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A SURPRISING COVERING OF THE REAL LINE

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Abstract. We construct an increasing sequence of Borel subsets of $\mathbb{R}$, such that their union is $\mathbb{R}$, but $\mathbb{R}$ cannot be covered with countably many translations of one set. The proof uses a random method.

1. Introduction

Consider the following problem: if S is a subset of the real line, then how many translated copies of S can cover $\mathbb{R}$? We say that the set S is κ-small if the real line cannot be covered with less than κ translates of S. For example, if S is of first category or S has zero Lebesgue measure, we clearly need more than countably many translates. For an overview of this problem see [3] and [4]. Here we mention a few of the newest instances and aspects of this topic:

Gruenhage has observed that for any given compact set S of positive Lebesgue measure, it is consistent with ZFC that S is not c-small. He has proved, on the other hand, that the Cantor set is c-small. Darji and Keleti have extended this result to every compact set of packing dimension less than 1 (see [2]). Abért and Keleti [1] have constructed an increasing sequence of subsets of $\mathbb{R}$ such that their union is $\mathbb{R}$, but every set is c-small. (We give a short proof in Proposition 1.) They have used this construction to represent every permutation of the plane as a product of finitely many very simple permutations. (Komjáth [5] has extended their result to every infinite Abelian group.) Their construction uses the Axiom of Choice, hence it does not give very nice (Borel, measurable, etc.) sets. Because of its paradoxical nature one would expect that there is no similar construction for Borel sets. However, in this paper we construct a sequence of ω_1-small Borel sets such that their union is $\mathbb{R}$. The proof is based on an elementary random method (strong law of large numbers).

2. Main result

We denote the continuum cardinality by c, and the real, rational and natural numbers by $\mathbb{R}, \mathbb{Q}$ and $\mathbb{N}$, respectively. The complement of the set S is denoted by

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$S^c = \mathbb{R} \setminus S$. By random digit we always mean a random variable which is zero or one with the same probability $\frac{1}{2}$.

Proposition 1. *There exists an increasing sequence of subsets of* $\mathbb{R}$, A_k, *such that* $\bigcup A_k = \mathbb{R}$, *but* $\mathbb{R}$ *cannot be covered with less than continuum translates of any* A_k.

Proof. Take a Hamel base h_α ($\alpha \in c$). For every $k \in \mathbb{N}$ let

$$A_k = \Big\{ \sum_\alpha q_\alpha h_\alpha : q_\alpha \in \mathbb{Q}, |q_\alpha| < k, q_\alpha = 0 \text{ except of finitely many } \alpha \Big\}.$$

Clearly $\bigcup A_k = \mathbb{R}$, and $A_k \subset A_{k+1}$. Fix a $k \in \mathbb{N}$ and consider the numbers of the form $3kh_\alpha$. At most one of these numbers can be covered by a translate of A_k, and the cardinality of these numbers is a continuum. □

The next lemma is about a sequence of zero-one digits. We denote by S_h the average of the first h digits.

Lemma 2. *There exists a sequence* $\{p_l\}$ *converging to zero such that the following holds. For any zero-one sequence of* l *digits if we continue the sequence with independent random digits* (0 *or* 1 *with the same probability*), *then*

$$P\Big(\sup_{i>0} S_{l+i} > 1-t\Big) < p_l.$$

Proof. Take a sequence of independent random digits. Denote the average of the first i by ζ_i. By the strong law of large numbers ζ_i converges to $\frac{1}{2}$ with probability 1, hence for every fixed $\varepsilon > 0$,

$$P\left(\sup_{m>l} \zeta_m > \frac{1}{2} + \varepsilon\right) \to 0.$$

Whenever $S_{l+i} > 1-t$ the average of the last i digits must be at least $1-t$, and $i > l$ since every digit is at most 1 and $S_l < 1-2t$. Since $1-t > \frac{3}{4}$ we have

$$P\left(\sup_i S_{l+i} > 1-t\right) < P\left(\sup_{m>l} \zeta_m > 1-t\right) < P\left(\sup_{m>l} \zeta_m > \frac{3}{4}\right) =: p_l,$$

and this converges to zero as we have seen. □

Theorem 3. *There exists an increasing sequence of Borel subsets of* $\mathbb{R}$, $A_1 \subset A_2 \subset \dots$, *such that* $\bigcup A_k = \mathbb{R}$, *but* $\mathbb{R}$ *cannot be covered with countably many translates of any* A_k.

