(Here we use the assumption that F is a face.) Since $\Phi = \varphi$ on a closed set F and $\Phi = \varphi(F)$ on the complement of F, it is easy to verify that Φ is even lower semicontinuous, i.e., given an open set $U \subset [0,1]^{\mathbb{N}}$, the set

$$\Phi^{-1}(U) = \{x \in X : \Phi(x) \cap U \neq \emptyset\}$$

is open in X.

Let ρ be an invariant metric on $\mathbb{R}^{\mathbb{N}}$ which is compatible with the pointwise topology on $[0,1]^{\mathbb{N}}$. Although $(\mathbb{R}^{\mathbb{N}}, \rho)$ is not complete, its completion Y is a Fréchet space and $[0,1]^{\mathbb{N}}$ is a compact and consequently closed subset of Y.

Now we are ready to use Lazar's selection theorem (see [5, Theorem 3.1]) to obtain an affine continuous mapping $\psi : X \to [0,1]^{\mathbb{N}}$ such that $\psi(x) \in \Phi(x)$ for all $x \in X$.

Set $h_n := \pi_n \circ \psi$, $n \in \mathbb{N}$, where $\pi_n : [0,1]^{\mathbb{N}} \to \mathbb{R}$ is the projection to the n–th coordinate. Clearly, each h_n is a continuous affine function on X.

Since $\psi = \varphi$ on F, $h_n \restriction_F = f_n \restriction_F$ for all $n \in \mathbb{N}$. If $x \notin F$, $\psi(x) = \varphi(y)$ for some $y \in F$. If we denote this y as $r(x)$ and set $r(x) := x$ on F, we obtain the sought mapping r and conclude the proof. □

Lemma 3.2. *In the situation of Lemma* 3.1*, for every $f \in \mathcal{B}_\alpha(\mathcal{H})$ there exists a function $h \in \mathcal{B}_\alpha(\mathcal{A})$ such that $h(x) = f(r(x))$ for all $x \in X$.*

Proof. If $\alpha = 0$, i.e, $f = f_n$ for some $n \in \mathbb{N}$, take $h := h_n$.

Suppose that the lemma has been verified for all $\beta < \alpha$ where $\alpha \in (0, \omega_1)$. Pick $f \in \mathcal{B}_\alpha(\mathcal{H})$. Then there exist functions $f_n \in \mathcal{B}_{\alpha_n}(\mathcal{H})$, $n \in \mathbb{N}$, where $\alpha_n < \alpha$, such that $f_n \to f$ on F. According to the induction hypothesis, for every $n \in \mathbb{N}$ we are able to select $h_n \in \mathcal{B}_{\alpha_n}(\mathcal{A})$ such that $h_n(x) = f_n(r(x))$ for every $x \in X$. Since $f_n(r(x)) \to f(r(x))$, the function $h := \lim_n h_n$ is well defined, $h \in \mathcal{B}_\alpha(\mathcal{A})$ and $h(x) = f(r(x))$ for all $x \in X$. This finishes the proof. □

Lemma 3.3. *Let $\mathcal{F}$ be a bounded family of continuous functions on a compact set $K \subset \operatorname{ext} X$. Set $F := \overline{\operatorname{co}}\, K$.*

(i) *The set F is a closed face of X and* $\operatorname{ext} F = K$.

(ii) *The family* $\mathcal{H} := \{H^f \restriction_F : f \in \mathcal{F}\}$ *consists of affine continuous functions on* F.

(iii) *If* $f \in \mathcal{B}_\alpha(\mathcal{F})$, *then* $H^f \restriction_F \in \mathcal{B}_\alpha(\mathcal{H})$.

Proof. First note that $\operatorname{ext} F = K$ due to the Milman theorem (see [2, Theorem 1.3]). According to [2, Lemma 1.6], F is a closed face of X and, consequently, it is a simplex. Since $\operatorname{ext} F$ is a closed set, the mapping $x \mapsto \delta_x$, $x \in F$, is continuous on F (see [1, Theorem II.4.1]). From this fact assertion (ii) follows as well as (iii) after a use of transfinite induction. ☐

Proof of Theorem 1.1. By setting $F := \overline{\operatorname{co}}\, K$ we obtain a closed face in X (see Lemma 3.3 (i)). Given a function $f \in \mathcal{B}_\alpha^{bb}(K)$, find a countable family $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$ of continuous functions on K such that $f \in \mathcal{B}_\alpha(\mathcal{F})$ and $f_n(K) \subset \overline{\operatorname{co}}\, f(K)$ for every $n \in \mathbb{N}$.

Define $g_n := H^{f_n} \restriction_F$, $n \in \mathbb{N}$. According to Lemma 3.3 (ii), the family $\mathcal{H} := \{g_n : n \in \mathbb{N}\}$ consists of continuous affine functions on F. Moreover, the values of every function in $\mathcal{H}$ are contained in $\overline{\operatorname{co}}\, f(K)$. Lemma 3.1 provides a family $\mathcal{A} = \{h_n : n \in \mathbb{N}\}$ of continuous affine functions on X and a mapping $r : X \to F$ such that $r(x) = x$ on F and $h_n(x) = g_n(r(x))$ for each $x \in X$ and $n \in \mathbb{N}$. Since $H^f \restriction_F \in \mathcal{B}_\alpha(\mathcal{H})$ due to Lemma 3.3 (iii), an application of Lemma 3.2 finishes the proof. ☐

Remark 3.4. We briefly indicate an easier proof of Theorem 1.1 if K is supposed to be metrizable. In this case, $F := \overline{\operatorname{co}}\, K$ is a closed face of X (see Lemma 3.3 (i)) which is moreover metrizable.

Indeed, if b is the mapping assigning to each probability measure μ on K its barycenter (i.e., $b(\mu)$ is the unique point in X such that μ represents $b(\mu)$), then b is a continuous mapping of a metrizable compact set $\mathcal{M}^1(K)$ of all probability measures on K onto F. Since a continuous image of a compact metric space is metrizable (see [4, Chapter 4, Problem S]), the claim is proved.

Another corollary of Lazar's selection theorem (see [5, Theorem 3.6]) provides an affine continuous retraction $r : X \to F$. Thus the sought extension of a given function $f \in \mathcal{B}_\alpha^{bb}(K)$ can be defined as $H^f \restriction_F \circ r$.

Corollary 3.5. *Let* K *be a closed subset of a compact space* X. *Then for every bounded function* f *on* K *of Baire class* α *there exists a bounded function* h *on* X *of Baire class* α *such that* $f = h$ *on* K *and* $h(X) \subset \overline{\operatorname{co}}\, f(K)$.

Proof. If X is a compact space, the set $Y := \mathcal{M}^1(X)$ of all probability measures on X endowed with the weak-star topology is a simplex (see [1, Corollary II.4.2] or [2, Theorem 7.5]). The canonical embedding $\varepsilon : x \mapsto \varepsilon_x$, $x \in X$, serves as a homeomorphism of X onto $\operatorname{ext} Y$.

Given a bounded function f on K of Baire class α, Theorem 1.1 provides a function $h \in \mathcal{B}_\alpha^{bb}(\mathcal{A}(Y))$ such that $h = f \circ \varepsilon^{-1}$ on $\varepsilon(K)$ and $h(Y) \subset \overline{\operatorname{co}}\,(f \circ \varepsilon^{-1})(\varepsilon(K))$. Then the function $h \circ \varepsilon$ is the sought extension of f and we are done. ☐

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