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A SHORT PROOF OF A CONJECTURE ON THE CONNECTIVITY OF GRAPH COLORING COMPLEXES

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Abstract. The Hom–complexes were introduced by Lovász to study topological obstructions to graph colorings. It was conjectured by Babson and Kozlov, and proved by Čukić and Kozlov, that $\mathtt{Hom}(G, K_n)$ is $(n-d-2)$–connected, where d is the maximal degree of a vertex of G, and n the number of colors. We give a short proof of the conjecture.

Introduction

It was conjectured by Babson and Kozlov [1], and proved by Čukić and Kozlov [4], that $\mathtt{Hom}(G, K_n)$ is $(n-d-2)$–connected, where d is the maximal degree of a vertex of G, and n the number of colors. We give a shorter proof of this, by generalizing the proof of that $\mathtt{Hom}(K_m, K_n)$ is $(n-m-1)$–connected in Babson and Kozlov [1].

For defintions and basic theorems on Hom–complexes used in this text, see the papers mentioned above, or the survey by Kozlov [6].

1. An analogue of the chromatic number

An independent subset of vertices of a graph is a set, such that no vertices of it are adjacent. The minimal number of sets needed to partition the vertex set of a graph G into independent sets is the chromatic number $\chi(G)$.

Definition 1.1. A *covering* $I_1, I_2, \ldots, I_k$ of G is a sequence of independent subsets of $V(G)$ such that they partition $V(G)$, and I_i is a maximal independent set in the induced subgraph of G with vertex set $I_i \cup I_{i+1} \cup \ldots \cup I_k$, for all i, where $1 \leq i \leq k$.

A partition of G into $\chi(G)$ independent sets can always be transformed to a covering by ordering the independent sets and if needed enlarging them. But a covering can use more than $\chi(G)$ sets. Define $\dot{\chi}(G)$ to be the maximal number of sets in a covering of G. Clearly, $\dot{\chi}(G) \geq \chi(G)$.

Lemma 1.2. *If d is the maximal degree of a vertex of G, then $\dot{\chi}(G) \leq d + 1$.*

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Proof. Let $I_1, I_2, \dots, I_{\dot\chi(G)}$ be a covering of G, and $v \in I_{\dot\chi(G)}$. For each i, where $1 \le i < \dot\chi(G)$, there is a $w \in I_i$ adjacent to v, because otherwise I_i would not be a maximal independent set. Hence the degree of v is at least $\dot\chi(G) - 1$. The degree of v is at most d, thus $\dot\chi(G) \le d + 1$. □

Lemma 1.3. *If H is an induced subgraph of G, then $\dot\chi(H) \le \dot\chi(G)$.*

Proof. It suffices to prove this when H and G only differ by a vertex v of G. Let $I_1, I_2, \dots, I_{\dot\chi(H)}$ be a covering of H. If v is adjacent to a vertex in each of the sets I_i, then $\{v\}, I_1, I_2, \dots, I_{\dot\chi(H)}$ is a covering of G and $\dot\chi(H) + 1 \le \dot\chi(G)$. Otherwise, let I_j be the first set in the covering such that v is not adjacent to any vertex of I_j. Then $I_1, I_2, \dots, I_j \cup \{v\}, \dots, I_{\dot\chi(H)}$ is a covering of G, and $\dot\chi(H) \le \dot\chi(G)$. □

Lemma 1.4. *If I is a maximal independent set of G, then $\dot\chi(G) > \dot\chi(G \setminus I)$.*

Proof. Let $I_1, I_2, \dots, I_{\dot\chi(G\setminus I)}$ be a covering of $G \setminus I$. Then $I, I_1, I_2, \dots, I_{\dot\chi(G\setminus I)}$ is a covering of G with $1 + \dot\chi(G \setminus I)$ sets. □

2. Higher connectivity of $\mathtt{Hom}(G, K_n)$

Lemma 2.1. *If I is an independent set of G, and $I' \subset I$, then $\Delta = \{\eta \in \mathtt{Hom}(G, K_n) \mid n \in \eta(i) \Rightarrow i \in I\}$ collapses onto $\Delta' = \{\eta \in \mathtt{Hom}(G \setminus (I \setminus I'), K_n) \mid n \in \eta(i) \Rightarrow i \in I'\}$.*

