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THE P-LAPLACE EQUATION ON A CLASS OF GRUSHIN-TYPE SPACES

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ABSTRACT. We find the fundamental solution to the P-Laplace equation in Grushin-type spaces. The singularity occurs at the sub-Riemannian points, which naturally corresponds to finding the fundamental solution of a generalized Grushin operator in Euclidean space. We then use this solution to find an infinite harmonic function with specific boundary data and to compute the capacity of annuli centered at the singularity. A solution to the 2-Laplace equation in a wider class of spaces is presented.

1. GRUSHIN-TYPE SPACES

In this paper, we will find the fundamental solution to the P-Laplace equation for $1 < P < \infty$ in a class of Grushin-type spaces with singularities at the sub-Riemannian points. Before presenting the main theorem, we recall the construction of such spaces and their main properties. We begin with $\mathbb{R}^n$, possessing coordinates $(x_1, x_2, \ldots, x_n)$ and vector fields

$$X_i = \rho_i(x_1, x_2, \ldots, x_{i-1}) \frac{\partial}{\partial x_i}$$

for $i = 2, 3, \ldots, n$ where $\rho_i(x_1, x_2, \ldots, x_{i-1})$ is a (possibly constant) polynomial. We decree that $\rho_1 \equiv 1$, so that

$$X_1 = \frac{\partial}{\partial x_1}.$$

A quick calculation shows that when $i < j$, the Lie bracket is given by

$$X_{ij} \equiv [X_i, X_j] = \rho_i(x_1, x_2, \ldots, x_{i-1}) \frac{\partial \rho_j(x_1, x_2, \ldots, x_{j-1})}{\partial x_i} \frac{\partial}{\partial x_j}. \tag{1.1}$$

Because the ρ_i's are polynomials, at each point there is a finite number of iterations of the Lie bracket so that $\frac{\partial}{\partial x_i}$ has a nonzero coefficient. This is easily seen for X_1 and X_2, and the result is obtained inductively for X_i. (It is noted that the number of iterations necessary is a function of the point.) It follows that Hörmander's condition is satisfied by these vector fields.

We may further endow $\mathbb{R}^n$ with an inner product (singular where the polynomials vanish) so that the collection $\{X_i\}$ forms an orthonormal basis. This produces a sub-Riemannian manifold that we will call g_n, which is also the tangent space

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to a generalized Grushin-type space G_n. Points in G_n will also be denoted by $p = (x_1, x_2, \dots, x_n)$.

Though G_n is not a Lie group, it is a metric space with the natural metric being the Carnot-Carathéodory distance, which is defined for points p and q as follows:

$$d_C(p,q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| \, dt.$$

Here Γ is the set of all curves γ such that $\gamma(0) = p, \gamma(1) = q$ and

$$\gamma'(t) \in \operatorname{span}\{\{X_i(\gamma(t))\}_{i=1}^n\}.$$

By Chow's theorem (see, for example, [1]) any two points can be joined by such a curve, which means $d_C(p,q)$ is an honest metric. Using this metric, we can define a Carnot-Carathéodory ball of radius r centered at a point p_0 by

$$B_C(p_0, r) = \{p \in G_n : d_C(p, p_0) < r\}$$

and similarly, we shall denote a bounded domain in G_n by Ω. The Carnot-Carathéodory metric behaves differently at points where the polynomials ρ_i vanish. Fixing a point p_0, consider the n-tuple $r_{p_0} = (r_{p_0}^1, r_{p_0}^2, \dots, r_{p_0}^n)$ where $r_{p_0}^i$ is the minimal number of Lie bracket iterations required to produce

$$[X_{j_1}, [X_{j_2}, [\cdots [X_{j_{r_{p_0}^i}}, X_i] \cdots]]](p_0) \neq 0.$$

Note that though the minimal length is unique, the iteration used to obtain that minimum is not. Note also that

$$\rho_i(p_0) \neq 0 \leftrightarrow r_{p_0}^i = 0.$$

Using Theorem 7.34 from [1] we obtain the local estimate at p_0,

$$d_C(p_0, p) \sim \sum_{i=1}^n |x_i - x_i^0|^{\frac{1}{1+r_{p_0}^i}}. \tag{1.2}$$

Given a smooth function f on G_n, we define the horizontal gradient of f as

$$\nabla_0 f(p) = (X_1 f(p), X_2 f(p), \dots, X_n f(p))$$

and the symmetrized second order (horizontal) derivative matrix by

$$((D^2 f(p))^\star)_{ij} = \frac{1}{2}(X_i X_j f(p) + X_j X_i f(p))$$

for $i, j = 1, 2, \dots, n$.

