

## LINKS IN AN OPEN BOOK DECOMPOSITION AND IN THE STANDARD CONTACT STRUCTURE

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*Dedicated to Professor Yukio Matsumoto on his sixtieth birthday*

ABSTRACT. We study a relationship between arc presentations of links in  $\mathbb{R}^3$  and Legendrian links in  $\mathbb{R}^3$  with the standard tight contact structure. We determine the arc indices of torus knots.

### 1. INTRODUCTION

A relationship between closed braid representatives of links in  $\mathbb{R}^3$  and transverse and Legendrian links in  $\mathbb{R}^3$  with the standard contact structure was first discovered by Bennequin in his pioneering work [1] in “contact topology.” He studied this relationship and proved in [1] that the standard contact structure on  $\mathbb{R}^3$  is tight, that is, there is no overtwisted disc in  $(\mathbb{R}^3, \xi_{\text{std}})$ . Since then, some other relationships have been studied; cf. [3], [4]. In this paper, we study a relationship between arc presentations of links in  $\mathbb{R}^3$  and Legendrian links in  $(\mathbb{R}^3, \xi_{\text{std}})$ . The definitions given below suggest that a closed braid representative of a link in  $\mathbb{R}^3$  is a transverse link in  $\mathbb{R}^3$  with the open book decomposition  $\{A(\theta)\}_{0 \leq \theta \leq 2\pi}$ , and that an arc presentation of a link in  $\mathbb{R}^3$  is a (piecewise) Legendrian link in  $\mathbb{R}^3$  with  $\{A(\theta)\}$ . In fact, we show in §2 that one arc presentation of a link in  $\mathbb{R}^3$ , with its topological link type  $\mathcal{L}$ , corresponds to one Legendrian link in  $(\mathbb{R}^3, \xi_{\text{std}})$ , with its topological link type  $\overline{\mathcal{L}}$ , and vice versa, where  $\overline{\mathcal{L}}$  denotes the mirror image of  $\mathcal{L}$ . Studying this correspondence, we determine the arc index of torus knots, Theorem 1.1.

Let  $A$  be the trivial knot in  $S^3$ . The complement  $S^3 \setminus A$  is then a product  $(\text{int } D^2) \times S^1$ . There exists a fibre projection  $p_A: S^3 \setminus A \rightarrow S^1$  whose fibres  $A(\theta) = (p_A)^{-1}(e^{i\theta})$  are discs for  $0 \leq \theta \leq 2\pi$ . An orientation of  $A$  induces an orientation of  $A(\theta)$ , and a positive direction of the fibration  $\{A(\theta)\}_{0 \leq \theta \leq 2\pi}$ . We may think of  $S^3$  as the union  $\mathbb{R}^3 \cup \{\infty\}$  and think of  $A$  as the union ( $z$ -axis)  $\cup \{\infty\}$ . If we use the polar coordinate, then the fibres  $A(\theta)$  are half-planes at angle  $\theta$ .

Let  $L$  be an oriented link in  $S^3$  with  $L \cap A = \emptyset$ . If we can choose a projection  $p_A$  so that  $L$  intersects each fibre  $A(\theta)$  of the fibration  $\{A(\theta)\}$  transversely and

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positively in exactly  $n$  points, then  $L$  is said to be represented as a *closed  $n$ -braid relative to an axis  $A$* .

A link  $L$  in  $S^3$  can be embedded in finitely many fibres  $A(\theta)$  so that  $L$  meets each fibre  $A(\theta)$  in a single simple arc. Such an embedding is called an *arc presentation* of  $L$ , and the number of fibres used to represent  $L$  is the *arc number* of  $L$ , denoted by  $\alpha(L)$ . The minimum number of fibres required to represent a given link  $L$  in this fashion is a knot invariant, called the *arc index* of  $L$ , and denoted by  $\bar{\alpha}(L)$ . Note that  $\bar{\alpha}(L) = \bar{\alpha}(\bar{L})$ , where  $\bar{L}$  denotes the mirror image of  $L$ . This definition of an arc presentation of a link was suggested by Birman and Menasco [2], and was given by Cromwell [6]. Lyon [10] had given a definition of a presentation of links in *square-bridge position*. Lyon's presentation of links is essentially the same as an arc presentation of links. Brunn [5] showed that every link in  $\mathbb{R}^3$  admits a projection onto  $\mathbb{R}^2$  with exactly one multiple point. It follows from Brunn's proof that every link in  $\mathbb{R}^3$  admits an arc presentation.

