

EQUIVALENCE OF DOMAINS ARISING FROM DUALITY OF ORBITS ON FLAG MANIFOLDS II

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ABSTRACT. S. Gindikin and the author defined a $G_{\mathbb{R}}\text{-}K_{\mathbb{C}}$ invariant subset $C(S)$ of $G_{\mathbb{C}}$ for each $K_{\mathbb{C}}$ -orbit S on every flag manifold $G_{\mathbb{C}}/P$ and conjectured that the connected component $C(S)_0$ of the identity would be equal to the Akhiezer-Gindikin domain D if S is of nonholomorphic type. This conjecture was proved for closed S in the works of J. A. Wolf, R. Zierau, G. Fels, A. Huckleberry and the author. It was also proved for open S by the author. In this paper, we prove the conjecture for all the other orbits when $G_{\mathbb{R}}$ is of non-Hermitian type.

1. INTRODUCTION

Let $G_{\mathbb{C}}$ be a connected complex semisimple Lie group and $G_{\mathbb{R}}$ a connected real form of $G_{\mathbb{C}}$. Let $K_{\mathbb{C}}$ be the complexification in $G_{\mathbb{C}}$ of a maximal compact subgroup K of $G_{\mathbb{R}}$. Let $X = G_{\mathbb{C}}/P$ be a flag manifold of $G_{\mathbb{C}}$ where P is an arbitrary parabolic subgroup of $G_{\mathbb{C}}$. Then there exists a natural one-to-one correspondence between the set of $K_{\mathbb{C}}$ -orbits S and the set of $G_{\mathbb{R}}$ -orbits S' on X given by the condition

$$(1.1) \quad S \leftrightarrow S' \iff S \cap S' \text{ is nonempty and compact}$$

([M4]). For each $K_{\mathbb{C}}$ -orbit S we defined in [GM1] a subset $C(S)$ of $G_{\mathbb{C}}$ by

$$C(S) = \{x \in G_{\mathbb{C}} \mid xS \cap S' \text{ is nonempty and compact}\},$$

where S' is the $G_{\mathbb{R}}$ -orbit on X given by (1.1).

Akhiezer and Gindikin defined a domain $D/K_{\mathbb{C}}$ in $G_{\mathbb{C}}/K_{\mathbb{C}}$ as follows ([AG]). Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{m}$ denote the Cartan decomposition of $\mathfrak{g}_{\mathbb{R}} = \text{Lie}(G_{\mathbb{R}})$ with respect to K . Let \mathfrak{t} be a maximal abelian subspace in \mathfrak{im} . Put

$$\mathfrak{t}^+ = \{Y \in \mathfrak{t} \mid |\alpha(Y)| < \frac{\pi}{2} \text{ for all } \alpha \in \Sigma\},$$

where Σ is the restricted root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to \mathfrak{t} . Then D is defined by

$$D = G_{\mathbb{R}}(\exp \mathfrak{t}^+)K_{\mathbb{C}}.$$

We conjectured the following in [GM1].

Conjecture 1.1. (Conjecture 1.6 in [GM1]) *Suppose that $X = G_{\mathbb{C}}/P$ is not $K_{\mathbb{C}}$ -homogeneous. Then we will have $C(S)_0 = D$ for all $K_{\mathbb{C}}$ -orbits S of nonholomorphic type on X . Here $C(S)_0$ is the connected component of $C(S)$ containing the identity.*

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(See [GM1, M6] for the definition of the $K_{\mathbb{C}}$ -orbits of nonholomorphic type. When $G_{\mathbb{R}}$ is of non-Hermitian type, all the $K_{\mathbb{C}}$ -orbits are defined to be of nonholomorphic type.)