Definition 1. The function $f : G_n \mapsto \mathbb{R}$ is said to be C^1_{sub} if $X_i f$ is continuous for all $i = 1, 2, \dots, n$. Similarly, the function f is C^2_{sub} if $X_i X_j f(p)$ is continuous for all $i, j = 1, 2, \dots, n$.

Using these derivatives, we consider two main operators on C^2_{sub} functions called the P-Laplacian

$$\Delta_P f = \operatorname{div}(\|\nabla_0 f\|^{P-2} \nabla_0 f) = \sum_{i=1}^n X_i(\|\nabla_0 f\|^{P-2} X_i f)$$

defined for $1 < P < \infty$ and the infinite Laplacian

$$\Delta_\infty f = \sum_{i,j=1}^n X_i f X_j f X_i X_j f = \langle \nabla_0 f, (D^2 f)^\star \nabla_0 f \rangle.$$

We may define Sobolev spaces in the natural way. Namely, for any open set $\mathcal{O} \subset G_n$, the function f is in the horizontal Sobolev space $W^{1,q}(\mathcal{O})$ if the functions $f,\ X_1 f, \ldots, X_n f$ lie in $L^q(\mathcal{O})$. Replacing $L^q(\mathcal{O})$ by $L^q_{loc}(\mathcal{O})$, the space $W^{1,q}_{loc}(\mathcal{O})$ is defined similarly. We may then use these Sobolev spaces to consider the above operators in the usual weak sense.

2. The co-area formula and measure theory

Given fixed numbers $m \in \mathbb{N}$, $a \in \mathbb{R}$, and $0 \neq c \in \mathbb{R}$, we consider the following vector fields:

$$\left\{ \begin{array}{lcl} X_1 & = & \dfrac{\partial}{\partial x_1}, \\ X_i & = & c(x_1 - a)^m \dfrac{\partial}{\partial x_i} \ \text{ for } i = 2 \text{ to } n. \end{array} \right. \tag{2.1}$$

Note that this choice corresponds to $\rho_i(x_1, x_2, \ldots, x_{i-1}) = c(x_1 - a)^m$ for $2 \leq i \leq n$. Note also that in local coordinates, the 2-Laplacian operator is given by

$$\frac{\partial^2}{\partial x_1^2} + c^2(x_1 - a)^{2m} \sum_{i=2}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Let $\Omega \subset G_n$ be a bounded domain, and let $\psi \in C^1_{\text{sub}}(\Omega)$ be a smooth, real-valued function which extends continuously to $\partial\Omega$. For convenience, we write ∇ for the Euclidean gradient on $G_n = \mathbb{R}^n$. In place of Fubini's Theorem for iterated integrals, we will make use of the following co-area formula in the Euclidian context [6], which was extended to the Grushin case via Theorem 4.2 in [7].

Theorem 2.1. *Under the hypotheses as above, then for any function $g \in L^1(\Omega)$,*

$$\iint_\Omega g \|\nabla \psi\| \, d\mathcal{L}_n \ = \ \int_0^\infty \int_{\psi^{-1}\{r\}} g \, d\mathcal{H} dr, \tag{2.2}$$

where $d\mathcal{L}_n$ denotes Lebesgue n-measure on Ω, and $d\mathcal{H}$ denotes Hausdorff $(n-1)$-measure on $\psi^{-1}(\{r\})$.

Remark 1. As above, the theorem also holds for continuous functions ψ which are smooth everywhere except at isolated points.

We now suppose the particular case where $p_0 \in G_n$ has coordinates $p_0 = (a, b_2, \ldots, b_n)$ and ψ is a nonnegative radial function with $\psi(p_0) = 0$. The following notation is suggestive for the inverse images of ψ:

$$\begin{array}{rcccl} B_R(p_0) & = & \psi^{-1}([0,R)) & = & \{p \in \Omega : \psi(p) < R\}, \\ \partial B_R(p_0) & = & \psi^{-1}(\{R\}) & = & \{p \in \Omega : \psi(p) = R\}. \end{array}$$

The p_0 is omitted when it is clear from the context. Now choose $g(x) := \|\nabla_0 \psi\|^P$. Since $\|\nabla_0 \psi\| \lesssim \|\nabla \psi\|$, we may apply the co-area formula to the function $g = (g/\|\nabla\psi\|) \cdot \|\nabla\psi\|$ to obtain the following proposition.