Proof. We write every real number in a binary system. If a number has two binary forms we choose the one that contains only zeros after some digits. Denote by $S_i(x)$ the average of the first i digits in the binary form of $x \in \mathbb{R}$. Let

$$A_k = \left\{ x \in \mathbb{R} : \limsup_i S_i(x) = 1 \text{ or} \limsup_i S_i(x) < 1 - \frac{1}{k} \right\}.$$

Clearly $A_k \subset A_{k+1}$ and $\bigcup A_k = \mathbb{R}$. The method used to prove that such type sets are Borel is rather well known. Consider the sets $B_m(r) = \Big\{ x \in \mathbb{R} : S_m(x) \le r \Big\}$. Each $B_m(r)$ is a countable union of intervals, and

$$A_k = \left(\bigcup_{d=1}^\infty \bigcup_{M=1}^\infty \bigcap_{m=M}^\infty B_m\left(1 - \frac{1}{k} - \frac{1}{d}\right)\right) \cup \left(\bigcap_{i=1}^\infty \bigcap_{M=1}^\infty \bigcup_{m=M}^\infty B_m\left(1 - \frac{1}{i}\right)^c \right).$$

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We should prove that $\mathbb{R}$ cannot be covered with countably many translates of any A_k. Fix an integer k and a sequence $\{x_n\}_{n=1}^{\infty}$. We construct an $x \in \mathbb{R}$ such that $x \notin A_k - x_n (n \in \mathbb{N})$. Take a sequence y_n that consists of the x_n's and contains every x_n infinitely many times. We shall construct a number $x \in \mathbb{R}$ and a sequence of integers $s(n) \to \infty$ such that

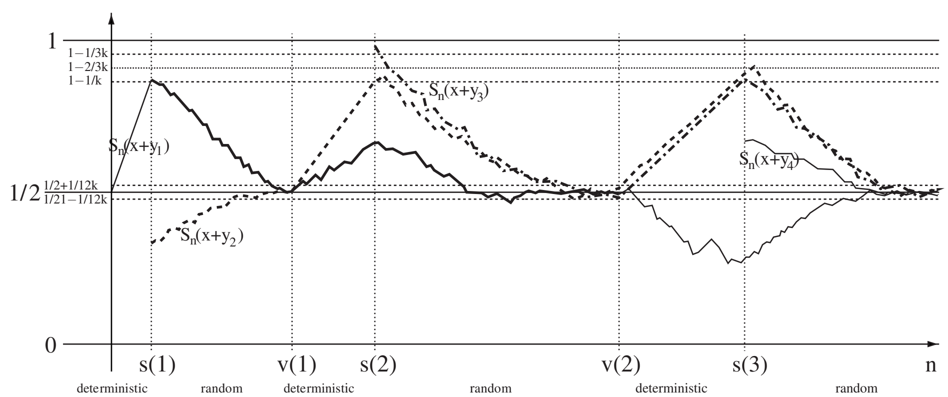
(i) $S_{s(n)}(x + y_n) \geq 1 - \frac{1}{k}$ for every $n \in \mathbb{N}$,

(ii) $S_i(x + y_n) < 1 - \frac{1}{3k}$ for every $i > s(n)$

hold. This would imply $x \notin \bigcup_n A_k - x_n$.

We construct the digits of x block by block, and we do not change the digits we have determined. The construction determines more and more digits and converges to a number $x \in \mathbb{R}$. We always determine a block of digits, one block by a deterministic method and the next by a random method. Let $v(n)$ and $s(n)$ (which will be chosen later) denote the place where the n^{th} random and deterministic block will end, respectively.

At the end of the n^{th} deterministic block we want to make the average $S_{s(n)}(x + y_n)$ bigger than $1 - \frac{1}{k}$. But we also want to keep $S_n(x + y_j)$ below $1 - \frac{1}{3k}$ for every $j < n$. For this we choose a very long random block before this deterministic block, and then we make every digit of $x + y_n$ to 1 in the deterministic block. This will indeed help since all of the averages $S_{v(n)}(x + y_j)$ $(j \leq n)$ will be very close to $\frac{1}{2}$ (and very close to each other), and $S_i(x + y_n)$ increases as fast as possible in the deterministic block. Therefore when the average $S_i(x + y_n)$ reaches $1 - \frac{1}{k}$ the other averages $S_i(x + y_j)$ remain less than $1 - \frac{2}{3k}$. We choose this i for $s(n)$.



