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DISTRIBUTION OF HECKE EIGENVALUES

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Abstract. We give two results concerning the distribution of Hecke eigenvalues of $SL(2,\mathbb{Z})$. The first result asserts that on certain average the Sato-Tate conjecture holds. The second result deals with the Gaussian central limit theorem for Hecke eigenvalues.

1. Introduction and main results

The aim of this paper is to give two results concerning the distribution of Hecke eigenvalues. In the first result (Theorem 1) the Sato-Tate distribution appears, and in the second result (Theorem 2) the Gaussian distribution appears. In the following, p and p_i $(i = 1, 2, \dots)$ stand for prime numbers, k is a positive even integer, and x is a positive real number. We denote by $\mathcal{F}_k$ the set of all normalized holomorphic Hecke eigen cusp forms of weight k with respect to $SL(2,\mathbb{Z})$. For $f \in \mathcal{F}_k$ we write $T_n f = \widetilde{\lambda}_f(n) f$, where T_n is the n-th Hecke operator, and put $\lambda_f(n) = \widetilde{\lambda}_f(n)/n^{\frac{k-1}{2}}$. Then we have

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right), \tag{1}$$

and it is proved by Deligne that $\lambda_f(p) \in [-2, 2]$ for p prime.

Inspired by the Sato-Tate conjecture for non-CM elliptic curves over $\mathbb{Q}$, Serre conjectured that for each $f \in \mathcal{F}_k$, $\lambda_f(p)$'s are uniformly distributed with respect to the Sato-Tate distribution, that is, for any continuous real function h on $[-2, 2]$ we have

$$\frac{1}{\pi(x)} \sum_{p \le x} h(\lambda_f(p)) \longrightarrow \frac{1}{2\pi} \int_{-2}^{2} h(t)\sqrt{4-t^2}\, dt, \qquad \text{as } x \to \infty, \tag{2}$$

where $\pi(x)$ denotes the number of all primes up to x (see e.g. [MM3]). We also refer to this conjecture as the Sato-Tate conjecture. This conjecture is equivalent to the fact that for each integer $m \ge 1$, $L(s, \mathrm{Sym}^m f)$ has an analytic continuation up to $\mathrm{Re}(s) \ge 1$ and does not vanish on the line $\mathrm{Re}(s) = 1$, where $L(s, \mathrm{Sym}^m f)$ is the m-th symmetric power L-function attached to f. Murty [Mu] showed that it is sufficient that for each $m \ge 1$, $L(s, \mathrm{Sym}^m f)$ has an analytic continuation up to $\mathrm{Re}(s) \ge 1$. Recently Kim and Shahidi [Ki], [KS] made remarkable progress

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concerning symmetric power L-functions. However, the Sato-Tate conjecture (2) has not yet been proved for any $f \in \mathcal{F}_k$.

The next theorem is one of two main results of this paper. This theorem asserts that on certain average the Sato-Tate conjecture (2) holds.

Theorem 1. *Suppose that $k = k(x)$ satisfies $\frac{\log k}{\log x} \to \infty$ as $x \to \infty$. Then for any continuous real function h on $[-2, 2]$, we have*

$$\frac{1}{\pi(x)\,\#\mathcal{F}_k} \sum_{\substack{p \le x \\ f \in \mathcal{F}_k}} h(\lambda_f(p)) \longrightarrow \frac{1}{2\pi} \int_{-2}^{2} h(t)\sqrt{4-t^2}\,dt$$

as $x \to \infty$.

In contrast to the Sato-Tate conjecture (2), in which we fix $f \in \mathcal{F}_k$ and take the sum of $\lambda_f(p)$ over the primes, the limit distribution when we fix a prime p and take the sum of $\lambda_f(p)$ over Hecke eigenforms $f \in \mathcal{F}_k$ was studied by Conrey, Duke, and Farmer [CDF, Theorem 1] and Serre [Se, Théorèm 1] (see also [Sa] for Maass wave forms of $SL(2, \mathbb{Z})$). For the case that $p \to \infty$ see [CDF, Theorem 2], and similar results were obtained in e.g. [Bi], [Na], [Yo].

The second main result of this paper is the following.