Let S_{op} denote the unique open $K_{\mathbb{C}}$ - B double coset in $G_{\mathbb{C}}$, where B is a Borel subgroup of $G_{\mathbb{C}}$ contained in P . It is shown in [H] and [M5] that $D \subset C(S_{\text{op}})_0$. (The opposite inclusion $D \supset C(S_{\text{op}})_0$ is proved in [B].) On the other hand the inclusion $C(S_{\text{op}})_0 \subset C(S)_0$ for every $K_{\mathbb{C}}$ -orbit S on $X = G_{\mathbb{C}}/P$ is shown in [GM1], Proposition 8.1 and Proposition 8.3. So we have the inclusion

$$(1.2) \quad D \subset C(S)_0.$$

We have only to prove the opposite inclusion.

For a simple root α with respect to B we can define a parabolic subgroup P_{α} by

$$P_{\alpha} = B \sqcup Bw_{\alpha}B,$$

where w_{α} is the reflection for the root α . Let S_0 be a closed $K_{\mathbb{C}}$ - B double coset in $G_{\mathbb{C}}$. Let S_1, \dots, S_{ℓ} ($\ell = \text{codim}_{\mathbb{C}} S_0$) be a sequence of $K_{\mathbb{C}}$ - B double cosets in $G_{\mathbb{C}}$ and $\alpha_1, \dots, \alpha_{\ell}$ a sequence of simple roots such that

$$S_k^{cl} = S_0 P_{\alpha_1} \cdots P_{\alpha_k}$$

and that

$$\dim_{\mathbb{C}} S_k = \dim_{\mathbb{C}} S_0 + k$$

for $k = 1, \dots, \ell$ (cf. [GM2], [M3], [Sp]). Especially $S_{\ell} = S_{\text{op}}$.

In this paper we first prove the following theorem.

Theorem 1.2. *Let x be an element of $G_{\mathbb{C}}$. If $I_0 = xS_0 \cap S'_{\text{op}}P_{\alpha_{\ell}} \cdots P_{\alpha_1}$ is connected, then*

$$I_k = xS_k^{cl} \cap S'_{\text{op}}P_{\alpha_{\ell}} \cdots P_{\alpha_{k+1}}$$

is connected for $k = 1, \dots, \ell$. (S'_{op} is the unique closed $G_{\mathbb{R}}$ - B double coset in $G_{\mathbb{C}}$ which corresponds to S_{op} by (1.1).)

Remark 1.3. The sets I_k ($k = 0, \dots, \ell$) are always nonempty because $xS_0P_{\alpha_1} \cdots P_{\alpha_{\ell}} = xS_{\text{op}}^{cl} = G_{\mathbb{C}} \supset S'_{\text{op}}$.

Let S be a $K_{\mathbb{C}}$ - P double coset in $G_{\mathbb{C}}$. Then we can write

$$S^{cl} = S_k^{cl} = S_0 P_{\alpha_1} \cdots P_{\alpha_k}$$

with some closed $K_{\mathbb{C}}$ - B double coset S_0 and a sequence $\alpha_1, \dots, \alpha_k$ of simple roots ([M3], [Sp]). Second we prove the following.

Theorem 1.4. (i) *If $x \in D^{cl}$, then I_k is connected.*

(ii) *If $x \in D^{cl} \cap C(S)$, then $I_k = xS \cap S'_k$.*

As a corollary we solve Conjecture 1.1 for non-Hermitian cases:

Corollary 1.5. *Let $G_{\mathbb{R}}$ be simple and of non-Hermitian type. Then $C(S)_0 = D$ for all the $K_{\mathbb{C}}$ -orbits $S \neq X$ on $X = G_{\mathbb{C}}/P$.*

Proof. When S is open in $G_{\mathbb{C}}$, the equality is proved in [M6]. So we may assume that S is not open. Let x be an element of $D^{cl} \cap C(S)$. Then we have only to show that $x \in D$. Since $S_k P_{\alpha_{k+1}} \cdots P_{\alpha_{\ell-1}} \cap S_{\text{op}} = \emptyset$, we have $S'_k P_{\alpha_{k+1}} \cdots P_{\alpha_{\ell-1}} \cap S'_{\text{op}} = \emptyset$ by the duality ([M2]), and therefore

$$S'_{\text{op}} P_{\alpha_{\ell-1}} \cdots P_{\alpha_{k+1}} \cap S'_k = \emptyset.$$