Proposition 2.2. *Let $\mathcal{V}$ be an absolutely continuous measure to $\mathcal{L}_n$ with Radon-Nikodym derivative $g = [d\mathcal{V}/d\mathcal{L}_n]$. Then for sufficiently small $R > 0$,*

$$\mathcal{V}(B_R) \ = \ \int_{B_R} d\mathcal{V} \ = \ \int_0^R \int_{\partial B_r} \frac{g}{\|\nabla\psi\|} \, d\mathcal{H} dr. \tag{2.3}$$

In light of the equality in (2.3), we see that the measure space $(G_n, \mathcal{V})$ is globally Ahlfors Q-regular with respect to balls centered at p_0. In particular, for $R > 0$,

$$\mathcal{V}(B_R) \;=\; \sigma_P R^Q \tag{2.4}$$

where $Q = (m+1)(n-1)+1$ and $\sigma_P = \mathcal{V}(B_1)$ is a fixed positive constant.

For technical purposes we proceed to study the boundary behavior of precompact domains Ω. This now motivates the following definition.

Definition 2. For small values $R \in R_\psi$, define a measure $\mathcal{S}$ on ∂B_R as

$$\mathcal{S}(\partial B_R) \;=\; \int_{\partial B_R} d\mathcal{S} \;=\; \int_{\partial B_R} \frac{g}{\|\nabla\psi\|}\, d\mathcal{H}.$$

In particular, S is absolutely continuous with respect to the Hausdorff $(n-1)$-measure $\mathcal{H}$. Using previous results, we now conclude:

Corollary 2.3. (1) *S is locally Ahlfors $(Q-1)$-regular and*

$$\mathcal{S}(\partial B_R) \;=\; Q\sigma_1 R^{Q-1}. \tag{2.5}$$

(2) *Let φ be a continuous and integrable function on B_R. Then as $R \to 0$,*

$$\frac{R^{1-Q}}{Q\sigma_P} \int_{\partial B_R} \varphi \, d\mathcal{S} \;\to\; \varphi(0). \tag{2.6}$$

Remark 2. (2.5) follows immediately from differentiating both (2.3) and (2.4). Since $\mathcal{S}$ is absolutely continuous with respect to Hausdorff $(n-1)$-measure $\mathcal{H}$, it follows that $\mathcal{S}$ is Borel regular. As a result, (2.6) is the analogue of the Lebesgue Density Theorem.

3. The P-Laplace equation

In this section, we compute the fundamental solution of the P-Laplacian for the previously-defined vector fields (2.1) and for $1 < P < \infty$. We then use these formulas to find the explicit formula for a solution to the Dirichlet problem with specific boundary data.

Theorem 3.1. *Let $p_0 = (a, b_2, b_3, \ldots, b_n)$ be an arbitrary fixed point. Consider the following quantities, for $1 < P < \infty$:*

$$\begin{aligned}
w &= \frac{Q-P}{(2m+2)(1-P)}, \\
\alpha &= \frac{Q-P}{1-P}, \\
h(x_1, x_2, \ldots, x_n) &= c^2(x_1-a)^{2m+2} + (m+1)^2 \sum_{i=2}^{n} (x_i - b_i)^2, \\
f(x_1, x_2, \ldots, x_n) &= [h(x_1, x_2, \ldots, x_n)]^w, \\
\psi(x_1, x_2, \ldots, x_n) &= [h(x_1, x_2, \ldots, x_n)]^{\frac{1}{2m+2}}, \\
\sigma_P &= \int_{B_1} \|\nabla_0 \psi\|^P \, d\mathcal{L}_n, \\
C_1 &= \alpha^{-1}(Q\sigma_P)^{\frac{1}{1-P}}, \\
C_2 &= (Q\sigma_P)^{\frac{1}{1-P}}.
\end{aligned}$$

Then, for the constants C_1 and C_2 as above,

$$\Delta_P C_1 f(x_1, x_2, \ldots, x_n) = \delta_{p_0} \quad \text{when } P \neq Q, \tag{3.1}$$

$$\Delta_P(C_2 \log \psi(x_1, x_2, \ldots, x_n)) = \delta_{p_0} \quad \text{when } P = Q, \tag{3.2}$$

in the sense of distributions.