We should deal with some technical problems. We must be careful since a digit sequence constructed as the binary form of some $x + y_n$ $(n \in \mathbb{N})$ should contain infinitely many zeros. (Every $x \in \mathbb{R}$ has a unique binary form containing infinitely many zeros, and the definition of the A_k's deals only with this binary form.) Sometimes (for example after the $(s(n) + 1)^{th}$ place) we make a zero digit of the number $x + y_n$, and this does not change the averages very much.

Suppose that we have chosen the first h digits of x to determine the first h digits of $x + y_j$. But taking the sum $x + y_j$ there can be a carry at the $(h + 1)^{th}$ digit changing what we have determined. To avoid this problem we choose the $(h + 1)^{th}$ digit of x to be the same as the $(h + 1)^{th}$ digit of y_j. Hence it does not depend on the latter digits whether there is a carry at this place. (But the choice of the digits

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in the block before depends on this carry.) The $(h+1)^{th}$ "defender digit" does not change the average very much since h is always very large.

Let $x^{(i)}$ denote the number that agrees with x in its first i digits, and all of its other digits are zeros. After the m^{th} step of the construction we need the following properties: we have integers $s(j)$ $(j \leq m)$ and the first $s(m)+1$ digits of x, such that

(1) $S_{s(j)}(x+y_j) > 1-\frac{1}{k}$ $(1 \leq j \leq m)$ independently of the latter digits of x.

(2) $S_{s(j)+i}(x+y_j) < 1-\frac{1}{3k}$ $\Big(1 \leq j \leq m, 0 \leq i \leq s(m)-s(j)+1\Big)$ independently of the latter digits of x.

(3) $S_{s(m)}(x+y_j) < 1-\frac{2}{3k}$ $(1 \leq j \leq m)$ independently of the latter digits of x.

(4) $s(m)$ is so large that if we continue $x^{(s(m))}$ using random digits, then

$$P\Big(\sup_i S_{s(m)+i}(x+y_j) > 1-\frac{1}{3k}\Big) < \frac{1}{4m} \qquad (1 \leq j \leq m).$$

These conditions for every m clearly imply (i) and (ii).

The 1^{st} step of the construction:

We succeed to have $1-\frac{1}{k} < S_{s(1)}(x+y_1) < 1-\frac{2}{3k}$ and $s(1)$ to be large enough; these imply conditions (2) and (4) by Lemma 2. (Conditions (1) and (3) are trivial.) We choose an $s(1)$ large enough and the digits of $x+y_1$ such that

$$1-\frac{1}{k} < S_{s(1)}(x+y_1) < 1-\frac{2}{3k},$$

and we make a "defender digit" at the $(s(1)+1)^{th}$ place.

The general, n^{th} step of the construction:

Suppose that we have ended the $(n-1)^{th}$ step; this means that we have conditions (1)-(4) for $m=n-1$. We shall choose $s(n)$ and the next $(s(n)-s(n-1))$ digits of x such that these four conditions will hold for $m=n$. Let $v(n) > s(n-1)$ be large enough for the next two conditions.

First by the strong law of large numbers, if we take random digits after the $(s(n-1)+1)^{th}$ digit until the $v(n-1)^{th}$ digit, then

$$P\Big(|S_{v(n-1)}(x^{(v(n-1))}+y_j)-\frac{1}{2}| \geq \frac{1}{12k}\Big) < \frac{1}{4n} \ (1 \leq j \leq n)$$

holds, if $v(n-1)$ is large enough. This implies

(5) $P\bigg((\exists j \leq n)\Big(|S_{v(n-1)}(x^{v(n-1)}+y_j)-\frac{1}{2}| \geq \frac{1}{12k}\Big)\bigg) < \frac{1}{4}$.

Second by Lemma 2 for $v(n-1)$ large enough if we have for some $s \geq v(n-1)$

(6) $S_s(x^{(s)}+y_j) < 1-\frac{2}{3k}$ $(1 \leq j \leq n)$,

then continuing x using random digits after the s^{th} place we get

$$P\Big(\sup_i S_{s+i}(x+y_j) > 1-\frac{1}{3k}\Big) < \frac{1}{4n} \quad (1 \leq j \leq n),$$

which yields

(7) $P\bigg((\exists j \leq n)\Big(\sup_i S_{s+i}(x+y_j) \geq 1-\dfrac{1}{3k}\Big)\bigg) < \dfrac{1}{4}$.