By Theorem 1.4(ii) we have

$$\begin{aligned} xS^{cl} \cap S'_{\text{op}} P_{\alpha_{\ell-1}} \cdots P_{\alpha_{k+1}} &= xS^{cl} \cap S'_{\text{op}} P_{\alpha_{\ell}} \cdots P_{\alpha_{k+1}} \cap S'_{\text{op}} P_{\alpha_{\ell-1}} \cdots P_{\alpha_{k+1}} \\ &= xS \cap S'_k \cap S'_{\text{op}} P_{\alpha_{\ell-1}} \cdots P_{\alpha_{k+1}} = \emptyset. \end{aligned}$$

Hence

$$xS_{\ell-1}^{cl} \cap S'_{\text{op}} = xS^{cl} P_{\alpha_{k+1}} \cdots P_{\alpha_{\ell-1}} \cap S'_{\text{op}} = \emptyset.$$

For the orbit $S_{\ell-1}$ we defined the following domain Ω in [GM2]:

$$\Omega = \{x \in G_{\mathbb{C}} \mid xS_{\ell-1}^{cl} \cap S'_{\text{op}} = \emptyset\}_0.$$

It is shown in [FH], Theorem 5.2.6 and [M6], Corollary 1.8 that

$$\Omega = D$$

when $G_{\mathbb{R}}$ is of non-Hermitian type. Hence $x \in D$. \square

Remark 1.6. Recently [M7] proved Conjecture 1.1 for all nonclosed $K_{\mathbb{C}}$ -orbits in Hermitian cases using Theorem 1.4. Thus the conjecture is now completely solved affirmatively.

2. $G_{\mathbb{R}}$ -ORBITS ON THE FULL FLAG MANIFOLD

The full flag manifold \mathcal{F} of $G_{\mathbb{C}}$ is the set of the Borel subgroups of $G_{\mathbb{C}}$. If we take a Borel subgroup B_0 of $G_{\mathbb{C}}$, then the factor space $G_{\mathbb{C}}/B_0$ is identified with \mathcal{F} by the map

$$G_{\mathbb{C}}/B_0 \ni gB_0 \mapsto gB_0g^{-1} \in \mathcal{F}.$$

It is known that every $G_{\mathbb{R}}$ -orbit ($G_{\mathbb{R}}$ -conjugacy class) on \mathcal{F} contains a Borel subgroup of the form

$$B = B(\mathfrak{j}, \Sigma^+) = \exp \left(\sum_{\alpha \in \Sigma^+ \sqcup \{0\}} \mathfrak{g}_{\mathbb{C}}(\mathfrak{j}, \alpha) \right),$$

where \mathfrak{j} is a θ -stable Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$, Σ^+ is a positive system of the root system Σ of the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ and $\mathfrak{g}_{\mathbb{C}}(\mathfrak{j}, \alpha) = \{X \in \mathfrak{g}_{\mathbb{C}} \mid [Y, X] = \alpha(Y)X \text{ for all } Y \in \mathfrak{j}\}$ ([A], [M1], [R]).

Roots in Σ are usually classified as follows.

- (i) If $\theta(\alpha) = \alpha$ and $\mathfrak{g}_{\mathbb{C}}(\mathfrak{j}, \alpha) \subset \mathfrak{k}_{\mathbb{C}}$, then α is called a “compact root”.
- (ii) If $\theta(\alpha) = \alpha$ and $\mathfrak{g}_{\mathbb{C}}(\mathfrak{j}, \alpha) \subset \mathfrak{m}_{\mathbb{C}}$, then α is called a “noncompact root”.
- (iii) If $\theta(\alpha) = -\alpha$, then α is called a “real root”.
- (iv) If $\theta(\alpha) \neq \pm\alpha$, then α is called a “complex root”.