Proof. We first comment that for the sake of rigor, we should invoke the regularization of h given by

$$h_\varepsilon(x_1, x_2, \ldots, x_n) = c^2((x_1-a)^2+\varepsilon^2)^{m+1} + (m+1)^2 \sum_{i=2}^{n}(x_i - b_i)^2$$

for $\varepsilon > 0$ and letting $\varepsilon \to 0$. However, we shall proceed formally. Suppressing the variables $(x_1, x_2, \ldots, x_n)$, we compute for $P \neq Q$:

$$\begin{aligned}
X_1 f &= c^2 w h^{w-1}(2m+2)(x_1-a)^{2m+1}, \\
X_j f &= 2cwh^{w-1}(m+1)^2(x_j-b_j)(x_1-a)^m, \\
\|\nabla_0 f\|^2 &= c^2w^2h^{2w-2}4(m+1)^2(x_1-a)^{2m} \\
&\quad \times \left(c^2(x_1-a)^{2m+2} + (m+1)^2\sum_{i=2}^{n}(x_i-b_i)^2\right) \\
&= 4c^2w^2h^{2w-1}(m+1)^2(x_1-a)^{2m}, \\
\|\nabla_0 f\|^{P-2} &= (2(m+1))^{P-2}|cw|^{P-2}h^{(w-\frac{1}{2})(P-2)}|x_1-a|^{mP-2m}.
\end{aligned}$$

To proceed, we require the following quantities:

$$\begin{aligned}
\|\nabla_0 f\|^{P-2}X_1 f &= (2(m+1))^{P-1}|x_1-a|^{mP}(x_1-a)c^2w|cw|^{P-2}h^{wP-w-\frac{P}{2}}, \\
\|\nabla_0 f\|^{P-2}X_i f &= (2(m+1))^{P-1}(m+1)|x_1-a|^{mP-2m}(x_1-a)^m \\
&\quad \times (x_i-b_i)cw|cw|^{P-2}h^{wP-w-\frac{P}{2}},
\end{aligned}$$

where $2 \leq i \leq n$. Writing

$$D_P \equiv \frac{\Delta_P f}{(2(m+1))^{P-1}|cw|^{P-2}cw}$$

we can then compute

$$\begin{aligned}
D_P &= c(mP+1)|x_1-a|^{mP}h^{wP-w-\frac{P}{2}} \\
&\quad + |x_1-a|^{mP}(x_1-a)(wP-w-\frac{P}{2})h^{wP-w-\frac{P}{2}-1}(2m+2)c^3(x_1-a)^{2m+1} \\
&\quad + \sum_{i=2}^{n} c|x_1-a|^{mP}(m+1)h^{wP-w-\frac{P}{2}} \\
&\quad + \sum_{i=2}^{n} 2c(m+1)^3(x_i-b_i)^2(wP-w-\frac{P}{2})h^{wP-w-\frac{P}{2}-1}|x_1-a|^{mP} \\
&= (wP-w-\frac{P}{2})h^{wP-w-\frac{P}{2}-1}|x_1-a|^{mP}(2m+2) \\
&\quad \times \left[c^3(x_1-a)^{2m+2} + c(m+1)^2\sum_{i=2}^{n}(x_i-b_i)^2\right] \\
&\quad + h^{wP-w-\frac{P}{2}}|x_1-a|^{mP}c[mP+1+(m+1)(n-1)]
\end{aligned}$$

$$\begin{aligned} &= h^{wP-w-\frac{P}{2}}|x_1-a|^{mP}c\Big[(wP-w-\frac{P}{2})(2m+2)+mP+Q\Big] \\ &= h^{wP-w-\frac{P}{2}}|x_1-a|^{mP}c\Big[P-Q-P(m+1)+mP+Q\Big] \\ &= 0. \end{aligned}$$