A *contact structure* on a 3-manifold  $M$  is a completely non-integrable plane field  $\xi$  in the tangent bundle of  $M$ . Since  $\xi$  is a plane field,  $\xi_x$  is a two-dimensional subspace of  $T_x M$  at each point  $x$  in  $M$ . There is a locally defined 1-form  $\alpha$  such that  $\xi_x = \ker(\alpha_x)$ . The plane field  $\xi$  is completely non-integrable if any 1-form  $\alpha$  defining  $\xi$  satisfies  $\alpha \wedge d\alpha \neq 0$ . The *standard contact structure*  $\xi_{\text{std}}$  on  $\mathbb{R}^3$  is given by  $\text{span}\{\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\}$ , which is also given by the kernel of the 1-form  $\alpha = dz - ydx$  defined on  $\mathbb{R}^3$ .

A *transverse link*  $T$  in  $(\mathbb{R}^3, \xi_{\text{std}})$  is a union of embedded circles that are always transverse to  $\xi$ , that is,  $T_x T \oplus \xi_x = T_x M$  for every point  $x$  on  $T$ . A *Legendrian link*  $L$  in  $(\mathbb{R}^3, \xi_{\text{std}})$  is a union of embedded circles that are always tangent to  $\xi$ , that is,  $T_x L \subset \xi_x$  for every point  $x$  on  $L$ .

Let  $T^2$  denote a standardly embedded torus in  $S^3$ , that is,  $T^2$  gives a genus one Heegaard splitting  $S^3 = V_0 \cup_{T^2} V_1$  of  $S^3$ , where each of  $V_0$  and  $V_1$  is a solid torus. Let  $\mu$  be the unique closed curve on  $T^2$  that bounds a disc in  $V_0$ , and  $\lambda$  the unique closed curve on  $T^2$  that bounds a disc in  $V_1$ . Give an orientation to  $\mu$  arbitrarily, and give an orientation to  $\lambda$  so that  $\mu$  and  $\lambda$  form a positive basis for  $H_1(T^2)$ , where  $T^2$  is oriented as the boundary of  $V_0$ . Up to homotopy, every closed curve on  $T^2$  can be described as  $p\mu + q\lambda$ , where  $p$  and  $q$  are integers. If this closed curve is embedded on  $T^2$ , then the closed curve is denoted by  $T_{(p,q)}$ . If  $p$  and  $q$  are mutually prime, then  $T_{(p,q)}$  is a *torus knot of type  $(p, q)$* . Placing  $\{\infty\} \in S^3 = \mathbb{R}^3 \cup \{\infty\}$  in the interior of  $V_1$ , we may assume that  $T_{(p,q)}$  is embedded in  $\mathbb{R}^3$ .

We prove the following theorem in §3.

**Theorem 1.1.** *The arc index of  $T_{(p,q)}$ ,  $\bar{\alpha}(T_{(p,q)})$ , is equal to  $|p| + |q|$ .*

After the completion of this work, Peter Cromwell informed the author that Elisabetta Beltrami independently had obtained Theorem 1.1 by similar methods as ours.