Theorem 2. *Suppose that $k = k(x)$ satisfies $\frac{\log k}{\log x} \to \infty$ as $x \to \infty$. Then for any bounded continuous real function h on $\mathbb{R}$, we have*

$$\frac{1}{\#\mathcal{F}_k} \sum_{f \in \mathcal{F}_k} h\left(\frac{\sum_{p \le x} \lambda_f(p)}{\sqrt{\pi(x)}}\right) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-\frac{t^2}{2}}\,dt \tag{3}$$

as $x \to \infty$.

Theorem 2 is a *central limit theorem* for Hecke eigenvalues and was inspired by, for example, Erdős-Kac's theorem ([EK], [El]). See also [MM1], [MM2]. We note that usual theory concerning the central limit theorem (see e.g. [B, Theorem 27.1]) is not applicable for the proof of Theorem 2, because the distributions of $\lambda_f(p)$ in the case of varying $f \in \mathcal{F}_k$ are different for each other prime p, according to [CDF] and [Se] mentioned above.

We remark that the Sato-Tate distribution appears in the central limit theorem of free probability theory (see e.g. [HP, Theorem 2.3.2]).

Let $\mathcal{F}_{\le K} := \{f \in \mathcal{F}_k | k \le K\} = \bigcup_{k \le K} \mathcal{F}_k$ for a positive integer K. By the arguments in this paper, we can also obtain the same results as Theorem 1 and Theorem 2 for $\mathcal{F}_{\le K}$, in place of $\mathcal{F}_k$, under the assumption that $K = K(x)$ satisfies $\frac{\log K}{\log x} \to \infty$ as $x \to \infty$.

2. Lemmas

In this section we give some facts which are used in the proof of our theorems. By using the trace formula for the Hecke operator T_n, the next lemma is proved in [Se, Section 4].

Lemma 1. *We have*

$$\sum_{f \in \mathcal{F}_k} \lambda_f(n) = \delta_n \frac{k-1}{12} \frac{1}{n^{\frac{1}{2}}} + O\left(n^c\right),$$

where the implied constant and $c\,(> 0)$ are absolute, and $\delta_n = 1$ if n is a square and $\delta_n = 0$ otherwise.

In particular, taking $n = 1$, we get

$$\#\mathcal{F}_k \sim \frac{k-1}{12} \sim \frac{k}{12} \qquad \text{as } k \to \infty. \tag{4}$$

The next lemma is proved in [CDF, Lemma 3].

Lemma 2. *Suppose p is prime. Then for an integer $n \geq 1$ we have*

$$\lambda_f(p)^n = \sum_{j=0}^{n} h_n(j)\lambda_f(p^j),$$

where

$$h_n(j) := \frac{2^{n+1}}{\pi} \int_0^{\pi} \cos^n\theta \sin(j+1)\theta \sin\theta d\theta. \tag{5}$$

We note that by calculation,

$$h_n(j) = 0 \tag{6}$$

if n is odd and j is even, or if n is even and j is odd.

3. Proof of Theorem 1

This section is devoted to proving Theorem 1.

Since $\frac{\log k}{\log x} \to \infty$ as $x \to \infty$, for an arbitrary positive real number a we have $\log k > a \log x$ and hence

$$k > x^a \tag{7}$$

if x is sufficiently large.

The Weierstrass approximation theorem implies that for a continuous real function $h(t)$ on the compact set $[-2, 2]$ and for any $\varepsilon > 0$, there exists a polynomial $p(t)$ such that $|h(t) - p(t)| < \varepsilon$ uniformly on $[-2, 2]$. So it is sufficient for getting Theorem 1 to prove that for each integer $r \geq 0$,

$$\frac{1}{\pi(x)\,\#\mathcal{F}_k} \sum_{\substack{p \leq x \\ f \in \mathcal{F}_k}} \lambda_f(p)^r \longrightarrow \frac{1}{2\pi} \int_{-2}^{2} t^r \sqrt{4-t^2}\, dt \qquad \text{as } x \to \infty. \tag{8}$$

Note that, by changing variables $u \mapsto 2\cos\theta$ and (5),

$$\frac{1}{2\pi} \int_{-2}^{2} t^r \sqrt{4-t^2}\, dt = \frac{2^{r+1}}{\pi} \int_0^{\pi} \cos^r\theta \sin^2\theta d\theta = h_r(0). \tag{9}$$