Note that these computations are valid wherever the function f is smooth and, in particular, these are valid away from the point p_0. We next note that

$$\|\nabla_0 f\|^{P-1} \sim \psi^{1-Q},$$

and so we conclude that $\|\nabla_0 f\|^{P-1}$ is locally integrable on G_n. We then consider $\phi \in C_0^\infty$ with compact support in the ball

$$B_R = \{q : \psi(q) < R\}.$$

Let $0 < r < R$ be given so that $B_r \subset B_R$. In the annulus $\mathcal{A} := B_R \setminus \overline{B_r}$ we have, via the Leibniz rule,

$$\begin{aligned} \operatorname{div}(\phi\|\nabla_0 f\|^{P-2}\nabla_0 f) &= \phi\operatorname{div}(\|\nabla_0 f\|^{P-2}\nabla_0 f) + \|\nabla_0 f\|^{P-2}\langle\nabla_0 f, \nabla_0\phi\rangle \\ &= 0 + \|\nabla_0 f\|^{P-2}\langle\nabla_0 f, \nabla_0\phi\rangle. \end{aligned}$$

Let $\mathcal{L}_n$ and $\mathcal{H}$ be the measures from (2.2). Applying Stokes' Theorem,

$$\begin{aligned} &\int_{\mathcal{A}} \|\nabla_0 f\|^{P-2}\langle\nabla_0 f, \nabla_0\phi\rangle d\mathcal{L}_n = \int_{\mathcal{A}} \operatorname{div}(\phi\|\nabla_0 f\|^{P-2}\nabla_0 f) d\mathcal{L}_n \\ &= \int_{\mathcal{A}} \Big(X_1[\phi\|\nabla_0 f\|^{P-2}X_1 f] + \sum_{j=2}^n c(x_1-a)^m \frac{\partial}{\partial x_j}[\phi\|\nabla_0 f\|^{P-2}X_j f]\Big) d\mathcal{L}_n \\ &= \int_{\mathcal{A}} \Big(X_1[\phi\|\nabla_0 f\|^{P-2}X_1 f] + \sum_{j=2}^n \frac{\partial}{\partial x_j}[c(x_1-a)^m\phi\|\nabla_0 f\|^{P-2}X_j f]\Big) d\mathcal{L}_n \\ &= \int_{\mathcal{A}} \operatorname{div}_{\text{eucl}} \begin{bmatrix} \phi\|\nabla_0 f\|^{P-2}X_1 f \\ c(x_1-a)^m\phi\|\nabla_0 f\|^{P-2}X_2 f \\ \vdots \\ c(x_1-a)^m\phi\|\nabla_0 f\|^{P-2}X_n f \end{bmatrix} d\mathcal{L}_n \\ &= \int_{\partial\mathcal{A}} \frac{1}{\|\nu\|}\Big(\phi\|\nabla_0 f\|^{P-2}X_1 f\nu_1 + \sum_{j=2}^n c(x_1-a)^m\phi\|\nabla_0 f\|^{P-2}X_j f\nu_j\Big) d\mathcal{H} \\ &= -\int_{\partial B_r} \frac{1}{\|\nu\|}\Big(\phi\|\nabla_0 f\|^{P-2}X_1 f\nu_1 + \sum_{j=2}^n c(x_1-a)^m\phi\|\nabla_0 f\|^{P-2}X_j f\nu_j\Big) d\mathcal{H} \end{aligned}$$

where ν is the outward Euclidean normal. Recalling that

$$\psi(x_1, x_2, \ldots, x_n) = [h(x_1, x_2, \ldots, x_n)]^{\frac{1}{2m+2}},$$

we have

$$\nu_j = \frac{\partial\psi}{\partial x_j}.$$

Proceeding with the computation,

$$\begin{aligned}
&\int_{\mathcal{A}} \|\nabla_0 f\|^{P-2}\langle \nabla_0 f, \nabla_0 \phi\rangle d\mathcal{L}_n \\
&= -\int_{\partial B_r} \frac{1}{\|\nu\|}\Big(\phi\|\nabla_0 f\|^{P-2}X_1 f\nu_1 + \sum_{j=2}^{n} c(x_1-a)^m \phi\|\nabla_0 f\|^{P-2}X_j f\nu_j\Big) d\mathcal{H} \\
&= -\int_{\partial B_r} \frac{\alpha\psi^{\alpha-1}}{\|\nu\|}\phi\|\nabla_0\psi^{\alpha}\|^{P-2}\Big((\frac{\partial\psi}{\partial x_1})^2 + \sum_{j=2}^{n} c^2(x_1-a)^{2m}(\frac{\partial\psi}{\partial x_j})^2\Big) d\mathcal{H} \\
&= -\int_{\partial B_r} \frac{\alpha\psi^{\alpha-1}}{\|\nu\|}\phi\|\nabla_0\psi\|^{P-2}|\alpha|^{P-2}\psi^{(P-2)(\alpha-1)}\Big(\|\nabla_0\psi\|^2\Big) d\mathcal{H} \\
&= -\int_{\partial B_r} \frac{|\alpha|^{P-2}\alpha\psi^{(P-1)(\alpha-1)}}{\|\nu\|}\phi\|\nabla_0\psi\|^{P} d\mathcal{H}.
\end{aligned}$$