## 2. FRONT PROJECTION AND ARC PRESENTATION

Let  $L_\ell$  be a Legendrian link with its topological link type  $\mathcal{L}$ . Let  $\Pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be a projection given by  $(x, y, z) \mapsto (x, z)$ . The image of  $L_\ell$  under the projection  $\Pi$ ,  $\Pi(L_\ell)$ , is called the *front projection* of  $L_\ell$ , denoted by  $F_\ell$ . From a front projection  $F_\ell$ , the Legendrian link  $L_\ell$  can be reconstructed uniquely by using the differential equation  $y = \frac{dz}{dx}$ . Letting the over arc at each double point of  $F_\ell$  be the one with

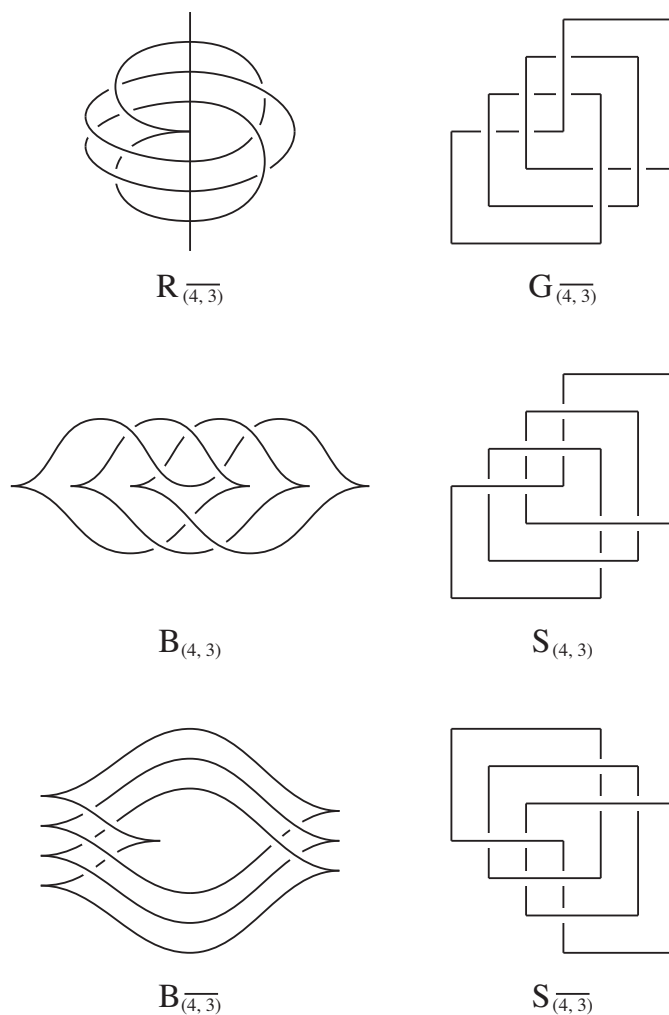


FIGURE 1.

the smaller slope  $\frac{dz}{dx} \in \mathbb{R} \setminus \{0\}$ , we obtain a *barber-pole diagram*  $B_\ell$ , which is a link diagram of the Legendrian link  $L_\ell$ ; cf. [9], [11]. In order to have the standard orientation on  $\mathbb{R}^3$ , we note that the positive  $y$ -axis goes into the page. Let  $w(F_\ell)$  be the writhe of the diagram  $B_\ell$ , and let  $rc(F_\ell)$  (respectively  $lc(F_\ell)$ ) be the number of right (resp. left) cusps of  $F_\ell$ . Note that  $rc(F_\ell) = lc(F_\ell)$ , and that the Thurston-Bennequin number of  $L_\ell$ ,  $tb(L_\ell)$ , is equal to  $w(F_\ell) - rc(F_\ell)$ .

Rotating  $B_\ell$  45 degrees counterclockwise, we may construct a diagram  $S_\ell$  in *square-bridge position*, defined by Lyon [10]. See the pair of diagrams in the middle of Figure 1. Each of these two diagrams represents a torus knot of type  $(4, 3)$ . The horizontal arc passes over the vertical arc at every crossing of the diagram  $S_\ell$ . It is observed by Rudolph [11] that the writhe of  $S_\ell$ ,  $w(S_\ell)$ , is equal to  $w(F_\ell)$ , and that one right (resp. left) cusp of  $F_\ell$  corresponds to one northeast (resp. southwest) corner of  $S_\ell$ , and vice versa.