For a simple root α of Σ^+ define the parabolic subgroup P_{α} as in Section 1. By the same arguments as in [V], Lemma 5.1, and [M3], Lemma 3, we can prove the following decomposition of $P_{\alpha}/B \cong P^1(\mathbb{C})$ into the $P_{\alpha} \cap G_{\mathbb{R}}$ -orbits.

Lemma 2.1. (i) If α is compact, then $P_{\alpha} = (P_{\alpha} \cap G_{\mathbb{R}})B$.

(ii) If α is noncompact or real, then $P_{\alpha}/B \cong P^1(\mathbb{C}) = \mathbb{C} \sqcup \{\infty\}$ is decomposed into the three $(P_{\alpha} \cap G_{\mathbb{R}})_0$ -orbits H_+ , H_- and H_0 which are diffeomorphic to the upper half plane, the lower half plane and $P^1(\mathbb{R}) = \mathbb{R} \sqcup \{\infty\}$, respectively. (Sometimes H_+ and H_- are in the same $P_{\alpha} \cap G_{\mathbb{R}}$ -orbit.)

(iii) If α is complex, then P_{α}/B is decomposed into the two $P_{\alpha} \cap G_{\mathbb{R}}$ -orbits consisting of a point yB and the complement $(P_{\alpha} - yB)/B$.

Remark 2.2. Concerning the $K_{\mathbb{C}}$ -action on $G_{\mathbb{C}}/B$, it is shown in [V], Lemma 5.1 (cf. [M3], Lemma 3, [GM1], Lemma 9.1), that:

- (i) If α is compact, then $P_{\alpha} = (P_{\alpha} \cap K_{\mathbb{C}})B$.
- (ii) If α is noncompact or real, then P_{α}/B is decomposed into three $(P_{\alpha} \cap K_{\mathbb{C}})_0$ -orbits consisting of two points and the complement.
- (iii) If α is complex, then P_{α}/B is decomposed into two $P_{\alpha} \cap K_{\mathbb{C}}$ -orbits consisting of a point and the complement.

As a corollary of Lemma 2.1 we have:

Corollary 2.3. *Let g be an arbitrary element of $G_{\mathbb{C}}$. Then every $(gP_{\alpha}g^{-1} \cap G_{\mathbb{R}})_0$ -invariant closed subset of gP_{α}/B is connected.*

Remark 2.4. On the contrary a $gP_{\alpha}g^{-1} \cap K_{\mathbb{C}}$ -invariant closed subset of gP_{α}/B may not be connected in view of Remark 2.2(ii).

3. PROOFS OF THE THEOREMS

Proof of Theorem 1.2. We will prove the theorem by induction on k . Suppose that I_{k-1} is connected. Then

$$\begin{aligned} I_{k-1}P_{\alpha_k} &= (xS_{k-1}^{cl} \cap S'_{\text{op}}P_{\alpha_{\ell}} \cdots P_{\alpha_k})P_{\alpha_k} \\ &= xS_k^{cl} \cap S'_{\text{op}}P_{\alpha_{\ell}} \cdots P_{\alpha_k} \\ &= (xS_k^{cl} \cap S'_{\text{op}}P_{\alpha_{\ell}} \cdots P_{\alpha_{k+1}})P_{\alpha_k} \\ &= I_kP_{\alpha_k} \end{aligned}$$

is connected. Suppose that $I_k = A_1 \sqcup A_2$ with some nonempty closed subsets A_1 and A_2 of I_k . Then we will get a contradiction. Since the Borel subgroup B is connected, A_1 and A_2 are right B -invariant. Since $A_1P_{\alpha_k}$ and $A_2P_{\alpha_k}$ are closed and since

$$A_1P_{\alpha_k} \cup A_2P_{\alpha_k} = I_kP_{\alpha_k}$$

is connected, we have $A_1P_{\alpha_k} \cap A_2P_{\alpha_k} \neq \emptyset$. Take an element g of $A_1P_{\alpha_k} \cap A_2P_{\alpha_k}$. Then $gP_{\alpha_k} \cap I_k$ is decomposed as