Recall that by definition, $\psi \equiv r$ on ∂B_r. We then have

$$\int_{\mathcal{A}} \|\nabla_0 f\|^{P-2}\langle \nabla_0 f, \nabla_0 \phi\rangle d\mathcal{L}_n = -|\alpha|^{P-2}\alpha r^{1-Q}\int_{\partial B_r} \frac{\phi\|\nabla_0\psi\|^P}{\|\nu\|}\, d\mathcal{H}.$$

Letting $r \to 0$, we apply (2.6) and obtain

$$\int_{\mathcal{A}} \|\nabla_0 f\|^{P-2}\langle \nabla_0 f, \nabla_0 \phi\rangle d\mathcal{L}_n \;\to\; -|\alpha|^{P-2}\alpha(Q\sigma_P)\phi(p_0). \tag{3.3}$$

We then obtain the case for $P \neq Q$. The case of $P = Q$ is similar and left to the reader. □

It was shown in [2] that for our class of Grushin-type spaces, viscosity infinite harmonic functions are limits of weak P-harmonic functions as P tends to infinity. This motivates the following corollary.

Corollary 3.2. *The function ψ, as defined above, is infinite harmonic in the space $G_n \setminus \{p_0\}$.*

Proof. We use the formula that for a smooth function u,

$$\Delta_\infty u = \nabla_0 u \cdot \nabla_0 \|\nabla_0 u\|^2.$$

Computing as in the theorem, we have

$$\|\nabla_0\psi\|^2 = c^2(x_1-a)^{2m}h^{\frac{-m}{m+1}}.$$

Thus, we obtain

$$\begin{aligned}
\nabla_0\|\nabla_0\psi\|^2 &= c^2\nabla_0[(x_1-a)^{2m}h^{\frac{-m}{m+1}}] \\
&= 2m(m+1)c^2(x_1-a)^{2m-1}h^{\frac{-2m-1}{m+1}}\begin{bmatrix} (m+1)\sum_{i=2}^{n}(x_i-b_i)^2 \\ -c(x_1-a)^{m+1}(x_2-b_2) \\ -c(x_1-a)^{m+1}(x_3-b_3) \\ \vdots \\ -c(x_1-a)^{m+1}(x_n-b_n) \end{bmatrix}.
\end{aligned}$$

Combining the calculations above, we have

$$\begin{aligned}\Delta_\infty \psi &= \left[2(m+1)^2\sum_{i=2}^{n}(x_i-b_i)^2-\sum_{i=2}^{n}(x_i-b_i)^2 2(m+1)^2\right] \\ &\quad \times h^{\frac{-6m-3}{2m+2}} 2mc^4(x_1-a)^{4m} \\ &= 0.\end{aligned}$$

The corollary then follows. □

Using the existence-uniqueness of viscosity infinite harmonic functions [2], the fact that absolute minimizers are viscosity infinite harmonic [4], [10], and the equivalence of both to comparison with cones [3], we conclude the following corollary.

Corollary 3.3. *Define the function $g : G_n \mapsto \mathbb{R}$ by*

$$g(x_1, x_2, \dots, x_n) = \begin{cases} 1 & on \quad \partial B(p_0, 1), \\ 0 & at \quad p_0. \end{cases}$$

Then ψ is the unique absolute minimizer of g into the ball $B(p_0, 1)$. In addition, ψ enjoys comparison with cones.

4. Spherical capacity

In this section, we will use previous results to compute the capacity of spherical rings centered at the point $p_0 = (a, b_2, b_3, \dots, b_n)$. We first recall the definition of P-capacity.