Rotating  $S_\ell$  90 degrees counterclockwise, and switching the over/under information at every crossing, we obtain a diagram  $S_{\bar{\ell}}$  in square-bridge position. The diagram  $S_{\bar{\ell}}$  represents a link  $\bar{L}$  with its topological link type  $\bar{\mathcal{L}}$ . Rotating  $S_{\bar{\ell}}$  45 degrees clockwise, we may obtain a barber-pole diagram  $B_{\bar{\ell}}$  which represents a Legendrian link  $L_{\bar{\ell}}$  with its topological link type  $\bar{\mathcal{L}}$ . See the pair of diagrams at the bottom of Figure 1. Each of these two diagrams represents a torus knot of type  $(-4, 3)$ , the mirror image of a torus knot of type  $(4, 3)$ .

Let  $G_{\bar{\ell}}$  (resp.  $G_\ell$ ) be the diagram which is obtained from  $S_\ell$  (resp.  $S_{\bar{\ell}}$ ) by switching the over/under information at every crossing. So the vertical arc passes over the horizontal arc at every crossing of the diagrams  $G_{\bar{\ell}}$  and  $G_\ell$ . Note that  $G_{\bar{\ell}}$  (resp.  $G_\ell$ ) is also obtained from  $S_{\bar{\ell}}$  (resp.  $S_\ell$ ) by rotating it 90 degrees clockwise (resp. counterclockwise). We may regard  $G_{\bar{\ell}}$  (resp.  $G_\ell$ ) as a *rectangular diagram* of a link  $\bar{L}$  (resp.  $L$ ) with its topological link type  $\bar{\mathcal{L}}$  (resp.  $\mathcal{L}$ ); cf. [6], [7]. From  $G_{\bar{\ell}}$  (resp.  $G_\ell$ ), we may construct an arc presentation  $R_{\bar{\ell}}$  (resp.  $R_\ell$ ) of  $\bar{L}$  (resp.  $L$ ). See the pair of diagrams at the top of Figure 1. Each of these two diagrams represents a torus knot of type  $(-4, 3)$ . This construction shows that the number of vertical arcs of  $G_{\bar{\ell}}$  (resp.  $G_\ell$ ) is equal to the arc number  $\alpha(R_{\bar{\ell}})$  (resp.  $\alpha(R_\ell)$ ) of  $R_{\bar{\ell}}$  (resp.  $R_\ell$ ). Since  $G_{\bar{\ell}}$  (resp.  $G_\ell$ ) is a diagram of embedded circles in  $\mathbb{R}^3$ , the number of vertical arcs of  $G_{\bar{\ell}}$  (resp.  $G_\ell$ ) is equal to that of horizontal arcs of  $G_{\bar{\ell}}$  (resp.  $G_\ell$ ), and the number of corners of  $G_{\bar{\ell}}$  (resp.  $G_\ell$ ) is equal to that of the union of vertical arcs and horizontal arcs of  $G_{\bar{\ell}}$  (resp.  $G_\ell$ ). The above constructions of  $G_{\bar{\ell}}$  and  $G_\ell$  show that the number of vertical (resp. horizontal) arcs of  $G_{\bar{\ell}}$  is equal to that of horizontal (resp. vertical) arcs of  $G_\ell$ . It follows that  $\alpha(R_{\bar{\ell}}) = \alpha(R_\ell)$ , and that the number of corners of each of  $G_{\bar{\ell}}$  and  $G_\ell$  is equal to  $\alpha(R_{\bar{\ell}}) + \alpha(R_\ell) = 2\alpha(R_{\bar{\ell}}) = 2\alpha(R_\ell)$ .