In the case $r = 0$, (8) holds. We now fix an integer $r \geq 1$. By Lemma 2 we have

$$\begin{aligned} S_{x,r} &:= \sum_{p \leq x} \sum_{f \in \mathcal{F}_k} \lambda_f(p)^r \\ &= \sum_{p \leq x} \sum_{f \in \mathcal{F}_k} \sum_{j=0}^{r} h_r(j)\lambda_f(p^j) \\ &= \sum_{p \leq x} \sum_{j=0}^{r} h_r(j) \sum_{f \in \mathcal{F}_k} \lambda_f(p^j) \\ &= \sum_{p \leq x} \left(h_r(0) \sum_{f \in \mathcal{F}_k} \lambda_f(1) + \sum_{j=1}^{r} h_r(j) \sum_{f \in \mathcal{F}_k} \lambda_f(p^j) \right). \end{aligned}$$

This, Lemma 1 and $\lambda_f(1) = 1$ yield

$$\begin{aligned} S_{x,r} &= \sum_{p\le x} h_r(0)\,\#\mathcal{F}_k + \sum_{p\le x}\sum_{j=1}^{r} h_r(j)\left(\delta_{p^j}\frac{k-1}{12}\frac{1}{p^{\frac{j}{2}}} + O\left(p^{jc}\right)\right) \\ &= h_r(0)\,\#\mathcal{F}_k\,\pi(x) + \sum_{j=1}^{r} h_r(j)\left(\frac{k-1}{12}\sum_{p\le x}\frac{\delta_{p^j}}{p^{\frac{j}{2}}} + \sum_{p\le x} O\left(p^{jc}\right)\right). \end{aligned} \tag{10}$$

Since the formula $\sum_{p\le x} 1/p \sim \log\log x$ holds, we have for $1 \le j \le r$,

$$\frac{k-1}{12}\sum_{p\le x}\frac{\delta_{p^j}}{p^{\frac{j}{2}}} \le \frac{k-1}{12}\sum_{p\le x}\frac{1}{p} \sim \frac{k-1}{12}\log\log x.$$

This, (4) and the prime number theorem $\pi(x) \sim x/\log x$ give

$$\frac{1}{\pi(x)\,\#\mathcal{F}_k}\frac{k-1}{12}\sum_{p\le x}\frac{\delta_{p^j}}{p^{\frac{j}{2}}} \longrightarrow 0 \qquad \text{as } x \to \infty, \tag{11}$$

for $1 \le j \le r$. Since

$$\sum_{p\le x} O\left(p^{jc}\right) \ll \sum_{p\le x} O\left(p^{rc}\right) \ll x^{rc}\pi(x),$$

we also have, by (4) and (7),

$$\frac{1}{\pi(x)\,\#\mathcal{F}_k}\sum_{p\le x} O\left(p^{jc}\right) \ll \frac{x^{rc}}{\#\mathcal{F}_k} \longrightarrow 0 \qquad \text{as } x \to \infty. \tag{12}$$

Combining (10), (11) and (12), we obtain

$$\frac{1}{\pi(x)\,\#\mathcal{F}_k} S_{x,r} \longrightarrow h_r(0) \qquad \text{as } x \to \infty.$$

This and (9) give (8) for fixed $r \ge 1$. Hence we complete the proof of Theorem 1.

4. Proof of Theorem 2

This section is devoted to proving Theorem 2. As in (7), for an arbitrary positive real number a we have

$$k > x^a \tag{13}$$

if x is sufficiently large. Since prime p is an integer, it is sufficient for getting (3) to prove it only when x runs through the positive integers. Moreover, according to Theorem 25.8 and Theorem 30.2 (the method of moments) in [B], it is sufficient to prove that for each integer $r \ge 1$,

$$U_{x,r} := \frac{1}{\#\mathcal{F}_k}\sum_{f\in\mathcal{F}_k}\left(\frac{\sum_{p\le x}\lambda_f(p)}{\sqrt{\pi(x)}}\right)^r \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^r e^{-\frac{t^2}{2}}\,dt \tag{14}$$

as a positive integer x goes to ∞. It is known that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^r e^{-\frac{t^2}{2}}\,dt = \begin{cases} \frac{r!}{2^{\frac{r}{2}}\left(\frac{r}{2}\right)!}, & \text{if } r \text{ is even,} \\ 0, & \text{if } r \text{ is odd.} \end{cases} \tag{15}$$