Definition 3. Let $\Omega \subset G_n$ be a bounded, open set, and $K \subset \Omega$ a compact subset. For $1 \leq P < \infty$ we define the P-capacity as

$$\operatorname{cap}_P(K, \Omega) = \inf\left\{\int_\Omega \|\nabla_0 u\|^P : u \in C_0^\infty(\Omega),\ u|K = 1\right\}.$$

We note that although the definition is valid for $P = 1$, we will consider only $1 < P < \infty$, as in the previous sections. Because P-harmonic functions are minimizers to the energy integral

$$\int_{G_n} \|\nabla_0 f\|^P,$$

it is natural to consider P-harmonic functions when computing the capacity. In particular, an easy calculation similar to the previous section shows

$$u(p) = \begin{cases} \dfrac{\psi(p)^\alpha - R^\alpha}{r^\alpha - R^\alpha} & \text{when} \quad P \neq Q, \\[2ex] \dfrac{\log \psi(p) - \log R}{\log r - \log R} & \text{when} \quad P = Q \end{cases}$$

is a smooth solution to the Dirichlet problem

$$\begin{cases} \Delta_P u = 0 & \text{in} \quad B(p_0, R) \setminus \overline{B}(p_0, r), \\ u = 1 & \text{on} \quad \partial B(p_0, r), \\ u = 0 & \text{on} \quad \partial B(p_0, R), \end{cases}$$

for $1 < P < \infty$.

We state the following theorem, which follows from the computations of the previous section.

Theorem 4.1. *Let $0 < r < R$ and $1 < P < \infty$. Then we have*

$$\operatorname{cap}_P\big(B(p_0,r),B(p_0,R)\big) = \begin{cases} \alpha^{P-1}Q\sigma_P\big(r^\alpha - R^\alpha\big)^{1-P} & \textit{when} \quad 1<P<Q, \\ Q\sigma_Q[\log R - \log r]^{1-Q} & \textit{when} \quad P=Q, \\ \alpha^{P-1}Q\sigma_P\big(R^\alpha - r^\alpha\big)^{1-P} & \textit{when} \quad P>Q. \end{cases}$$

5. An extension concerning the 2-Laplacian

We extend the results of Section 3 for the 2-Laplace operator to a wider class of Grushin-type spaces. In particular, we consider the following vector fields in $\mathbb{R}^n$:

$$\begin{cases} X_1 &= \dfrac{\partial}{\partial x_1}, \\ X_i &= c_i(x_1-a)^{m_i}\dfrac{\partial}{\partial x_i} \text{ for } i=2 \text{ to } n, \end{cases} \tag{5.1}$$

where $0 \neq c_i \in \mathbb{R}$ and $m_i \in \mathbb{N}\cup\{0\}$. Note that the difference between these vector fields and those of the previous sections is that the leading coefficients c_i and exponents m_i may vary for each X_i. In addition, we do not exclude the case where $m_i = 0$.

In local coordinates, the 2-Laplacian operator is given by

$$\Delta_2 = \frac{\partial^2}{\partial x_1^2} + \sum_{i=2}^{n} c_i^2(x_1-a)^{2m_i}\frac{\partial^2}{\partial x_i^2}.$$

Using the linearity of this operator and computing as in Theorem 3.1, we obtain the following corollary.

Corollary 5.1. *Let $p_0 = (a, b_2, \ldots, b_n)$, $p = (z_1, \ldots, z_n)$, and consider the function*

$$\begin{aligned}\Upsilon(x_1,x_2,\ldots,x_n) &= \sum_{\substack{2\le i\le n\\ m_i\neq 0}} \Big(c_i^2(x_1-a)^{2m_i+2} + (m_i+1)^2(x_i-b_i)^2\Big)^{\frac{-m_i}{2m_i+2}} \\ &\quad + \sum_{\substack{2\le i\le n\\ m_i=0}} \log\Big(c_i^2(x_1-a)^{2m_i+2} + (m_i+1)^2(x_i-b_i)^2\Big)^{\frac{1}{2m_i+2}}.\end{aligned}$$

Then $\Delta_2\Upsilon(p) = 0$, unless $z_1 = a$ and there is an i so that $z_i = b_i$.

Remark 3. In the case where $c = c_i$ and $m = m_i$ for all $2 \le i \le n$, this reduces to the previous environment. However, this solution differs from that of Theorem 3.1, whereas for $n = 2$ and $m = 0$ this solution agrees with the fundamental solution in Euclidean $\mathbb{R}^2$. Because the singular set has positive $(n-2)$-Lebesgue measure, Υ^α is not a logical candidate for the fundamental solution to the P-Laplacian for any exponent α.

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