A SHORT PROOF OF A CONJECTURE ON THE CONNECTIVITY OF GRAPH COLORING COMPLEXES

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Abstract. The $\text{Hom}$-complexes were introduced by Lovász to study topological obstructions to graph colorings. It was conjectured by Babson and Kozlov, and proved by Ćukić and Kozlov, that $\text{Hom}(G, K_n)$ is $(n - d - 2)$-connected, where $d$ is the maximal degree of a vertex of $G$, and $n$ the number of colors. We give a short proof of the conjecture.

Introduction

It was conjectured by Babson and Kozlov [1], and proved by Ćukić and Kozlov [4], that $\text{Hom}(G, K_n)$ is $(n - d - 2)$-connected, where $d$ is the maximal degree of a vertex of $G$, and $n$ the number of colors. We give a shorter proof of this, by generalizing the proof of that $\text{Hom}(K_m, K_n)$ is $(n - m - 1)$-connected in Babson and Kozlov [1].

For definitions and basic theorems on $\text{Hom}$-complexes used in this text, see the papers mentioned above, or the survey by Kozlov [6].

1. An analogue of the chromatic number

An independent subset of vertices of a graph is a set, such that no vertices of it are adjacent. The minimal number of sets needed to partition the vertex set of a graph $G$ into independent sets is the chromatic number $\chi(G)$.

Definition 1.1. A covering $I_1, I_2, \ldots, I_k$ of $G$ is a sequence of independent subsets of $V(G)$ such that they partition $V(G)$, and $I_i$ is a maximal independent set in the induced subgraph of $G$ with vertex set $I_i \cup I_{i+1} \cup \ldots \cup I_k$, for all $i$, where $1 \leq i \leq k$.

A partition of $G$ into $\chi(G)$ independent sets can always be transformed to a covering by ordering the independent sets and if needed enlarging them. But a covering can use more than $\chi(G)$ sets. Define $\hat{\chi}(G)$ to be the maximal number of sets in a covering of $G$. Clearly, $\hat{\chi}(G) \geq \chi(G)$.

Lemma 1.2. If $d$ is the maximal degree of a vertex of $G$, then $\hat{\chi}(G) \leq d + 1$. 

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Proof. Let $I_1, I_2, \ldots, I_{\tilde{\chi}(G)}$ be a covering of $G$, and $v \in I_{\tilde{\chi}(G)}$. For each $i$, where $1 \leq i < \tilde{\chi}(G)$, there is a $w \in I_i$ adjacent to $v$, because otherwise $I_i$ would not be a maximal independent set. Hence the degree of $v$ is at least $\tilde{\chi}(G) - 1$. The degree of $v$ is at most $d$, thus $\tilde{\chi}(G) \leq d + 1$. \qed

Lemma 1.3. If $H$ is an induced subgraph of $G$, then $\tilde{\chi}(H) \leq \tilde{\chi}(G)$.

Proof. It suffices to prove this when $H$ and $G$ only differ by a vertex $v$ of $G$. Let $I_1, I_2, \ldots, I_{\tilde{\chi}(H)}$ be a covering of $H$. If $v$ is adjacent to a vertex in each of the sets $I_i$, then $\left\{v\right\}, I_1, I_2, \ldots, I_{\tilde{\chi}(H)}$ is a covering of $G$ and $\tilde{\chi}(H) + 1 \leq \tilde{\chi}(G)$. Otherwise, let $I_I$ be the first set in the covering such that $v$ is not adjacent to any vertex of $I_j$. Then $I_1, I_2, \ldots, I_I \cup \left\{v\right\}, \ldots, I_{\tilde{\chi}(H)}$ is a covering of $G$, and $\tilde{\chi}(H) \leq \tilde{\chi}(G)$. \qed

Lemma 1.4. If $I$ is a maximal independent set of $G$, then $\tilde{\chi}(G) > \tilde{\chi}(G \setminus I)$.

Proof. Let $I_1, I_2, \ldots, I_{\tilde{\chi}(G \setminus I)}$ be a covering of $G \setminus I$. Then $I, I_1, I_2, \ldots, I_{\tilde{\chi}(G \setminus I)}$ is a covering of $G$ with $1 + \tilde{\chi}(G \setminus I)$ sets. \qed  

2. Higher connectivity of $\mathrm{Hom}(G, K_n)$

Lemma 2.1. If $I$ is an independent set of $G$, and $I' \subset I$, then $\Delta = \left\{ \eta \in \mathrm{Hom}(G, K_n) \mid n \notin \eta(i) \Rightarrow i \in I \right\}$ collapses onto $\Delta' = \left\{ \eta \in \mathrm{Hom}(G \setminus \left( I \setminus I' \right), K_n) \mid n \in \eta(i) \Rightarrow i \in I' \right\}$.