We have, by the multinomial formula,

$$\left(\sum_{p\le x}\lambda_f(p)\right)^r \tag{16}$$
$$=\sum_{u=1}^{r}\sum\nolimits^{(1)}_{(r_1,\dots,r_u)}\frac{r!}{r_1!r_2!\cdots r_u!}\frac{1}{u!}\sum\nolimits^{(2)}_{(p_1,\dots,p_u)}\lambda_f(p_1)^{r_1}\lambda_f(p_2)^{r_2}\cdots\lambda_f(p_u)^{r_u},$$

where $\sum^{(1)}_{(r_1,\dots,r_u)}$ means the sum over the u-tuples $(r_1,r_2,\dots,r_u)$ of positive integers satisfying $r_1+r_2+\cdots+r_u=r$, and $\sum^{(2)}_{(p_1,\dots,p_u)}$ means the sum over the u-tuples $(p_1,p_2,\dots,p_u)$ of distinct primes not greater than x. For u-tuples $(r_1,\dots,r_u)$ and $(p_1,\dots,p_u)$ as in (16) we define

$$B(r_1,\dots,r_u;p_1,\dots,p_u):=\sum_{f\in\mathcal{F}_k}\lambda_f(p_1)^{r_1}\cdots\lambda_f(p_u)^{r_u},$$
$$A(r_1,\dots,r_u):=\sum\nolimits^{(2)}_{(p_1,\dots,p_u)}B(r_1,\dots,r_u;p_1,\dots,p_u).$$

Then (16) gives

$$U_{x,r}=\sum_{u=1}^{r}\sum\nolimits^{(1)}_{(r_1,\dots,r_u)}\frac{r!}{r_1!r_2!\cdots r_u!}\frac{1}{u!}\frac{1}{\pi(x)^{\frac{r}{2}}}\frac{1}{\#\mathcal{F}_k}A(r_1,\dots,r_u). \tag{17}$$

By Lemma 2 and (1),

$$\begin{aligned}
&B(r_1,\dots,r_u;p_1,\dots,p_u)\\
&=\sum_{f\in\mathcal{F}_k}\left(\sum_{j_1=0}^{r_1}h_{r_1}(j_1)\lambda_f(p_1^{j_1})\right)\cdots\left(\sum_{j_u=0}^{r_u}h_{r_u}(j_u)\lambda_f(p_u^{j_u})\right)\\
&=\sum_{f\in\mathcal{F}_k}\sum_{\substack{0\le j_1\le r_1,\\ \vdots\\ 0\le j_u\le r_u}}h_{r_1}(j_1)\cdots h_{r_u}(j_u)\lambda_f(p_1^{j_1}\cdots p_u^{j_u})\\
&=\sum_{\substack{0\le j_1\le r_1,\\ \vdots\\ 0\le j_u\le r_u}}h_{r_1}(j_1)\cdots h_{r_u}(j_u)\sum_{f\in\mathcal{F}_k}\lambda_f(p_1^{j_1}\cdots p_u^{j_u}).
\end{aligned} \tag{18}$$

Under this setting, we have the following lemmas.

Lemma 3. *Let $k=k(x)$ satisfy $\frac{\log k}{\log x}\to\infty$ as $x\to\infty$. Assume that an u-tuple $(r_1,\dots,r_u)$ in (17) satisfies the condition that r_ℓ is odd for some ℓ. Then*

$$\frac{1}{\pi(x)^{\frac{r}{2}}}\frac{1}{\#\mathcal{F}_k}A(r_1,\dots,r_u)\longrightarrow 0$$

as $x\to\infty$.