We need (7) for condition (4) at the end of the n^{th} step. After the $v(n-1)^{th}$ place we take n "defender digits" to defend the first $v(n-1)$ digits of the sums $x+y_j$ $(1 \leq j \leq n)$. The "defender digits" do not change the distribution of the averages of the previous digits, since they remain random digits. Hence (5) is true for x

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instead of $x^{(v(n-1))}$, and we have with at least $1-\frac{1}{4}$ probability

(8) $|S_{v(n-1)}(x+y_j)-\frac{1}{2}|<\frac{1}{12k} \quad (1\le j\le n)$.

Condition (4) for $m=n-1$ implies that the averages remain less than $1-\frac{1}{3k}$ with at least $1-\frac{1}{4}$ probability during the random choice:

(9) $S_l(x+y_j)<1-\frac{1}{3k} \qquad \big(1\le j\le n, s(n-1)+1\le l\le v(n-1)\big)$.

We choose the digits in this random block such that (8) and (9) will hold; we can do this since it has a probability greater than $\frac{1}{2}$. Now we determine $s(n)$. After the $(v(n-1)+n)^{th}$ place we choose every digit of $x+y_n$ to be 1 until the average of the digits is more than $1-\frac{1}{k}$ at a place; let $s(n)$ be the first such place. The next is a "defender digit". (First we choose the "defender digit", and the choice of the previous digits depends on whether there is a carry at this place.) We have

– $S_{s(n)}(x+y_j)>1-\frac{1}{k}$ $(1\le j\le n)$ independent of the latter digits of x, (1) holds for $m=n$.

The other averages $x+y_j$ $(1\le j\le n-1)$ cannot increase much faster in this deterministic block. If $S_{v(n)}(x+y_j)\le S_{v(n)}(x+y_n)$, then $S_{s(n)}(x+y_j)\le S_{s(n)}(x+y_n)$. If $S_{v(n)}(x+y_j)>S_{v(n)}(x+y_n)$, then $S_{s(n)}(x+y_j)-S_{s(n)}(x+y_n)\le S_{v(n)}(x+y_j)-S_{v(n)}(x+y_n)<\frac{1}{6k}$ by (8). These yield

– $S_{s(n)}(x+y_j)<1-\frac{2}{3k}$ $(1\le j\le n-1)$, hence (3) holds for $m=n$.

– (9) implies (2) after the n^{th} step.

– (3) is true at the end of the n^{th} step; this implies (6) for $s=s(n)$.

Hence (7) holds for $s=s(n)$, which means that we have (4) after the n^{th} step. □

Corollary 4. *There is a decreasing sequence of Borel subsets $B_n\subset\mathbb{R}$ such that $\bigcap B_n=\emptyset$, but for every k and countable set S, a translate of S is in B_k.*

Proof. Take the sets A_n from Theorem 3, and let $B_n:=A_n^c$ $(n\in\mathbb{N})$. Clearly $\bigcap B_n=\emptyset$. Choose a set B_k and a countable set S. By Theorem 3 $A_k-S\ne\mathbb{R}$; there is an $x\notin A_k-S$. Hence $x+S\subset A_k^c=B_k$. □

References

[1] M. Abért and T. Keleti, *Shuffle the plane*, Proc. Amer. Math. Soc. **130**, (2002), 549–553. MR1862136 (2003g:03070)

[2] Udayan B. Darji and Tamás Keleti, *Covering $\mathbb{R}$ with translates of a compact set*, Proc. Amer. Math. Soc. **131** (2003), 2593–2596. MR1974660 (2004d:03100)

[3] P. Erdős, *Some remarks on set theory*, Annals of Math. **44**, (1943), 643–646. MR0009614 (5:173c)

[4] Z. Ruzsa, *Euklideszi terek fedése kis halmazokkal* (in Hungarian), Thesis for the Master's Degree (2001).

[5] P. Komjáth, *Five degrees of separation*, Proc. Amer. Math. Soc. **130**, (2002), 2413–2417. MR1897467 (2003c:03082)

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