Proof. It suffices to prove this when $I \setminus I' = \left\{v\right\}$. Let $\eta_1, \eta_2, \ldots, \eta_k$ be an ordering of $\left\{ \eta \in \Delta \mid n \notin \eta(v) \right\}$ such that if $\eta(w) \geq \eta'(w)$, for all $w \in V(G)$, then $\eta$ is not after $\eta'$. Define $\eta_i^* = \eta_i(w) = \eta_i(v)$ for $w \neq v$, and $\eta_i^*(v) = \eta_i(v) \cup \left\{n\right\}$. Each successive removal of $\eta_i^*$ together with $\eta_i$ from $\Delta$ for $i = 1, 2, \ldots, k$ is a collapse step. The cells left are $\Delta'' = \left\{ \eta \in \Delta \mid \eta(v) = \left\{n\right\} \right\}$. Finally, there is a bijection between the face posets of $\Delta'$ and $\Delta''$ by extending each $\eta \in \Delta'$ with $\eta(v) = \left\{n\right\}$.

The main use of Lemma 2.1 is when $I' = \emptyset$. Then $n \notin \eta(w)$ for all $\eta \in \Delta'$ and $w \in V(G) \setminus I$, so $\Delta' = \mathrm{Hom}(G \setminus I, K_{n-1})$. Another way to prove the lemma is to use discrete Morse theory [3].

Lemma 2.2 (Nerve Lemma, [2] Theorem 10.6(ii), [3]). Let $\Delta$ be a regular cell complex, and $\left(\Delta_j\right)_{j \in J}$ a family of subcomplexes such that $\Delta = \bigcup_{j \in J} \Delta_j$, and every nonempty finite intersection $\Delta_{j_1} \cap \Delta_{j_2} \cap \cdots \cap \Delta_{j_t}$ is $(m - t + 1)$-connected. Then $\Delta$ is $m$-connected if and only if the nerve $\mathcal{N}(\Delta_j)$ is $m$-connected.

Assume that $m \geq 0$. We will construct a family of subcomplexes such that $\Delta = \bigcup_{j \in J} \Delta_j$, all $\Delta_j$ are $m$-connected, and every intersection $\Delta_{j_1} \cap \Delta_{j_2} \cap \cdots \cap \Delta_{j_t}$ is $(m - t + 1)$-connected for $t \geq 2$. If a complex is $(m - 1)$-connected, then it is $(m - t + 1)$-connected for $t \geq 2$. Since all intersections are nonempty, the nerve is a simplex, and $\Delta$ is $m$-connected.

Theorem 2.3. $\mathrm{Hom}(G, K_n)$ is $(n - \tilde{\chi}(G) - 1)$-connected.

Proof. We use induction on $\tilde{\chi}(G)$ and on $n - \tilde{\chi}(G)$. When $\tilde{\chi}(G) = 1$, $G$ has no edges, so $\mathrm{Hom}(G, K_n)$ is contractible, and in particular, $(n - \tilde{\chi}(G) - 1)$-connected. If $n - \tilde{\chi}(G) = 0$, then $n \geq \tilde{\chi}(G)$ so $\mathrm{Hom}(G, K_n)$ is nonempty, and $(n - \tilde{\chi}(G) - 1)$-connected.

For all $I \in \mathcal{I}$, let $\Delta_j = \left\{ \eta \in \mathrm{Hom}(G, K_n) \mid n \in \eta(i) \Rightarrow i \in I \right\}$, where $\mathcal{I}$ is the family of maximal independent subsets of $G$. Clearly, $\mathrm{Hom}(G, K_n) = \bigcup_{I \in \mathcal{I}} \Delta_j$. By
Lemma 2.1, the complex $\Delta_I$ is homotopy equivalent to $\text{Hom}(G \setminus I, K_{n-1})$, which is $((n-1) - (\chi(G) - 1) - 1)$-connected by Lemma 1.4 and induction. If $I \supseteq I' \neq \emptyset$, then $\bigcap_{I \in I'} \Delta_I = \{ \eta \in \text{Hom}(G, K_n) \mid n \in \eta(i) \Rightarrow i \in \bigcap_{I \in I'} I \}$ is homotopy equivalent to $\text{Hom}(G \setminus (\bigcap_{I \in I'} I), K_{n-1})$ by Lemma 2.1 and $((n - 1) - \chi(G) - 1)$-connected by Lemma 1.3 and induction. By the Nerve Lemma we are done. \hfill $\square$

**Corollary 2.4.** $\text{Hom}(G, K_n)$ is $(n - d - 2)$-connected.

**Proof.** Lemma 1.2 states that $\chi(G) \leq d + 1$. \hfill $\square$

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