Let  $\text{ne}(S_\ell)$  (resp.  $\text{nw}(S_\ell)$ ,  $\text{se}(S_\ell)$ ,  $\text{sw}(S_\ell)$ ) be the number of northeast (resp. northwest, southeast, southwest) corners of the diagram  $S_\ell$ . Since there is a one-to-one correspondence between right (resp. left) cusps of  $F_\ell$  and northeast (resp. southwest) corners of  $S_\ell$ , we have  $rc(F_\ell) = \text{ne}(S_\ell)$  and  $lc(F_\ell) = \text{sw}(S_\ell)$ . The equality  $\text{ne}(S_\ell) = \text{sw}(S_\ell)$  follows from the equality  $rc(F_\ell) = lc(F_\ell)$ . Since  $S_\ell$  is a diagram of embedded circles in  $\mathbb{R}^3$ , we obtain the equality  $\text{nw}(S_\ell) = \text{se}(S_\ell)$ . The equality  $\text{ne}(S_\ell) + \text{nw}(S_\ell) + \text{sw}(S_\ell) + \text{se}(S_\ell) = 2\alpha(R_\ell)$  follows from the observation that the number of corners of  $S_\ell$  is equal to that of  $G_\ell$ , which is equal to  $2\alpha(R_\ell)$ . Then the Thurston-Bennequin number  $tb(L_\ell)$  of  $L_\ell$  is equal to  $w(F_\ell) - rc(F_\ell) = w(S_\ell) - \text{ne}(S_\ell) = w(S_\ell) - \frac{\text{ne}(S_\ell) + \text{sw}(S_\ell)}{2}$ .

The diagram  $S_{\bar{\ell}}$  in square-bridge position corresponds to a Legendrian link  $L_{\bar{\ell}}$  with its topological link type  $\bar{\mathcal{L}}$ . Let  $F_{\bar{\ell}}$  be a front projection of  $L_{\bar{\ell}}$  obtained from  $S_{\bar{\ell}}$  by rotating it 45 degrees clockwise. The construction of  $S_{\bar{\ell}}$  shows that  $\text{ne}(S_\ell) = \text{nw}(S_{\bar{\ell}})$ ,  $\text{nw}(S_\ell) = \text{sw}(S_{\bar{\ell}})$ ,  $\text{sw}(S_\ell) = \text{se}(S_{\bar{\ell}})$  and  $\text{se}(S_\ell) = \text{ne}(S_{\bar{\ell}})$ , and that  $w(S_\ell) = -w(S_{\bar{\ell}})$ . Combining the above equalities, we obtain the following lemma.

**Lemma 2.1.** *The Thurston-Bennequin number  $tb(L_{\bar{\ell}})$  of  $L_{\bar{\ell}}$  is equal to  $-tb(L_\ell) - \alpha(R_\ell)$ .*

### 3. PROOF OF THEOREM 1.1

Changing the roles of  $T_{(-p,-q)}$  and  $T_{(p,q)}$ , and  $V_0$  and  $V_1$  defined in §1, if necessary, and noticing  $\bar{\alpha}(L) = \bar{\alpha}(\bar{L})$ , it suffices to prove the theorem for torus knots of type  $(p, q)$  with  $p > q > 0$ .

Let  $R_{(\overline{p},\overline{q})}$  be an arc presentation of a torus knot of type  $(-p, q)$  realizing its arc index, that is,  $\alpha(R_{(\overline{p},\overline{q})}) = \overline{\alpha}(T_{(-p,q)}) = \overline{\alpha}(T_{(p,q)})$ . Let  $G_{(\overline{p},\overline{q})}$  be a rectangular diagram of  $T_{(-p,q)}$  corresponding to  $R_{(\overline{p},\overline{q})}$ . Switching the over/under information of  $G_{(\overline{p},\overline{q})}$  at every crossing, we obtain a diagram  $S_{(p,q)}$  of  $T_{(p,q)}$  in square-bridge position.