Proof. By (18), (6) and the assumption that r_ℓ is odd, we have

$$B(r_1,\dots,r_u;p_1,\dots,p_u)=\sum\nolimits^{(3)}_{(j_1,\dots,j_u)}h_{r_1}(j_1)\cdots h_{r_u}(j_u)\sum_{f\in\mathcal{F}_k}\lambda_f(p_1^{j_1}\cdots p_u^{j_u}), \tag{19}$$

where $\sum^{(3)}_{(j_1,\dots,j_u)}$ denotes the sum over the u-tuples $(j_1,\dots,j_u)$ of integers satisfying that $0 \le j_i \le r_i$ for all $i = 1,\dots,u$ and that j_ℓ is odd. Since j_ℓ is odd, $p_1^{j_1}\cdots p_u^{j_u}$ is not a square, so that by Lemma 1

$$\begin{aligned} B(r_1,\dots,r_u;p_1,\dots,p_u) &= \sum\nolimits^{(3)}_{(j_1,\dots,j_u)} h_{r_1}(j_1)\cdots h_{r_u}(j_u)\, O\left((p_1^{j_1}\cdots p_u^{j_u})^c\right) \\ &= \sum\nolimits^{(3)}_{(j_1,\dots,j_u)} O_r\left((p_1^{j_1}\cdots p_u^{j_u})^c\right). \end{aligned}$$

This and $u \le r$ give

$$\begin{aligned} &A(r_1,\dots,r_u) = \sum\nolimits^{(2)}_{(p_1,\dots,p_u)} \sum\nolimits^{(3)}_{(j_1,\dots,j_u)} O_r\left((p_1^{r_1}\cdots p_u^{r_u})^c\right) \\ &= \sum\nolimits^{(2)}_{(p_1,\dots,p_u)} \sum\nolimits^{(3)}_{(j_1,\dots,j_u)} O_r((p_1^{r}\cdots p_u^{r})^c) \\ &= \sum\nolimits^{(2)}_{(p_1,\dots,p_u)} \sum\nolimits^{(3)}_{(j_1,\dots,j_u)} O_r(x^{urc}) = \sum\nolimits^{(2)}_{(p_1,\dots,p_u)} O_r(x^{urc}) \\ &\le \pi(x)^u O_r(x^{urc}) \ll_r \pi(x)^r x^{r^2c} \ll x^{r+r^2c}. \end{aligned}$$

Hence by this, (4) and (13), we obtain

$$\frac{1}{\pi(x)^{\frac{r}{2}}}\frac{1}{\#\mathcal{F}_k}A(r_1,\dots,r_u) \longrightarrow 0 \qquad \text{as } x \to \infty.$$

□

Lemma 4. *Let $k = k(x)$ satisfy $\frac{\log k}{\log x} \to \infty$ as $x \to \infty$. Assume that an u-tuple $(r_1,\dots,r_u)$ in (17) satisfies the condition that all r_i $(i = 1,\dots,u)$ are even. If $r_i = 2$ for all $i = 1,\dots,u$, then we have*

$$\frac{1}{\pi(x)^{\frac{r}{2}}}\frac{1}{\#\mathcal{F}_k}A(r_1,\dots,r_u) \longrightarrow 1 \qquad \text{as } x \to \infty; \tag{20}$$

if not, then we have

$$\frac{1}{\pi(x)^{\frac{r}{2}}}\frac{1}{\#\mathcal{F}_k}A(r_1,\dots,r_u) \longrightarrow 0 \qquad \text{as } x \to \infty. \tag{21}$$

Proof. First we consider the case $(r_1, r_2,\dots,r_u) = (2,2,\dots,2)$ and will prove (20). Recalling $r_1 + \cdots + r_u = r$, we have

$$u = \frac{r}{2}. \tag{22}$$

From (18), (6) and the fact $h_2(0) = h_2(2) = 1$, it follows that

$$\begin{aligned} B(r_1,\dots,r_u;p_1,\dots,p_u) &= \sum_{\substack{j_1,\dots,j_u \\ =0 \text{ or } 2}} \sum_{f\in\mathcal{F}_k} \lambda_f(p_1^{j_1}\cdots p_u^{j_u}) \\ &= \sum_{f\in\mathcal{F}_k} \lambda_f(1) + \sum\nolimits^{(4)}_{(j_1,\dots,j_u)} \sum_{f\in\mathcal{F}_k} \lambda_f(p_1^{j_1}\cdots p_u^{j_u}), \end{aligned} \tag{23}$$

where $\sum^{(4)}_{(j_1,\dots,j_u)}$ denotes the sum over the u-tuples $(j_1,\dots,j_u)$ satisfying that for each $i = 1,\dots,u$, j_i is equal to 0 or 2 and that $(j_1,\dots,j_u) \ne (0,\dots,0)$. Then since