$$gP_{\alpha_k} \cap I_k = (gP_{\alpha_k} \cap A_1) \sqcup (gP_{\alpha_k} \cap A_2)$$

with two nonempty closed subsets $gP_{\alpha_k} \cap A_1$ and $gP_{\alpha_k} \cap A_2$. But this contradicts Corollary 2.3 because $gP_{\alpha_k} \cap I_k = gP_{\alpha_k} \cap S'_{\text{op}}P_{\alpha_{\ell}} \cdots P_{\alpha_{k+1}}$ is $gP_{\alpha_k}g^{-1} \cap G_{\mathbb{R}}$ -invariant. □

Lemma 3.1. (i) S_k is relatively closed in $S_{\text{op}}P_{\alpha_{\ell}} \cdots P_{\alpha_{k+1}}$.
 (ii) S'_k is relatively open in $S'_{\text{op}}P_{\alpha_{\ell}} \cdots P_{\alpha_{k+1}}$.

Proof. By the duality for the closure relation ([M3]) we have only to show (i). Let \tilde{S} be a $K_{\mathbb{C}}$ - B double coset contained in the boundary of S_k . Then

$$\text{codim}_{\mathbb{C}} \tilde{S} > \text{codim}_{\mathbb{C}} S_k = \ell - k.$$

Hence \tilde{S} cannot be contained in $S_{\text{op}}P_{\alpha_{\ell}} \cdots P_{\alpha_{k+1}}$ by [V], Lemma 5.1 (cf. [GM1], Lemma 9.1). □

Proof of Theorem 1.4. (i) Since S_0/B is compact and S'_0/B is open, we see that

$$\begin{aligned} C(S_0) &= \{x \in G_{\mathbb{C}} \mid xS_0 \cap S'_0 \text{ is nonempty and closed in } G_{\mathbb{C}}\} \\ &= \{x \in G_{\mathbb{C}} \mid xS_0 \subset S'_0\}. \end{aligned}$$

Hence $C(S_0)_0$ is the cycle space for S'_0 defined in [WW]. Since $D \subset C(S_0)_0$ by (1.2), it follows that

$$x \in D \implies xS_0 \subset S'_0.$$

Suppose that $x \in D^{cl}$. Then we have

$$xS_0 \subset S_0^{cl} \subset S'_{\text{op}} P_{\alpha_\ell} \cdots P_{\alpha_1}$$

and hence $I_0 = xS_0$ is connected. By Theorem 1.2 the intersection

$$I_k = xS^{cl} \cap S'_{\text{op}} P_{\alpha_\ell} \cdots P_{\alpha_{k+1}}$$

is connected.

(ii) By Lemma 3.1 S'_k is relatively open in $S'_{\text{op}} P_{\alpha_\ell} \cdots P_{\alpha_{k+1}}$. On the other hand S is also relatively open in S^{cl} . Hence

$$(3.1) \quad xS \cap S'_k \text{ is relatively open in } I_k = xS^{cl} \cap S'_{\text{op}} P_{\alpha_\ell} \cdots P_{\alpha_{k+1}}.$$

Suppose that $x \in C(S)$. Then $xS \cap S'$ is nonempty and closed in $G_{\mathbb{C}}$ by definition. Since S'_k is relatively closed in S' , it follows that

$$(3.2) \quad xS \cap S'_k \text{ is closed in } G_{\mathbb{C}}.$$

Since $xS \cap S' = (xS \cap S'_k)P$, it also follows that

$$(3.3) \quad xS \cap S'_k \text{ is nonempty.}$$

Suppose moreover that $x \in D^{cl}$. Then I_k is connected by (i). Hence it follows from (3.1), (3.2) and (3.3) that

$$I_k = xS \cap S'_k. \quad \square$$

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