Let  $L_{(p,q)}$  be a Legendrian knot corresponding to  $S_{(p,q)}$ . Theorem 4.1 and Lemma 4.8 in [8] show that  $L_{(p,q)}$  is obtained from the unique Legendrian torus knot of type  $(p, q)$  with its maximal Thurston-Bennequin number  $\overline{tb}(T_{(p,q)})$  by performing stabilizations  $n$  times for  $n \geq 0$ . Then the Thurston-Bennequin number  $tb(L_{(p,q)})$  of  $L_{(p,q)}$  is equal to  $\overline{tb}(T_{(p,q)}) - n = pq - p - q - n$ . We define  $S_{(\overline{p},\overline{q})}$ ,  $L_{(\overline{p},\overline{q})}$  and  $R_{(p,q)}$  in the same way as in §2. Note that the topological knot type of  $L_{(\overline{p},\overline{q})}$  is  $T_{(-p,q)}$ . Lemma 2.1 shows that the Thurston-Bennequin number  $tb(L_{(\overline{p},\overline{q})})$  of  $L_{(\overline{p},\overline{q})}$  is equal to  $-tb(L_{(p,q)}) - \alpha(R_{(p,q)}) = -tb(L_{(p,q)}) - \alpha(R_{(\overline{p},\overline{q})}) = -(pq - p - q - n) - \overline{\alpha}(T_{(p,q)})$ . A theorem of Etnyre and Honda (Theorem 4.1 in [8]) shows that the maximal Thurston-Bennequin number  $\overline{tb}(T_{(-p,q)})$  of a torus knot of type  $(-p, q)$  is equal to  $-pq$ , where  $p > q > 0$ . Therefore, we have the inequality  $tb(L_{(\overline{p},\overline{q})}) = -pq + p + q + n - \overline{\alpha}(T_{(p,q)}) \leq -pq$ , so we have  $\overline{\alpha}(T_{(p,q)}) \geq p + q + n$ .

As observed by Cromwell on p. 39 in [6], we have an arc presentation  $R'_{(p,q)}$  of  $T_{(p,q)}$  with  $\alpha(R'_{(p,q)}) = p + q$ . So we have  $\overline{\alpha}(T_{(p,q)}) \leq p + q$ . Thus, the above two inequalities show that  $n = 0$  and  $\overline{\alpha}(T_{(p,q)}) = p + q$ .

#### 4. SOME REMARKS

Let  $L_\ell$  be a Legendrian link in  $(\mathbb{R}^3, \xi_{\text{std}})$ . Each point of the intersection of  $L_\ell$  with the plane  $\{y = 0\}$  corresponds to a point on  $F_\ell$  where the tangent line to  $F_\ell$  is parallel to the  $x$ -axis, and each of these points of  $F_\ell$  corresponds to a corner of  $S_\ell$ . The *contact bridge number*  $b_\xi(L_\ell)$ , of a Legendrian link  $L_\ell$  is defined to be half of the number of intersections of  $L_\ell$  with the plane  $\{y = 0\}$ . The *contact bridge index*  $\overline{b}_\xi(\mathcal{L})$  is the minimum of  $b_\xi(L_\ell)$  among all Legendrian links  $L_\ell$  representing the topological link type  $\mathcal{L}$ . Note that the contact bridge index is a topological link invariant. The definitions of the arc index and the contact bridge index show the following.

**Proposition 4.1.** *For every topological link type  $\mathcal{L}$ , we have  $\overline{b}_\xi(\mathcal{L}) = \overline{\alpha}(\mathcal{L})$ .*

We make another remark on a relationship between knots in an open book decomposition and knots in the standard contact structure. Let  $K$  be an oriented knot. As described by Cromwell in Proposition on p. 34 of [6], two closed braid representatives  $K^+$  and  $-K^-$  of  $K$  are constructed from one arc presentation  $K^r$  of  $K$ , where  $-K^-$  denotes  $K^-$  with the opposite orientation. A closed braid representative  $K^+$  (resp.  $-K^-$ ) is obtained from  $K^r$  (resp.  $-K^r$ ) by tilting each arc so that the tilted arcs intersect the fibration  $\{A(\theta)\}$  positively, and by connecting these arcs with short arcs on  $\partial N(A)$ , each of which intersects  $\{A(\theta)\}$  positively. From one Legendrian knot  $K^\ell$ , a positive transverse knot  $T_+(K^\ell)$  and a negative transverse knot  $T_-(K^\ell)$  are constructed as positive and negative transverse push-offs of  $K^\ell$ . It is clear from these constructions that  $K^+$  corresponds to  $T_+(K^\ell)$ , and  $K^-$  to  $T_-(K^\ell)$ .

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