$p_1^{j_1}\cdots p_u^{j_u}$ in (23) is a square, Lemma 1 gives

$$\begin{aligned}&B(r_1,\dots,r_u;p_1,\dots,p_u)\\ &= \#\mathcal{F}_k + \sum\nolimits^{(4)}_{(j_1,\dots,j_u)}\left(\frac{k-1}{12}\frac{1}{(p_1^{j_1}\cdots p_u^{j_u})^{\frac12}} + O((p_1^{j_1}\cdots p_u^{j_u})^c)\right).\end{aligned}$$

So

$$\begin{aligned}&A(r_1,\dots,r_u)\\ &= \#\mathcal{F}_k\sum\nolimits^{(2)}_{(p_1,\dots,p_u)}1 + \frac{k-1}{12}\sum\nolimits^{(2)}_{(p_1,\dots,p_u)}\sum\nolimits^{(4)}_{(j_1,\dots,j_u)}\frac{1}{(p_1^{j_1}\dots p_u^{j_u})^{\frac12}}\\ &\quad+\sum\nolimits^{(2)}_{(p_1,\dots,p_u)}\sum\nolimits^{(4)}_{(j_1,\dots,j_u)}O((p_1^{j_1}\dots p_u^{j_u})^c).\end{aligned}\tag{24}$$

We have

$$\begin{aligned}\sum\nolimits^{(2)}_{(p_1,\dots,p_u)}1 &= \sum_{\substack{(p_1,\dots,p_u):\\ p_1,\dots,p_u\le x}}1 - \sum_{\substack{(p_1,\dots,p_u):\\ p_1,\dots,p_u\le x,\\ p_i=p_j \text{ for some } i,j\ (i\ne j)}}1\\ &= \pi(x)^u + O_u\left(\pi(x)^{u-1}\right) = \pi(x)^{\frac r2} + O_r\left(\pi(x)^{\frac r2-1}\right)\end{aligned}\tag{25}$$

by (22). Since for an u-tuple $(j_1,\dots,j_u)$ in $\sum^{(4)}_{(j_1,\dots,j_u)}$, j_i's are 0 or 2, and the formula $\sum_{p\le x}\frac1p\sim\log\log x$ holds, we have

$$\begin{aligned}&\sum\nolimits^{(2)}_{(p_1,\dots,p_u)}\sum\nolimits^{(4)}_{(j_1,\dots,j_u)}\frac{1}{(p_1^{j_1}\cdots p_u^{j_u})^{\frac12}}\\ &=\sum\nolimits^{(4)}_{(j_1,\dots,j_u)}\sum\nolimits^{(2)}_{(p_1,\dots,p_u)}\frac{1}{(p_1^{j_1}\cdots p_u^{j_u})^{\frac12}}\\ &\le\sum\nolimits^{(4)}_{(j_1,\dots,j_u)}\sum_{p_1,\dots,p_u\le x}\frac{1}{(p_1^{j_1}\cdots p_u^{j_u})^{\frac12}}\\ &=\sum\nolimits^{(4)}_{(j_1,\dots,j_u)}\pi(x)^{\alpha(j_1,\dots,j_u)}\left(\sum_{p\le x}\frac1p\right)^{u-\alpha(j_1,\dots,j_u)}\\ &\ll_u\sum\nolimits^{(4)}_{(j_1,\dots,j_u)}\pi(x)^{\alpha(j_1,\dots,j_u)}(\log\log x)^{u-\alpha(j_1,\dots,j_u)},\end{aligned}\tag{26}$$

where for an u-tuple $(j_1,\dots,j_u)$ we put $\alpha(j_1,\dots,j_u):=\#\{1\le i\le u\mid j_i=0\}$. For any u-tuple $(j_1,\dots,j_u)$ in $\sum^{(4)}_{(j_1,\dots,j_u)}$, we have $\alpha(j_1,\dots,j_u)\le u-1$. This, (26) and (22) yield