Proof. It suffices to prove this when $I \setminus I' = \{v\}$. Let $\eta_1, \eta_2, \dots, \eta_k$ be an ordering of $\{\eta \in \Delta \mid n \notin \eta(v)\}$ such that if $\eta(w) \supseteq \eta'(w)$, for all $w \in V(G)$, then η is not after η'. Define η_i^* as $\eta_i^*(w) = \eta_i(w)$ for $w \neq v$, and $\eta_i^*(v) = \eta_i(v) \cup \{n\}$. Each successive removal of η_i^* together with η_i from Δ for $i = 1, 2, \dots, k$ is a collapse step. The cells left are $\Delta'' = \{\eta \in \Delta \mid \eta(v) = \{n\}\}$. Finally, there is a bijection between the face posets of Δ' and Δ'' by extending each $\eta \in \Delta'$ with $\eta(v) = \{n\}$. □

The main use of Lemma 2.1 is when $I' = \emptyset$. Then $n \notin \eta(w)$ for all $\eta \in \Delta'$ and $w \in V(G) \setminus I$, so $\Delta' = \mathtt{Hom}(G \setminus I, K_{n-1})$. Another way to prove the lemma is to use discrete Morse theory [5].

Lemma 2.2 (Nerve Lemma, [2, Theorem 10.6(ii)], [3])**.** *Let Δ be a regular cell complex, and $(\Delta_j)_{j\in J}$ a family of subcomplexes such that $\Delta = \bigcup_{j\in J} \Delta_j$ and every nonempty finite intersection $\Delta_{j_1} \cap \Delta_{j_2} \cap \dots \cap \Delta_{j_t}$ is $(m - t + 1)$–connected. Then Δ is m–connected if and only if the nerve $\mathcal{N}(\Delta_j)$ is m–connected.*

Assume that $m \ge 0$. We will construct a family of subcomplexes such that $\Delta = \bigcup_{j\in J} \Delta_j$, all Δ_j are m–connected, and every intersection $\Delta_{j_1} \cap \Delta_{j_2} \cap \dots \cap \Delta_{j_t}$ is $(m - 1)$–connected for $t \ge 2$. If a complex is $(m - 1)$–connected, then it is $(m - t + 1)$–connected for $t \ge 2$. Since all intersections are nonempty, the nerve is a simplex, and Δ is m–connected.

Theorem 2.3. *$\mathtt{Hom}(G, K_n)$ is $(n - \dot\chi(G) - 1)$–connected.*

Proof. We use induction on $\dot\chi(G)$ and on $n - \dot\chi(G)$. When $\dot\chi(G) = 1$, G has no edges, so $\mathtt{Hom}(G, K_n)$ is contractible, and in particular, $(n - \dot\chi(G) - 1)$–connected. If $n - \dot\chi(G) = 0$, then $n \ge \chi(G)$ so $\mathtt{Hom}(G, K_n)$ is nonempty, and $(n - \dot\chi(G) - 1)$–connected.

For all $I \in \mathcal{I}$, let $\Delta_I = \{\eta \in \mathtt{Hom}(G, K_n) \mid n \in \eta(i) \Rightarrow i \in I\}$, where $\mathcal{I}$ is the family of maximal independent subsets of G. Clearly, $\mathtt{Hom}(G, K_n) = \bigcup_{I\in\mathcal{I}} \Delta_I$. By

Lemma 2.1, the complex Δ_I is homotopy equivalent to $\mathtt{Hom}(G \setminus I, K_{n-1})$, which is $((n-1)-(\dot{\chi}(G)-1)-1)$–connected by Lemma 1.4 and induction. If $\mathcal{I} \supseteq \mathcal{I}' \neq \emptyset$, then $\bigcap_{I \in \mathcal{I}'} \Delta_I = \{\eta \in \mathtt{Hom}(G, K_n) \mid n \in \eta(i) \Rightarrow i \in \bigcap_{I \in \mathcal{I}'} I\}$ is homotopy equivalent to $\mathtt{Hom}(G \setminus (\bigcap_{I \in \mathcal{I}'} I), K_{n-1})$ by Lemma 2.1, and $((n-1) - \dot{\chi}(G) - 1)$–connected by Lemma 1.3 and induction. By the Nerve Lemma we are done. □

Corollary 2.4. $\mathtt{Hom}(G, K_n)$ *is* $(n-d-2)$*–connected.*

Proof. Lemma 1.2 states that $\dot{\chi}(G) \leq d+1$. □

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