$$\begin{aligned}&\sum\nolimits^{(2)}_{(p_1,\dots,p_u)}\sum\nolimits^{(4)}_{(j_1,\dots,j_u)}\frac{1}{(p_1^{j_1}\cdots p_u^{j_u})^{\frac12}}\\ &\ll_u\sum\nolimits^{(4)}_{(j_1,\dots,j_u)}\pi(x)^{u-1}(\log\log x)^u\\ &\ll_r\pi(x)^{u-1}(\log\log x)^u=\pi(x)^{\frac r2-1}(\log\log x)^{\frac r2}.\end{aligned}\tag{27}$$

We have

$$\begin{aligned}
&\sum\nolimits^{(2)}_{(p_1,\dots,p_u)} \sum\nolimits^{(4)}_{(j_1,\dots,j_u)} O((p_1^{j_1}\dots p_u^{j_u})^c) \\
&\ll \sum\nolimits^{(2)}_{(p_1,\dots,p_u)} \sum\nolimits^{(4)}_{(j_1,\dots,j_u)} x^{urc} \\
&\ll_r \sum\nolimits^{(2)}_{(p_1,\dots,p_u)} x^{urc} \le \pi(x)^u x^{urc} = \pi(x)^{\frac{r}{2}} x^{\frac{1}{2}r^2c}.
\end{aligned} \tag{28}$$

Combining (24), (25), (27) and (28), we obtain

$$\begin{aligned}
A(r_1,\dots,r_u) &= \#\mathcal{F}_k \left(\pi(x)^{\frac{r}{2}} + O_r\left(\pi(x)^{\frac{r}{2}-1}\right)\right) \\
&\quad + \frac{k-1}{12} O_r\left(\pi(x)^{\frac{r}{2}-1}(\log\log x)^{\frac{r}{2}}\right) + O_r\left(\pi(x)^{\frac{r}{2}} x^{\frac{1}{2}r^2c}\right),
\end{aligned}$$

so that, by (4), (13) and the formula $\pi(x) \sim x/\log x$,

$$\frac{1}{\pi(x)^{\frac{r}{2}}} \frac{1}{\#\mathcal{F}_k} A(r_1,\dots,r_u) \longrightarrow 1$$

as $x \to \infty$. Therefore (20) is proved.

Next we shall consider the case $(r_1, r_2, \dots, r_u) \ne (2, 2, \dots, 2)$ and prove (21). In this case, using the assumption and recalling $r_1 + \dots + r_u = r$, we have

$$u \le \frac{r}{2} - 1. \tag{29}$$

By (18), (6) and Lemma 1,

$$\begin{aligned}
&B(r_1,\dots,r_u;p_1,\dots,p_u) \\
&= \sum\nolimits^{(5)}_{(j_1,\dots,j_u)} h_{r_1}(j_1)\cdots h_{r_u}(j_u) \sum_{f\in\mathcal{F}_k} \lambda_f(p_1^{j_1}\cdots p_u^{j_u}) \\
&= \sum\nolimits^{(5)}_{(j_1,\dots,j_u)} h_{r_1}(j_1)\cdots h_{r_u}(j_u) \left(\frac{k-1}{12}\frac{1}{(p_1^{j_1}\cdots p_u^{j_u})^{\frac{1}{2}}} + O((p_1^{j_1}\cdots p_u^{j_u})^c)\right),
\end{aligned}$$

where $\sum^{(5)}_{(j_1,\dots,j_u)}$ denotes the sum over the u-tuples $(j_1,\dots,j_u)$ of even integers satisfying that $0 \le j_i \le r_i$ for each $i = 1,\dots,u$. Hence

$$\begin{aligned}
A(r_1,\dots,r_u) &= \frac{k-1}{12} \sum\nolimits^{(2)}_{(p_1,\dots,p_u)} \sum\nolimits^{(5)}_{(j_1,\dots,j_u)} \frac{h_{r_1}(j_1)\cdots h_{r_u}(j_u)}{(p_1^{j_1}\cdots p_u^{j_u})^{\frac{1}{2}}} \\
&\quad + \sum\nolimits^{(2)}_{(p_1,\dots,p_u)} \sum\nolimits^{(5)}_{(j_1,\dots,j_u)} h_{r_1}(j_1)\cdots h_{r_u}(j_u)\, O((p_1^{j_1}\cdots p_u^{j_u})^c).
\end{aligned}$$

Using (29), we have

$$\begin{aligned}
&\sum\nolimits^{(2)}_{(p_1,\dots,p_u)} \sum\nolimits^{(5)}_{(j_1,\dots,j_u)} \frac{h_{r_1}(j_1)\cdots h_{r_u}(j_u)}{(p_1^{j_1}\cdots p_u^{j_u})^{\frac{1}{2}}} \\
&= \sum\nolimits^{(2)}_{(p_1,\dots,p_u)} \sum\nolimits^{(5)}_{(j_1,\dots,j_u)} O_r(1) \\
&= \sum\nolimits^{(2)}_{(p_1,\dots,p_u)} O_r(1) \ll_r \pi(x)^u \le \pi(x)^{\frac{r}{2}-1}
\end{aligned}$$

and

$$\begin{aligned}
&\sum\nolimits^{(2)}_{(p_1,\dots,p_u)} \sum\nolimits^{(5)}_{(j_1,\dots,j_u)} h_{r_1}(j_1)\cdots h_{r_u}(j_u)\, O((p_1^{j_1}\cdots p_u^{j_u})^c)\\
&= \sum\nolimits^{(2)}_{(p_1,\dots,p_u)} \sum\nolimits^{(5)}_{(j_1,\dots,j_u)} O_r(x^{urc}) = \sum\nolimits^{(2)}_{(p_1,\dots,p_u)} O_r(x^{urc})\\
&\ll_r \pi(x)^u x^{urc} \le \pi(x)^{\frac{r}{2}-1} x^{(\frac{r}{2}-1)rc},
\end{aligned}$$

so that

$$A(r_1,\dots,r_u) \ll_r \frac{k-1}{12}\pi(x)^{\frac{r}{2}-1} + \pi(x)^{\frac{r}{2}-1}x^{(\frac{r}{2}-1)rc}.$$

This, (4) and (13) give

$$\frac{1}{\pi(x)^{\frac{r}{2}}}\frac{1}{\#\mathcal{F}_k}A(r_1,\dots,r_u) \ll_r \frac{1}{\pi(x)}\frac{\frac{k-1}{12}}{\#\mathcal{F}_k} + \frac{\pi(x)^{-1}x^{(\frac{r}{2}-1)rc}}{\#\mathcal{F}_k} \longrightarrow 0$$

as $x \to \infty$. Therefore we obtain (21). □

From Lemma 3 and Lemma 4 it follows that for an u-tuple $(r_1,\dots,r_u)$ in (17),

$$\frac{1}{\pi(x)^{\frac{r}{2}}}\frac{1}{\#\mathcal{F}_k}A(r_1,\dots,r_u) \longrightarrow \begin{cases} 1 & \text{if } (r_1,\dots,r_u) = (2,\dots,2),\\ 0 & \text{otherwise,}\end{cases} \tag{30}$$

as $x \to \infty$. Note that if $(r_1,\dots,r_u) = (2,\dots,2)$, then $u = \frac{r}{2}$, as in (22).

We now consider the sum $U_{x,r}$ in (17) and let x be a positive integer. Suppose that r is odd. Then by $r_1 + \cdots + r_u = r$ and (30), all terms in (17) go to 0 as $x \to \infty$, and hence $U_{x,r}$ goes to 0. Next suppose that r is even. Then by (30) we have that as $x \to \infty$, all terms in (17) go to 0 except the term corresponding to $u = \frac{r}{2}$ and $r_1 = \cdots = r_u = 2$, which goes to $r!/(r_1!\cdots r_u!u!) = \frac{r!}{2^{\frac{r}{2}}\,(\frac{r}{2})!}$. Hence $U_{x,r}$ goes to $\frac{r!}{2^{\frac{r}{2}}\,(\frac{r}{2})!}$.

This, (14) and (15) complete the proof of Theorem 2.

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